Reducibility and Strong Normalisation for the Computational Metalanguage

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Overview

We prove strong normalisation for $\lambda_{ML}$, a lambda-calculus with types that distinguish computations from values. This leads to a general method to lift notions defined on values up to computations.

Outline of talk:

- Background and motivation: $\lambda_{ML}$, computation types.
- Strong normalisation by translation and some combinatorics.
- Strong normalisation by Girard-Tait reducibility.

The challenge for reducibility is to apply this semantic notion to terms of computation type *whether or not we know what counts as a “computation”*. 
Background

Moggi’s *computational metalanguage* $\lambda_{\text{ML}}$ provides a way to explicitly describe computations with side-effects within a pure typed lambda-calculus. The central feature is a new type constructor:

For any type $A$ of values there is a type $TA$ of computations that return an answer in $A$.

Examples of computational effects include non-termination, exceptions, I/O, state, nondeterminism and jumps.
Types and terms of $\lambda_{\text{ML}}$

Types $A, B, C ::= 0 \mid A \to B \mid TA$

Terms $M, N, P ::= x:A \mid \lambda x:A.M \mid MN$

$\mid [M] \mid \text{let } x:A \Leftarrow M \text{ in } N$

$\Gamma \vdash M : A \quad \Gamma \vdash M : TA \quad \Gamma, x:A \vdash N : TB$

$\Gamma \vdash [M] : TA \quad \Gamma \vdash \text{let } x:A \Leftarrow M \text{ in } N : TB$

The type constructor $T$ acts as a categorical strong monad.
Some applications of $\lambda_{ML}$

- Denotational semantics: adapt pure models (domains, categories) uniformly to handle computational effects.
- Haskell: monads for mixing functional and stateful code, programming interactions with the real world.
- Compilers: MLj and SML.NET use a monadic intermediate language to carry out type-preserving compilation.
Reduction in $\lambda_{ML}$

$$(\beta) \quad (\lambda x. M) N \rightarrow M[N/x]$$

$$(\eta) \quad \lambda x. M x \rightarrow M$$

$$(\text{let } \beta) \quad \text{let } x \leftarrow [V] \text{ in } N \rightarrow N[V/x]$$

$$(\text{let } \eta) \quad \text{let } x \leftarrow M \text{ in } [x] \rightarrow M$$

$$(\text{let assoc}) \quad \text{let } x \leftarrow (\text{let } y \leftarrow M \text{ in } N) \text{ in } P$$

$$\quad \rightarrow \quad \text{let } y \leftarrow M \text{ in } (\text{let } x \leftarrow N \text{ in } P) \quad y \notin \text{fn}(P)$$

**Theorem.** $\lambda_{ML}$ is strongly normalising: no term $M \in \lambda_{ML}$ has an infinite reduction sequence $M \rightarrow M_1 \rightarrow \cdots$
First proof — translation

\[ \begin{align*}
\phi(0) &= 0 \quad & \phi(x) &= x \\
\phi(TA) &= \phi(A) \quad & \phi(MN) &= \phi(M)\phi(N) \\
\phi(A \to B) &= \phi(A) \to \phi(B) \quad & \phi(\lambda x.M) &= \lambda x.\phi(M) \\
\phi([M]) &= \phi(M) \quad & \phi(\text{let } x \Leftarrow M \text{ in } N) &= (\lambda x.\phi(N))\phi(M)
\end{align*} \]

Interpret \( T \) as the identity type constructor, with no computational effects.
Standard lambda-calculus reductions are unchanged: $\beta$ to $\beta$, $\eta$ to $\eta$.

\[
\phi(\text{let } \beta) \quad (\lambda x. N) M \rightarrow N[M/x]
\]

\[
\phi(\text{let } \eta) \quad (\lambda x. x) M \rightarrow M
\]

\[
\phi(\text{let assoc}) \quad (\lambda x. P)((\lambda y. N) M) \rightarrow (\lambda y. (\lambda x. P) N) M \quad y \notin \text{fn}(P)
\]

This last rule is a strict extension of $\lambda_{\beta\eta}$, although it is admissible and a known “administrative” reduction from continuation-passing work.
Strong normalisation for $\lambda_{\beta\eta\text{assoc}}$

The following asymmetric measure decreases under $\eta$ and (assoc).

$$s(x) = 1 \quad s(\lambda x. M) = s(M) \quad s(MN) = s(M) + 2s(N)$$

It may increase under $\beta$, so in addition we define $b(M) = (\max \# \beta\text{-reductions of } M)$ and use $\langle b(M), s(M) \rangle$ ordered lexicographically.

**Lemma.** $b((\lambda x. P)((\lambda y. N)M)) \geq b((\lambda y.(\lambda x. P)N)M)$

**Proof.** Explicit matching of $\beta$-reduction sequences on the right with others on the left, with some careful carrying and borrowing. \(\square\)

Thus $\lambda_{\beta\eta\text{assoc}}$ is strongly normalising, hence $\lambda_{\text{ML}}$ is also.
Second proof — reducibility

Translation works, but only because we happen to have a result for \(\lambda_{\beta\eta}\) to hand. What can we do working with \(\lambda_{ML}\) directly?

