Low-Discrepancy Curves and Efficient Coverage of Space

Subramanian Ramamoorthy[1,3], Ram Rajagopal[2], Qing Ruan[3], and Lothar Wenzel[3]

[1] Department of Electrical and Computer Engineering, The University of Texas at Austin, s.ramamoorthy@mail.utexas.edu

[2] Department of Electrical Engineering and Computer Science, The University of California at Berkeley, ramr@eecs.berkeley.edu

[3] National Instruments Corp., {qing.ruan,lothar.wenzel}@ni.com

Abstract: We introduce the notion of low-discrepancy curves and use it to solve the problem of optimally covering space. In doing so, we extend the notion of low-discrepancy sequences in such a way that sufficiently smooth curves with low discrepancy properties can be defined and generated. Based on a class of curves that cover the unit square in an efficient way, we define induced low discrepancy curves in Riemannian spaces. This allows us to efficiently cover an arbitrarily chosen abstract surface that admits a diffeomorphism to the unit square. We demonstrate the application of these ideas by presenting concrete examples of low-discrepancy curves on some surfaces that are of interest in robotics.

1 Introduction

Uniform coverage of space is an important requirement in several applications in robotics and allied areas involving motion planning. As noted in a recent survey article [1], coverage path planning is critical in applications such as robotic de-mining, spray painting, machine milling and non-destructive evaluation of complex industrial parts, to name just a few.

In a more general setting, uniform coverage is a natural requirement in problems involving search in abstract spaces. Common abstract spaces of interest include configuration spaces of mechanical systems, parameter spaces such as the space of coefficients of a rational transfer function or a probability distribution, etc. Often, one is interested in generating continuous curves that cover such spaces. This is a natural requirement for physical robots that move in a continuous world. However, such a need also arises in several other settings involving, e.g., continuous adaptation and learning in a dynamical system.

Motivated by such application needs, we are investigating the problem of generating low-discrepancy curves that provide an assurance of uniform coverage of space in an incremental setting, such that the length of the search path correlates well with the quality of coverage.

Our approach to solving this problem involves two steps. First, we generate curves that uniformly and incrementally cover a model space, such as the unit square. We generalize the well established theory of low-discrepancy sequences in such a way that sufficiently smooth curves with low discrepancy properties can be defined and generated. In addition to the types of curves that we present in this paper, one may also tap into a sizeable literature on ergodic theory [2] to construct alternate curves with different coverage properties. Based on such curves, induced low discrepancy curves in Riemannian spaces may be constructed. This is achieved through the definition and determination of an area and fairness-preserving diffeomorphism. Given a suitable parametrization of the space to be covered, this procedure yields a curve that can cover it uniformly, optimally in a low-discrepancy sense, and incrementally. This second step ensures that our algorithm is applicable in a wide variety of applications, requiring only that we have a description of the abstract space in the form of a suitable Riemannian metric.

Low discrepancy point sets and sequences [3] have a successful history within robotics. They have been successfully used in sampling based motion planning and area coverage applications. This work has been covered well in the past proceedings of the Workshop on the Algorithmic Foundations of Robotics, so we will not extensively survey it here. [4] contains an excellent discussion on the use of low-discrepancy sampling techniques in motion planning. In more recent work, [5], [6], techniques have been proposed for generating sequences in an incremental fashion, which is often a very important requirement. However, on the one hand, the generation of these sequences is based on a computationally expensive search for an optimal ordering [5] while on the other hand, even though some of these computational efficiency problems may be addressed by better algorithm design [6], we often seek the stronger result of a continuous curve in an abstract space.

The notion of using a diffeomorphism to induce a low-discrepancy curve in the abstract space bears a methodological resemblance to some prior work in robotics, e.g., [7], where sphere worlds are mapped to arbitrary convex spaces. However, we are not aware of prior attempts to define diffeomorphisms that address fairness of space coverage. This is crucial for our work.

The plan of the rest of the paper is as follows. In section 2, we begin with an overview of low-discrepancy sets and sequences. In section 3 we generalize the idea of low-discrepancy sequences to low-discrepancy curves. We show that such curves do exist. Section 4 deals with low-discrepancy curves on abstract surfaces and in Riemannian spaces. These curves can be derived from low-discrepancy curves in unit cubes. In Section 5, we apply the developed methods to the problem of scanning various surfaces. In a certain sense to be defined, we will show that the proposed scanning procedures are optimal. Finally, we conclude with some comments regarding future directions and open questions.

2 On the Notion of Low-Discrepancy Point Sets and Sequences

The definition of discrepancy of a finite set X was introduced to quantify the homogeneity of finite-dimensional point sets [8]:

$$D(X) = \sup_R |m(R) - p(R)| \tag{1}$$

In equation (1), R runs over all d-dimensional rectangles $[0, r]^d$ with $0 \leq r \leq 1$, $m(R)$ stands for the Lebesgue measure of R and $p(R)$ is the ratio of the number of points of X in R and the number of all points of X. The lower the discrepancy the better or more homogeneous is the distribution of the point set. The discrepancy of an infinite sequence $X = \{x_1, x_2, x_3, ..., x_n, ...\}$ is a new sequence of positive real numbers $D(X_n)$, where X_n stands for the first n elements of X.

