Strong Normalization for the $\lambda$-calculus with Computational Monads

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Overview

We are interested in general methods for reasoning about $\lambda_{ML}$, a lambda-calculus with types that distinguish computations from values. As an example, we prove strong normalization in two different ways.

Outline of talk:

- Background and motivation: $\lambda_{ML}$, computation types, MLj.
- Strong normalization by translation
- Strong normalization by reducibility
Background

Moggi’s *computational metalanguage* $\lambda_{ML}$ provides a way to explicitly describe computations with side-effects within a pure typed lambda-calculus. The central feature is a new type constructor:

For any type $A$ of values there is a type $TA$ of computations that return an answer in $A$.

Examples of computational effects include non-termination, exceptions, I/O, state, nondeterminism and jumps.
Types and terms of $\lambda_{ML}$

Types

\[ A, B, C ::= O \mid A \rightarrow B \mid TA \]

Terms

\[ M, N, P ::= x:A \mid \lambda x:A.M \mid MN \]
\[ \mid [M] \mid \text{let } x:A \triangleleft M \text{ in } N \]

\[ \Gamma \vdash M : A \quad \Gamma \vdash M : TA \quad \Gamma, x:A \vdash N : TB \]
\[ \Gamma \vdash [M] : TA \quad \Gamma \vdash \text{let } x:A \triangleleft M \text{ in } N : TB \]

The type constructor $T$ acts as a categorical strong monad.
Motivation

The MLj and SML.NET compilers use a monadic intermediate language (MIL) to manage the translation from a higher-order functional language (Standard ML) into an imperative object-oriented bytecode (JVM / .NET).

Typed SML source code
↓
Complex MIL
↓
Simplified MIL
↓
Verifiable bytecode

MIL is $\lambda_{ML}$ extended with datatypes, exceptions, effects, etc.

This is *type-preserving* compilation, carrying types right through compilation to guide optimisation and help generate verifiable code.
Reduction in $\lambda_{ML}$

(\beta) \quad (\lambda x. M) N \rightarrow M[N/x]

(\eta) \quad \lambda x. Mx \rightarrow M

(let \(\beta\)) \quad \text{let } x \leftarrow [V] \text{ in } N \rightarrow N[V/x]

(let \(\eta\)) \quad \text{let } x \leftarrow M \text{ in } [x] \rightarrow M

(let assoc) \quad \text{let } x \leftarrow (\text{let } y \leftarrow M \text{ in } N) \text{ in } P
\quad \rightarrow \text{ let } y \leftarrow M \text{ in } (\text{let } x \leftarrow N \text{ in } P) \quad y \notin \text{fn}(P)

Theorem. $\lambda_{ML}$ is strongly normalizing: no term $M \in \lambda_{ML}$ has an infinite reduction sequence $M \rightarrow M_1 \rightarrow \cdots$
First proof — translation

\[
\begin{align*}
[O] &= O \\
[x] &= x \\
[[M]] &= [M] \\
[T\,A] &= [A] \\
[M\,N] &= [M]\,[N] \\
[let\ x \leftarrow M\ in\ N] &= (\lambda x.[N])[M] \\
[A \rightarrow B] &= [A] \rightarrow [B] \\
[\lambda x.\,M] &= \lambda x.[M]
\end{align*}
\]

Interpret \(T\) as the identity type constructor, with no computational effects.
Reductions translated

Standard lambda-calculus reductions are unchanged: $\beta$ to $\beta$, $\eta$ to $\eta$.

\[
\begin{align*}
\text{let } \beta & : (\lambda x. N) M \rightarrow N[M/x] \\
\text{let } \eta & : (\lambda x. x) M \rightarrow M \\
\text{let assoc} & : (\lambda x. P)((\lambda y. N) M) \rightarrow (\lambda y. (\lambda x. P) N)) M \quad y \notin \text{fn}(P)
\end{align*}
\]

This last rule is a strict extension of $\lambda_{\beta\eta}$, although it is known in work on continuation-passing.
Strong normalization for $\lambda_{\beta\eta\text{assoc}}$

The following asymmetric measure decreases under $\eta$ and ($\lambda$assoc).

$$s(x) = 1 \quad s(\lambda x. M) = s(M) \quad s(MN) = s(M) + 2s(N)$$

It may increase under $\beta$, so in addition we take $b(M) = \max \# \beta$-reductions of $M$ and use $\langle b(M), s(M) \rangle$ ordered lexicographically.