For example, Tait’s method for \(\lambda_{\beta\eta}\), as presented in [GLT89]:

- Define *reducibility* of terms, by induction on types.
- Show useful properties of reducibility by induction on types; in particular that all reducible terms are strongly normalising.
- Show that all terms are reducible, by induction on term structure.
Reducibility for $\lambda_{\beta\eta}$

The definition of reducibility is by induction on types:

- A ground term $M : 0$ is reducible iff $M$ is strongly normalising.
- A product term $M : A \times B$ is reducible iff $\text{fst}(M)$ and $\text{snd}(M)$ are both reducible.
- A function term $M : A \rightarrow B$ is reducible iff for all reducible $N : A$ the application $MN : B$ is reducible.
Properties of reducibility

(CR1) If \( M \) is reducible then it is strongly normalising.

(CR2) If \( M \) is reducible and \( M \rightarrow M' \) then \( M' \) is reducible.

(CR3) If \( M \) is neutral (a variable or an application), and for all \( M \rightarrow M' \) we have \( M' \) reducible, then \( M \) is reducible too.

Theorem. All terms are reducible.

Corollary. All terms are strongly normalising.
Non-definitions of reducibility at computation types

(Bad 1) Term $M$ of type $TA$ is reducible if for all reducible $N$ of type $TB$, the term let $x \leftarrow M$ in $N$ is reducible.

Not inductive over types.

(Bad 2) Term $M$ of type $TA$ is reducible if for all strongly normalising $N$ of type $TB$, the term let $x \leftarrow M$ in $N$ is strongly normalising.

Inductive, but not strong enough.
Continuations

- A *term abstraction* \((x)N\) is a computation term \(N\) with a distinguished free variable \(x\).
- A *continuation* is a list of term abstractions:
  
  \[
  K ::= \text{Id} \mid K \circ (x)N
  \]
  
  - We apply continuations as nested \texttt{let}-sequence:
    
    \[
    \text{Id}@M = M
    \]
    
    \[
    (K \circ (x)N)@M = K@(\text{let } x \Leftarrow M \text{ in } N)
    \]
  
  - Continuations reduce: \(K \rightarrow K’\) iff \(\forall M. K@M \rightarrow K’@M\).
Reducibility at computation types

(Good 1) Term $M$ of type $TA$ is reducible if for all reducible continuations $K$, the application $K@M$ is strongly normalising.

(Good 2) Continuation $K$ taking terms of type $TA$ is reducible if for all reducible $V$ of type $A$, the application $K@[V]$ is strongly normalising.

Moving from $TA$ to $A$ avoids circularity, and we have a definition inductive over types. The characterisation is strong enough to follow through the standard results on reducibility and strong normalisation.
General “leap-frog” technique

Given a property $Q_A$ defined by induction on the structure of type $A$, define some further properties as follows:

\[ K \vdash M \iff K@M \text{ is strongly normalising} \]

**Values**

\[ V \in Q_A \]

**Continuations**

\[ K \in Q^T_A \iff \forall V \in Q_A . \ K \vdash [V] \]

**Computations**

\[ M \in Q^{TT}_A \iff \forall K \in Q^T_A . \ K \vdash M \]

Take $Q_{TA} = Q^{TT}_A$

This jump over continuations pushes any concept on values $A$ up to one on computations $TA$, whether or not we know the nature of $T$. 
Summary of results

\( \lambda_{\beta\eta_{\text{assoc}}} \) is strongly normalising, building on the fact that \( \lambda_{\beta\eta} \) is.

\( \lambda_{\text{ML}} \) is strongly normalising, by translation to \( \lambda_{\beta\eta_{\text{assoc}}} \).

\( \lambda_{\text{ML}} \) is strongly normalising, by reducibility.

“Leapfrog” allows us to define reducibility for computations without knowing any specific details of the type constructor \( T \).
Some related work

Normalisation in the computational metalanguage:

- Benton, Bierman and de Paiva (1998) give a modal logic corresponding to $\lambda_{ML}$, with accompanying proof normalisation.
- Filinski (2001) performs normalisation by evaluation for $\lambda_C$, which is equivalent to a proper subsystem of $\lambda_{ML}$.

Extending reasoning methods from values to computations:

- Pitts and Stark (1998) leapfrog a relation for proving operational equivalences between functional programs with local state.
Intermediate $\lambda_{\text{ML}}$

The MLj and SML.NET compilers use a monadic intermediate language (MIL) to manage the translation from a higher-order functional language (Standard ML) into an imperative object-oriented bytecode (JVM / .NET).

Typed SML source code
  ↓
Complex MIL
  ↓
Simplified MIL
  ↓
Verifiable bytecode

MIL is $\lambda_{\text{ML}}$ extended with datatypes, exceptions, effects, etc.

This is type-preserving compilation, carrying types right through compilation to guide optimisation and help generate verifiable code.