There exists a point set of given length that realizes the lowest discrepancy. It is known (the Roth bound [9]) that the following inequality holds true for all finite sequences X_n of length n in the d-dimensional unit cube.

$$D(X_n) \geq B_d \frac{(\log n)^{\frac{d-1}{2}}}{n} \tag{2}$$

B_d depends only on d. Except for the trivial case $d = 1$, it is unknown whether the theoretical lower bound is attainable. Many schemes to build finite sequences X_n of length n do exist that deliver a slightly worse limit,

$$D(X_n) \geq B_d \frac{(\log n)^d}{n} \tag{3}$$

There are also infinite sequences X with the above lower bound, equation (3), for all subsequences consisting of the first n elements. The latter result leads to the definition of low-discrepancy infinite sequences X. The inequality (3) must be valid for all sub-sequences of the first n elements, where B_d is an appropriate constant.

Many low-discrepancy sequences in d-dimensional unit cubes can be constructed as combinations of 1-dimensional low-discrepancy sequences. Popular low-discrepancy sequences are based on schemes introduced by Corput [10], Halton [11], Sobol [12], and Niederreiter [8].

One of the primary motivations for investigations into these sequences arises from high-dimensional function approximation and Monte-Carlo integration. In this setting, there is a well-known [8] relationship between integrals I, approximations I_n, and an infinite sequence $X = \{x_1, x_2, ..., x_n, ...\}$ in d-dimensions, known as the Koksma-Hlawka inequality.

$$|I(f) - I_n(f)| \leq V(f) D(X_n) \tag{4}$$

$$I(f) = \int_0^1 f(x)dx \tag{5}$$

$$I_n(f) = \frac{1}{n} \sum_{i=1}^{n} f(x_i) \tag{6}$$

where $V(f)$ is the variation of the function in the sense of Hardy and Krause.

3 Low-Discrepancy Curves in the Unit Square

One of the earliest known quasi-random sequences is the Richtmyer sequence [13], [14], which illustrates a simple but general result in ergodic dynamics [15], [16]. Let $x_n = \{n\alpha\}$ (i.e., $[n\alpha] \mod 1$) and $X = \{x_1, x_2, ..., x_n, ...\}$, where $\alpha = (\alpha_1, ..., \alpha_d)$ is irrational and $\alpha_1, ..., \alpha_d$ are linearly independent over the rational numbers. Then for almost all α in $\Re^d$ and for all positive ϵ, with exception of a set of points that has zero Lebesgue measure,

$$D(X_n) = O(\frac{\log^{d+1+\epsilon} n}{n}) \tag{7}$$

The Richtmeyer sequence is probably the only quasi-random sequence based on a linear congruential algorithm [17]. This is useful because it suggests a natural extension to the generation of curves. We will now provide such an extension.

Let C be a given piecewise smooth and finite curve in the unit square S. Furthermore, let R be an arbitrary aligned rectangle in S with lower left corner $(0, 0)$. Let L be the length of the given curve in S and l be the length of the sub-curve of C that lies in R. In case of well-distributed curves, the ratio l/L should represent the area $A(R)$ of R reasonably well. This gives rise to the following definition of discrepancy of a given finite piecewise smooth curve in S:

$$D(C) = \sup_R |\frac{l}{L} - \frac{A(R)}{A(S)}| \tag{8}$$

It would be desirable to construct curves C with the property that the discrepancy is always small. More precisely, we will call an infinite and piecewise sufficiently smooth curve $C : \Re^+ \mapsto S$, in natural parametrization, a low-discrepancy curve if for all positive arc lengths L the curves $C_L = C/[0, L]$ satisfy the inequalities (the function F must be defined appropriately):

$$D(C_L) \leq F(L) \tag{9}$$

In fact, a piecewise smooth curve in natural parametrization generates sequences $\{x_1, x_2, ..., x_n, ...\}$ by setting $x_n = C_n(n\Delta)$ where Δ is a fixed positive number. The inequality 9 lets us hope for a similar formula for the derived sequence $\{x_1, x_2, ..., x_n, ...\}$. Because of equation 7, a realistic goal is:

$$F(L) = O(\frac{\log^{3+\epsilon} L}{L}), d = 2 \tag{10}$$

We have to show that this goal is attainable.

To this end, for $\alpha = (\alpha_1, \alpha_2)$, let $C_{\mathcal{A}}(\alpha)$ be the piecewise linear curve $(t\alpha_1 \mod 1, t\alpha_2 \mod 1) = (\{t\alpha_1\}, \{t\alpha_2\})$ where t is in $\Re^+$.

In fact, we can define three classes of curves in the unit square, as shown in figure 1. For the simplest type of curves, let us call this class $C_{\mathcal{A}}$, the right-left and top-bottom edges are identified so that the curve jumps from one edge to the other upon hitting it. In this scheme, all curves are parallel and continue indefinitely when the square tiles the plane. In the second type of curves, class $C_{\mathcal{B}}$, we introduce reflections at the top and bottom edges but preserve the same identification between right and left edges. The third type of curves, class $C_{\mathcal{C}}$, involves reflections on all edges. The latter curve is continuous.

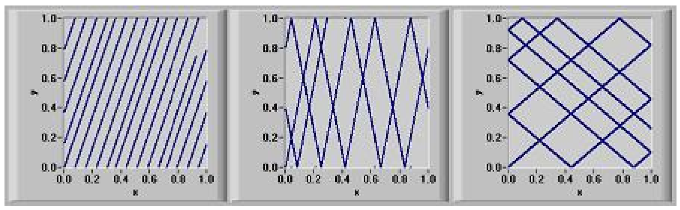


Fig. 1. Various low-discrepancy curves for the unit square: $C_{\mathcal{A}}, C_{\mathcal{B}}, C_{\mathcal{C}}$ from left to right.