**Lemma.** $b((\lambda x. P)((\lambda y. N) M)) \geq b((\lambda y.(\lambda x. P) N) M)$

**Proof.** Explicit matching of $\beta$-reductions on the right with others on the left, with some careful carrying and borrowing. $\square$

Thus $\lambda_{\beta\eta\text{assoc}}$ is strongly normalizing, hence $\lambda_{\text{ML}}$ is also.
Second proof — reducibility

By translating to $\lambda_{\beta\eta\text{assoc}}$, we are reusing strong normalization for $\beta$-reduction. Can we instead show this for $\lambda_{\text{ML}}$ directly?

For example, Tait’s method for $\lambda_{\beta\eta}$, as presented in [GLT89]:

- Define *reducibility* of terms, by induction on types.
- Show useful properties of reducibility (CR 1–3) by induction on types.
- Show that all terms are reducible, by induction on term structure.
Reducibility for $\lambda_{\beta\eta}$

The definition of reducibility is by induction on types:

- A ground term $M : O$ is reducible iff $M$ is strongly normalizing.
- A function term $M : A \to B$ is reducible iff for all reducible $N : A$ the application $MN : B$ is reducible.
Properties of reducibility

(CR1) If $M$ is reducible then it is strongly normalizing.

(CR2) If $M$ is reducible and $M \rightarrow M'$ then $M'$ is reducible.

(CR3) If $M$ is neutral (a variable or an application), and for all $M \rightarrow M'$ we have $M'$ reducible, then $M$ is reducible too.

Theorem. All terms are reducible.

Corollary. All terms are strongly normalizing.
Defining reducibility at computation types

- A *continuation* \((x)K : A \rightarrow TB\) is a computation term with a distinguished free variable \(x\) of type \(A\).

- A continuation \(K\) is defined as *let-reducible* if \((\text{let } x \leftarrow [V] \text{ in } K)\) is strongly normalizing for all reducible values \(V\).

- Define a computation \(M : TA\) to be reducible if \((\text{let } x \leftarrow M \text{ in } K)\) is strongly normalizing for all let-reducible continuations \(K\).

Now follow your nose to prove properties (CR1–3) and hence strong normalization for all of \(\lambda_{ML}\).
General technique

Given a property $Q_A$ defined by induction on the structure of type $A$, define some further properties as follows:

\[
M \perp K \iff \text{(let } x \leftarrow M \text{ in } K) \text{ is strongly normalizing}
\]

Value $V \in Q_A$

Continuation $K \in Q_A^\perp \iff \forall V \in Q_A . [V] \perp K$

Computation $M \in Q_A^{\perp\perp} \iff \forall K \in Q_A^\perp . M \perp K$

Take $Q_{TA} = Q_A^{\perp\perp}$

In situations without explicit computation types, this game of “leapfrog” can create a notion of property $Q$ on expressions from one on values only.
Summary of results

\( \lambda_{\beta\eta\text{assoc}} \) is strongly normalizing, building on the fact that \( \lambda_{\beta\eta} \) is.

\( \lambda_{\text{ML}} \) is strongly normalizing, by translation to \( \lambda_{\beta\eta\text{assoc}} \).

\( \lambda_{\text{ML}} \) is strongly normalizing, by reducibility.

“Leapfrog” allows us to define reducibility for computations without knowing any specific details of the type constructor T.
Some related work

Normalization in the computational metalanguage:

- Benton, Bierman and de Paiva (1998) give a modal logic corresponding to $\lambda_{ML}$, with accompanying proof normalization.

- Filinski (2001) performs normalization by evaluation for $\lambda_C$, which is equivalent to a proper subsystem of $\lambda_{ML}$.

Extending reasoning methods from values to computations:

- Pitts and Stark (1997) leapfrog a relation for proving operational equivalences between functional programs with local state.

- Pitts (1998) uses leapfrog in operational reasoning about parametric polymorphism, where the relevant computational effect is nontermination.