Theorem 1. *For almost all numbers α in $\Re^2$, $C_{\mathcal{A}}(\alpha)$ is a low-discrepancy curve in the sense of equations 9 and 10.*

Proof.
In order to prove this statement, we will establish that the ratio of the length of a curve segment to the total length is commensurate with the corresponding ratio of the area of an axis-aligned rectangle to the unit square that contains it, as suggested in equation 8.

Without loss of generality, we assume $\alpha_1, \alpha_2 > 0$. $C_{\mathcal{A}}(\alpha)$ intersects the axes at $(x = 0, y_n = \{\frac{n\alpha_2}{\alpha_1}\})$ and $(x_n = \{\frac{n\alpha_1}{\alpha_2}\}, y = 0)$, where n is an arbitrary natural number. For almost all α_1, α_2 all three of the quantities (α_1, α_2), $\frac{\alpha_1}{\alpha_2}$ and $\frac{\alpha_2}{\alpha_1}$ generate low-discrepancy sequences in the sense of equation 7, in $\Re^2$, $\Re$ and $\Re$ respectively. In other words, the aforementioned sequences x_n, y_n form low-discrepancy sequences in $[0, 1]$.

Now, for the class of curves $C_{\mathcal{A}}$, all curve segments between points of intersection with the edges of the square are parallel to each other. So, by reasoning about the distribution of these points of intersection, we may arrive at conclusions about the distribution of the curves themselves. With this in mind, we will define the average curve length, i.e., $A(R)$, in the form of integrals.

Let $\tan\phi = \frac{\alpha_2}{\alpha_1}$ and $[0, a] \times [0, b]$ be a rectangle with $0 < a, b < 1$. Depending on the relative values of $(\alpha_1, \alpha_2), a, b$ the integrals take on specific forms. We will explain the case when $\frac{b}{a} \leq \tan\phi$, $b < 1 - \tan\phi$ (see Figure 2) in some detail, the other cases being similar.

We divide the unit square into three parts, as shown in figure 2. Then, I_1, I_2 and I_3 are real numbers that represent the average length that the (α_1, α_2) lines

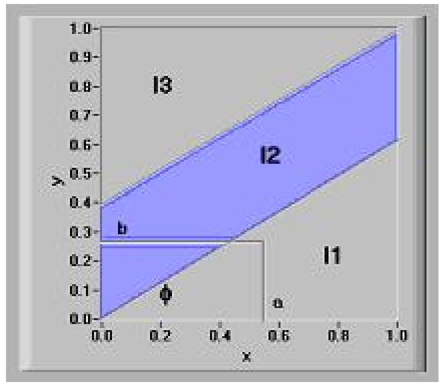


Fig. 2. Definition of the integrals I_1, I_2 and I_3

corresponding to these regions have in common with the rectangle $[0,a] \times [0,b]$. Asymptotically, as the curve length $L \to \infty$, these quantities may be represented as follows (based on simple geometric considerations),

$$I_1 = \int_0^{a-\frac{b}{\tan\phi}} \frac{b}{\sin\phi} dx + \int_{a-\frac{b}{\tan\phi}}^{a} \frac{a-x}{\cos\phi} dx = \frac{ab}{\sin\phi} - \frac{b^2\cos\phi}{2\sin^2\phi} \tag{11}$$

$$I_2 = \int_0^b \frac{b-y}{\sin\phi} dy = \frac{b^2}{2\sin\phi} \tag{12}$$

$$I_3 = 0 \tag{13}$$

$$\tag{14}$$

$$I_1 \sin\phi + (I_2 + I_3)\cos\phi = ab \tag{15}$$

The final term stands for the average length that (α_1, α_2) lines in $[0,1] \times [0,1]$ with slope $\tan\phi$ have in common with the rectangle $[0,a] \times [0,b]$. As expected, it is exactly the area of the rectangle.

In practice, with a finite length curve, what is the discrepancy? We can estimate this using the Koksma-Hlawka inequality 4. For instance, I_1 is approximated using finite length curve segments of the form,

$$l_1(x_i) = \begin{cases} \dfrac{b}{\sin\phi} & : \quad 0 \le x_i \le a - \dfrac{b}{\tan\phi} \\ \dfrac{a-x_i}{\cos\phi} & : \quad a - \dfrac{b}{\tan\phi} \le x_i \le a \\ 0 & : \quad a \le x_i \le 1 \end{cases}$$

Using a finite number, n, of such segments, we have the discrepancy as,

$$|I_1 - \sum_{i=1}^{n} l_1(x_i)| = O(\frac{\log^{2+\epsilon} n}{n}) \quad (16)$$

where $x_i = \{\frac{i\alpha_1}{\alpha_2}\}$

The sum stands for the length of that part of the given curve that lies in $[0, a] \times [0, b]$. The exact same argument may be made for the integrals I_2 and I_3. This means that a finitely generated curve segment of type $C_{\mathcal{A}}$ also has a very low discrepancy. Moreover, note that we are reasoning about a 1-dimensional sequence of points of intersection. So, the constant 3 (for $d = 2$) in equation 10 is now replaced with 2 ($d = 1$). ■

We can now develop two variations of Theorem 1.

Theorem 2. *For almost all numbers α in $\Re^2$, $C_{\mathcal{B}}(\alpha)$ and $C_{\mathcal{C}}(\alpha)$ are low-discrepancy curves in the sense of equations 9 and 10.*

Proof.

We begin with a remark. One can show that Theorem 1 is still valid when in the definition of low-discrepancy curves a much broader class of rectangles R is considered, i.e., rectangles where we replace $[0, a] \times [0, b]$ with the more generic $[c, a] \times [d, b]$. We refer the reader to [3] for several related proofs and expanded discussion on such ideas.

Curves of type $\mathcal{B}$:

Such a curve can be translated into an equivalent version acting in $[0, 1] \times [0, 2]$, by simply mirroring the square. To this end, reflections at the upper edge (see Figure 1) are ignored. What results is an equivalent scheme as type $\mathcal{A}$ in $[0, 1] \times [0, 2]$. For almost all choices of α, the resulting curve in $[0, 1] \times [0, 2]$ is low-discrepancy. The relation between the original space and the new one is straightforward. The original curve goes through a rectangle $R = [0, a] \times [0, b]$ if and only if the derived curve in $[0, 1] \times [0, 2]$ goes through $[0, a] \times [0, b]$ or through $[0, a] \times [2 - b, 2]$ (see the remark at the beginning of this proof). The latter implies that $C_{\mathcal{B}}(\alpha)$ satisfies equations 9 and 10.

Curves of type $\mathcal{C}$:

We essentially repeat the arguments from type $\mathcal{B}$. For almost all α, curves of type $\mathcal{B}$ in $[0, 2] \times [0, 1]$ are low-discrepancy. Such curves can be generated when reflections at the right edge are ignored. The mirrored version of this curve goes through a rectangle $R = [0, a] \times [0, b]$ if and only if the original curve in $[0, 2] \times [0, 1]$ goes through $[0, a] \times [0, b]$ or $[2 - a, 2] \times [0, b]$ (see the remark at the beginning of this proof). The latter implies that $C_{\mathcal{C}}(\alpha)$ satisfies 9 and 10. ■

Curves $C_{\mathcal{C}}(\alpha)$ can be regarded as first examples of continuous trajectories in a unit square that offer low-discrepancy behavior. In real area coverage scenarios they are highly efficient compared to alternate techniques, as we will demonstrate in section 5.

Finally, these results can be generalized to higher dimensions, using the same style of argument. We state this theorem without proof.

Theorem 3. *For almost all numbers α in $\Re^d$, $C_{\mathcal{A}}(\alpha), C_{\mathcal{B}}(\alpha)$ and $C_{\mathcal{C}}(\alpha)$ are low-discrepancy curves in the generalized sense of equations 9 and 10 in d-dimensional unit cubes.*

In practice, the question arises as to how these curves can be realized on digital computers. For this purpose, it is quite reasonable to assume that we have the ability to generate rational numbers of user-specified arbitrary precision. In this case, the error due to the rational approximation of α in theorem 1, 2 and 3 is correspondingly small so that finite lengths of the resulting curves generate discrepancies that are very low.

Theorem 4. *If $\{n\alpha\}$ generates a low-discrepancy sequence, for some irrational number α, then $\exists N \gg 1$ such that $\{n\frac{p}{q}\}$, $n = 1, \ldots, N$, also generates a low-discrepancy sequence, where $p, q \in \aleph$.*

Proof.
The Hurwitz theorem in number theory [18] states that there are infinitely many $\frac{p}{q}$ with the property $|\alpha - \frac{p}{q}| < \frac{1}{q^2}$.

Given any irrational number, it is possible to find a sufficiently large $q \in \aleph$ such that the error of the rational approximation is small. Let q_1 be such a number, with $q_1 > N \gg 1$. Then,

$$|n(\alpha - \frac{p_1}{q_1})| < q_1 . \frac{1}{q_1^2} = \frac{1}{q_1} < \frac{1}{N} \tag{17}$$

Now, assume that $\{(n\alpha) \mod 1\}$ generates a low-discrepancy point set, of discrepancy $D(\alpha)$ for $n = 1, \ldots, N$. When α is irrational, $(n\alpha) \mod 1$ never passes through the vertices of the unit square. Then, there always exists a neighborhood where $\{(nx) \mod 1\}$, $n = 1, \ldots, N$, is a continuous function of x. Correspondingly, this implies that $D(x)$ is a continuous function of x in this neighborhood.

By selecting suitable p_1, q_1 that approximate α well, to within $\epsilon = \frac{1}{q_1}$, we arrive at the bound, $|D(x) - D(\alpha)| < \delta$, where δ is a suitably small constant. This implies that a curve constructed using these sequences, according to the procedure shown in theorem 1, and using rational approximations to α, is still low-discrepancy. ■

4 Abstract Surfaces and Riemannian Spaces

In many common applications, we deal with spaces that are more complex than the unit square. Our constructions based on the unit square may be extended to more general spaces through the use of Riemannian geometry [19], [20]. In essence, Riemannian geometry allows us to deal with a space whose metric properties vary from point to point. This means that we can efficiently talk about the warping of a model space into a desired space, i.e., a manifold, in a principled way through the notion of metrics, line, area and volume elements, etc. Our approach in this paper will be to define this warping such that the low-discrepancy properties are preserved.

Given an abstract surface S with a Riemannian metric defined for (u, v) in $[0, 1]^2$,

$$ds^2 = E(u, v)du^2 + F(u, v)dudv + G(u, v)dv^2 \tag{18}$$

where $E(u, v), F(u, v), G(u, v)$ are differentiable functions in u and v and $EG-F^2$ is positive. The area element dA is defined by,

$$dA = \sqrt{E(u, v)G(u, v) - F^2(u, v)}du \wedge dv \tag{19}$$

The function,

$$\Psi(u, v) = \sqrt{E(u, v)G(u, v) - F^2(u, v)} \tag{20}$$

is nonnegative in $[0, 1]^2$ and $\Psi^2(u, v)$ is differentiable.

Let $\alpha = (\alpha_1, \alpha_2)$ be a given irrational vector (direction) in $\Re^2$. According to the definitions 18 and 19, line and area elements of S for a specific direction $(du, dv) = (\alpha_1 du, \alpha_2 du)$ (i.e., with $dv = \frac{\alpha_2}{\alpha_1} du$) satisfy,

$$\frac{ds}{du} = \sqrt{E(u, v)\alpha_1^2 + F(u, v)\alpha_1\alpha_2 + G(u, v)\alpha_2^2} \tag{21}$$

$$\frac{dA}{du^2} = \sqrt{E(u, v)G(u, v) - F^2(u, v)}\alpha_1\alpha_2 \tag{22}$$

From this, we define the quantity Q which describes our notion of fairness,

$$Q = \frac{\frac{ds}{du}}{\frac{dA}{du^2}} = \frac{\sqrt{E(u, v)\alpha_1^2 + F(u, v)\alpha_1\alpha_2 + G(u, v)\alpha_2^2}}{\sqrt{E(u, v)G(u, v) - F^2(u, v)}\alpha_1\alpha_2} \tag{23}$$

Definition 1

A piecewise smooth curve $C : \Re^+ \mapsto S$ lying on an abstract surface S is called a low-discrepancy curve based on a vector $\alpha = (\alpha_1, \alpha_2)$, iff

1. C is S-filling, i.e. C comes arbitrarily close to any point of S.
2. There is a parametrization of S where Q in equation 23 is constant for all (u, v).
3. In any regular point of C, the tangent vector is parallel to $\alpha = (\alpha_1, \alpha_2)$.

The following algorithm is based on Definition 1.

Algorithm 1

1. Find a parametrization of S that satisfies conditions 2 and 3 in Definition 1. See also Remark 1 below.
2. Generate a curve in S based on the image of a low-discrepancy curve in the unit square according to Theorem 2.

Remark 1: The parametrization in step 2 is not unique. In all examples an originally given natural parametrization is modified using replacements $u \mapsto h(u)$ or $v \mapsto h(v)$ where h is a smooth diffeomorphism.

An abstract d-dimensional surface is defined by,

$$ds^2 = \sum_{i,j=1}^{d} g_{ij}(u_1, u_2, ..., u_d) {du_i}^2 {du_j}^2 \tag{24}$$

where the matrix consisting of $g_{ij} : [0, d]^d \mapsto \Re$ is always symmetric, differentiable, and positive semi-definite. An embedding of an abstract space as in equation 24 in an m-dimensional Euclidean space is a diffeomorphism f of the hypercube $[0, 1]^d$ with $f_1(u_1, u_2, ..., u_d), f_2(u_1, u_2, ..., u_d), ..., f_m(u_1, u_2, ..., u_d) : \Re^d \mapsto \Re^m$ where the Riemannian metric of this embedding is described by equation 24. Usually, this definition is too restrictive. Instead, local diffeomorphisms, i.e., coordinate patches, should be used where these patches cover the whole space under consideration.

Let $\alpha = (\alpha_1, \alpha_2, ..., \alpha_d)$ be a given vector (direction) in $\Re^d$. According to equation 24, the line and volume/content elements for a specific direction $(du_1, du_2, ..., du_d) = (\alpha_1 du, \alpha_2 du, ..., \alpha_d du)$ are:

$$\frac{ds}{du} = \sqrt{\sum_{i,j=1}^{d} g_{ij}(u_1, u_2, ..., u_d)\alpha_i \alpha_j} \tag{25}$$

$$\frac{dV}{du^n} = \sqrt{\det(g_{ij}(u_1, u_2, ..., u_d))}\alpha_1 \alpha_2 ... \alpha_d \tag{26}$$

From this,

$$Q = \frac{\frac{ds}{du}}{\frac{dV}{du^n}} = \frac{\sqrt{\sum_{i,j=1}^{d} g_{ij}(u_1, u_2, ..., u_d)\alpha_i \alpha_j}}{\sqrt{\det(g_{ij}(u_1, u_2, ..., u_d))}\alpha_1 \alpha_2 ... \alpha_d} \tag{27}$$

Definition 2

A piecewise smooth curve $C : \Re^+ \mapsto S$ in the given Riemannian space S is called low-discrepancy curve based on a vector direction $\alpha = (\alpha_1, \alpha_2, ..., \alpha_d)$ iff

1. C is S-filling, i.e. C comes arbitrarily close to any point of S.
2. There is a parametrization of S where Q in equation 27 is constant for all $(u_1, u_2, ..., u_d)$.
3. In any regular point of C, the tangent vector is parallel to $\alpha = (\alpha_1, \alpha_2, ..., \alpha_d)$.

5 Examples of the Coverage of Various Spaces

In this section we will demonstrate the use of the proposed technique to cover various geometrical surfaces. These examples are chosen to correspond to shapes and surfaces that are commonly encountered in robotics. We will present visualizations in the Euclidean space. However, it should be clear that these same results apply to any equivalent space, in whatever domain or context, with the specified geometry.

5.1 Covering the surface of the unit cube

Consider the surface of a unit cube. This is the most natural generalization of the unit square in section 3. Figure 3 depicts an "opened out" version in the plane of the paper, a transformed version of the which can be used to tile the plane and a low-discrepancy curve drawn on this tiling.

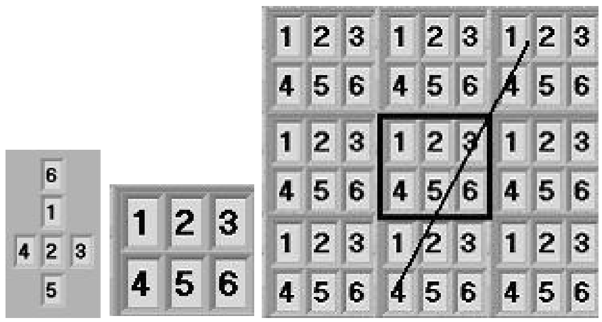


Fig. 3. Tiling of the plane and relation to the surface of the unit cube. Part a depicts the faces of an opened out cube (imagine a typical packing box). Part b depicts an equivalent version, rearranged in such a way that it is easier to tile the plane. Part c shows the low-discrepancy curve traversing the tiled and flattened cubes. The key point to note is that whenever two edges of a face in part c touch, they will also touch in the 3-dimensional cube. The depicted curve is a prescription of the sequence of faces, and points on that face, to visit.

5.2 Covering the annulus

Consider an annulus, i.e., a ring, that has a standard parametrization given by $x(u, v) = (u \cos v, u \sin v)$, where $0 \leq u_0 \leq u \leq u_1$ and $0 \leq v \leq 2\pi$.

Let us consider a diffeomorphism such that $u \mapsto g(u)$ and $v \mapsto v$. Then the ring can be re-parameterized by $x(u, v) = (g(u) \cos v, g(u) \sin v)$. Now, we can apply the method developed earlier. Let g be sufficiently smooth, where g maps $[u_0, u_1]$ onto $[u_0, u_1]$. Then, according to our proposed method, $g(u)$ must satisfy an ordinary differential equation (for $\alpha_1, \alpha_2 > 0$),

$$g'(u) = \frac{\alpha_2 g(u)}{\sqrt{c^2 g^2(u) \alpha_1^2 \alpha_2^2 - \alpha_1^2}} \tag{28}$$

where $g(u_0) = u_0$ and $g(u_1) = u_1$, and,

$$\frac{ds}{du} = \sqrt{g'^2(u)\alpha_1^2 + g^2(u)\alpha_2^2} \tag{29}$$

$$\frac{dA}{du^2} = g(u)g'(u)\alpha_1\alpha_2 \tag{30}$$

Equation 28 has the closed form solution,

$$\frac{\alpha_1}{\alpha_2}\{\sqrt{c^2g^2(u)\alpha_2^2-1}+\arctan(\frac{1}{\sqrt{c^2g^2(u)\alpha_2^2-1}})\}=u+D \tag{31}$$

where D is the unknown constant of integration. Using boundary conditions that arise from the domain and image constraints of the mapping, $g(u_0)=u_0$ and $g(u_1)=u_1$, it follows that,

$$\frac{\alpha_1}{\alpha_2}\{\sqrt{c^2u_0^2\alpha_2^2-1}+\arctan(\frac{1}{\sqrt{c^2u_0^2\alpha_2^2-1}})\}=u_0+D \tag{32}$$

$$\frac{\alpha_1}{\alpha_2}\{\sqrt{c^2u_1^2\alpha_2^2-1}+\arctan(\frac{1}{\sqrt{c^2u_1^2\alpha_2^2-1}})\}=u_1+D \tag{33}$$

We solve for c, D and then use equation 31 as an implicit definition of the required diffeomorphism. In general, obtaining such explicit solutions could become tedious and difficult. In such cases, one may solve the nonlinear differential equations numerically using shooting methods [21].

Figure 4 shows a resulting low-discrepancy curve filling the given ring for the specific case where $u_1 = 1, u_2 = 2$. The parameters α_1, α_2 and c were chosen appropriately by numerical experimentation.

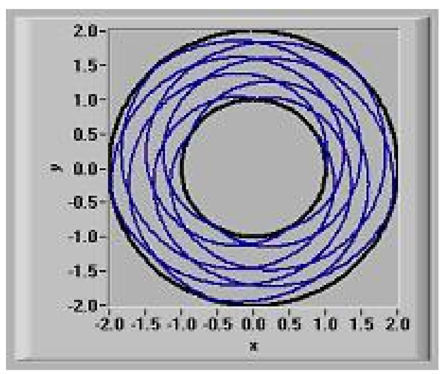


Fig. 4. A low-discrepancy curve in the annulus.

5.3 Covering the surface of a torus

The torus is one of the most important non-trivial surfaces that appear in robotics, especially when working with configuration spaces of mechanical systems. We will use our algorithm to generate a low-discrepancy curve for this surface. We use the diffeomorphism $u \mapsto u$ and $v \mapsto g(v)$. Given the $\Re^3$ embedding of a torus ($b < a$),

$$x(u,v) = ((a + b\cos(2\pi g(v)))\cos(2\pi u),$$
$$(a + b\cos(2\pi g(v)))\sin(2\pi u), b\sin(2\pi g(v))) \quad (34)$$

$$ds^2 = 4\pi^2(a + b\cos(2\pi g(v)))^2 du^2 + 4\pi^2 b^2 g'^2(v) dv^2 \quad (35)$$
$$dA^2 = 16\pi^4 b^2 (a + b\cos(2\pi g(v)))^2 g'^2(v) du^2 dv^2 \quad (36)$$

The function g maps $[0,1]$ onto $[0,1]$ and is sufficiently smooth. Constant ratio of $\frac{ds}{du}$ and $\frac{dA}{du^2}$ in α-direction can be achieved if the following equation holds true (c is a constant, $\alpha_1 > 0$):

$$\frac{(a + b\cos(2\pi g(v)))^2\alpha_1^2 + b^2 g'^2(v)\alpha_2^2}{4\pi^2 b^2 g'^2(v)(a + b\cos(2\pi g(v)))^2\alpha_1^2\alpha_2^2} = c \Rightarrow$$
$$g'(v) = \frac{(a + b\cos(2\pi g(v)))\alpha_1}{\sqrt{4c\pi^2 b^2 (a + b\cos(2\pi g(v)))^2\alpha_1^2\alpha_2^2 - b^2\alpha_2^2}} \quad (37)$$

The boundary conditions are $g(0) = 0$ and $g(1) = 1$. A solution of equation 37 guarantees $g'(0) = g'(1)$. As before, this equation also admits a closed form solution. However, the resulting expressions are long and cumbersome. For the purposes of this example, the parameter c is chosen numerically with the aid of a shooting method. Figure 5 shows a valid function g when $a = 2, b = 1$. The same figure depicts part of the resulting low-discrepancy curve lying on the surface of a torus. Because of $g'(0) = g'(1)$, the curve is smooth.

5.4 Covering the surface of a sphere

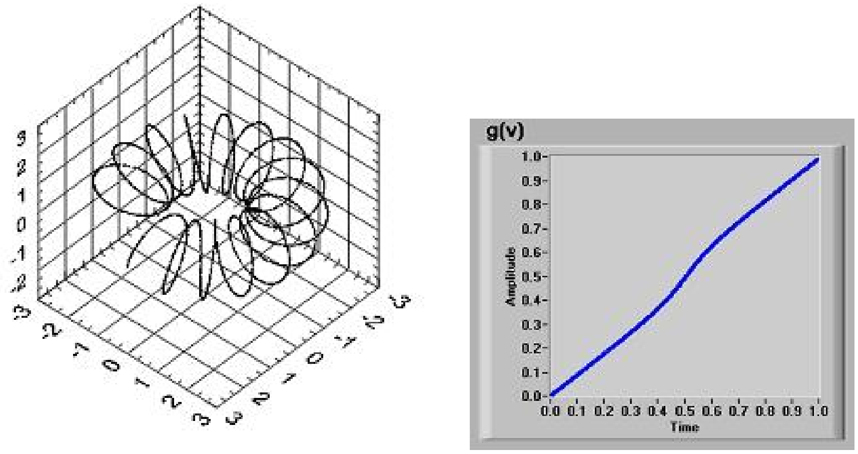
A more sophisticated example is a part of a sphere given as an abstract surface by $ds^2 = 4\pi^2\sin^2(\pi g(v))du^2 + \pi^2 dv^2$ where (u,v) is in $[0,1]\times[v_0,v_1]$ with $v_0 < v_1$ in $(0,1)$. A Euclidean embedding of this surface is given by,

$$x(u,v) = (\sin(2\pi u)\sin(\pi g(v)), \cos(2\pi u)\sin(\pi g(v)), \cos(\pi g(v))) \quad (38)$$

The function $g(v)$ is smooth and maps $[v_0,v_1]$ onto $[v_0,v_1]$. According to Definition 1 and equation 23, we have,

$$g'(v) = \frac{2\sin(\pi g(v))\alpha_1}{\sqrt{4c^2\pi^2\sin^2(\pi g(v))\alpha_1^2\alpha_2^2 - \alpha_2^2}} \quad (39)$$

The boundary conditions are $g(v_0) = v_0$ and $g(v_1) = v_1$. Figure 6 depicts the resulting low-discrepancy curve.



a *b*

Fig. 5. Low-discrepancy curves filling the surface of a torus, part a, based on an appropriately computed function g, part b.

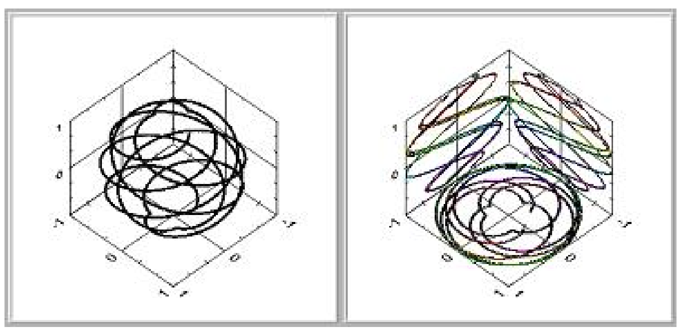


Fig. 6. A low-discrepancy curve on the surface of a sphere. The figure on the right shows the three 2-dimensional projections along the axis planes.

6 Discussion

Broadly speaking, there are three major steps in the procedure outlined above:

1. Define a criterion according to which the curve optimally covers space. This is a *fairness* requirement.
2. Define a diffeomorphism that carries the unit square to an abstract surface.
3. Based on the previous two steps, solve a differential equation whose solution yields the diffeomorphism that respects our stated criterion for fairness.

In this paper, we have defined fairness according to the requirement that any two arbitrary segments of the curve, if they have the same length, must cover the same amounts of area or volume. This is an intuitively obvious definition. There are several ways to extend the definition of fairness. For instance, we could ask that the diffeomorphism must be area preserving along several arbitrarily defined directions or for hyperplanes in n-dimensional space. Fairness requirements of this sort would be difficult to factor into the traditional techniques for generating low discrepancy sequences but do not substantially affect our proposed algorithm.

In this paper, for the purposes of exposition, we have considered somewhat regular and symmetric shapes. Our proposed algorithm would apply to other situations as well. For instance, consider a simple generalization to the surface defined in section 5.2, a generalized annulus defined in terms of the intersection of two planar surfaces with star-convex but otherwise arbitrarily shaped boundaries. Instead of $x(u,v) = (g(u)\cos v, g(u)\sin v)$, use the more general parametrization $x(u,v) = (M(u,v), N(u,v))$. Then our procedure for determining the diffeomorphism would generate a partial differential equation, of the structural form (ignoring constants), $(M_u N_v - M_v N_u)^2 = (M_u + M_v)^2 + (N_u + N_v)^2$. This differential equation would need to be solved subject to boundary conditions defined by the shape of the boundary of the planar surfaces. The mathematical structure of this equation bears a resemblance to Cauchy-Riemann equations and conformal mappings. However, our procedure applies to higher dimensions and handles other requirements including area preservation.

7 Conclusions

We have provided an extension of the theory of low discrepancy sequences to define curves that can uniformly cover space. In doing so, we take an approach that is more general than prior work that has depended on constructing and selecting points from lattices and grids. Our approach is applicable to any problem where the space to be covered can be described by an appropriate Riemannian metric. We have demonstrated the applicability of our proposed approach using examples that are of relevance to robotics.

In our current and future work, we are trying to extend these ideas in several directions. Firstly, as mentioned in section 6, we are studying the problem of covering star-convex but otherwise arbitrarily shaped spaces with obstacles. Our approach to solving these problems results in partial differential equations whose structure bears a resemblance to Cauchy-Riemann equations. Inspired by such connections, we are exploring ways to define generalized area preserving mappings that could have useful applications in robotics. In addition, we are also investigating the application of this technique to other abstract spaces, including parameter spaces of dynamical systems and shape spaces of reconfigurable robots.

Acknowledgment

The authors wish to thank Prof. Harald Niederreiter for reading an early version of the manuscript and for providing valuable feedback.

References

1. Choset, H. (2001) Coverage for robotics - A survey of recent results. *Annals of Mathematics and Artificial Intelligence* 31: 113-126
2. Zaslavsky, G.M. (2005) *Hamiltonian Chaos and Fractional Dynamics*. Oxford University Press, New York
3. Matoušek, J. (1999) *Geometric Discrepancy: An Illustrated Guide*. Springer.
4. LaValle, S. M., Branicky, M. S., Lindemann, S. R. (2004) On the relationship between classical grid search and probabilistic roadmaps. *International Journal of Robotics Research* 23(7-8):673–692
5. Lindemann, S. R., LaValle, S. M. (2003) Incremental low-discrepancy lattice methods for motion planning. In *Proc. IEEE International Conference on Robotics and Automation*
6. Lindemann, S. R., Yershova, A., LaValle, S. M. (2004) Incremental grid sampling strategies in robotics. In *Proc. Workshop on Algorithmic Foundations of Robotics*, pp. 297–312
7. Rimon, E., Koditschek, D.E. (1992) Exact robot navigation using artifical potential functions. *IEEE Trans. Robotics and Automation* 8(5):501–518
8. Niederreiter, H. (1992) *Random Number Generation and Quasi-Monte Carlo Methods*, CBMS-NSF Regional Conference Series in Applied Math., No.63, SIAM
9. Kocis L. and Whiten, W.J. (1997) Computational investigations of low-discrepancy sequences. *ACM Trans. Mathematical Software*, 23(2):266-294
10. van der Corput, J.G. (1935) *Vereilungsfunktionen I,II*, Nederl. Akad. Wetensch. Proc. Ser. B, 38:813-821,1058-1066
11. Halton J.H (1960) On the efficiency of certain quasi-random sequences of points in evaluating multi-dimensional integrals, *Numer. Math.* 2:84–90
12. Sobol', I.M. (1967) The distribution of points in a cube and the approximate evaluation of integrals, *Zh. Vychisl. Mat. i Mat. Fiz. (USSR Computational Mathematics and Mathematical Physics)*, 7:784-802 (In Russian)
13. Richtmyer, R. D. (1951) The evaluation of definite integrals, and quasi-Monte Carlo method based on the properties of algebraic numbers. Report LA-1342, Los Alamos Scientific Laboratory, Los Alamos, NM.
14. Richtmyer, R.D. (1958) A Non-random Sampling Method, Based on Congruences for Monte Carlo Problems, Report NYO-8674. New York: Institute of Mathematical Sciences, New York University.
15. Hammersly, J.M., Handscomb, D.C. (1964) *Monte Carlo Methods*. London: Methuen and Company.
16. Morokoff, W., Caflisch, R.E. (1994) Quasi-random sequences and their discrepancies, *SIAM J. Sci. Stat. Computing*, 15:1251-1279.
17. James, F., Hoogland, J., Kleiss, R. (1997) Multidimensional sampling for simulation and integration: measures, discrepancies, and quasi-random numbers. *Computer Physics Communications* 99: 180–220
18. Hardy, G. H., Wright, E. M. (1980) *An Introduction to the Theory of Numbers*. 5^{th} edition. Oxford University Press.
19. Gray, A. (1998) *Modern Differential Geometry Of Curves And Surfaces With Mathematica*. CRC Press.
20. McCleary, J. (1995) *Geometry from a Differentiable Viewpoint*. Cambridge University Press.
21. Press, W. H., Teukolsky, S. A., Vetterling, W. T., Flannery, B. P. (1992) *Numerical Recipes in C: The Art of Scientific Computing*. 2^{nd} edition. Cambridge University Press.