An Algebraic View of Bigraphs

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Motivation and programme

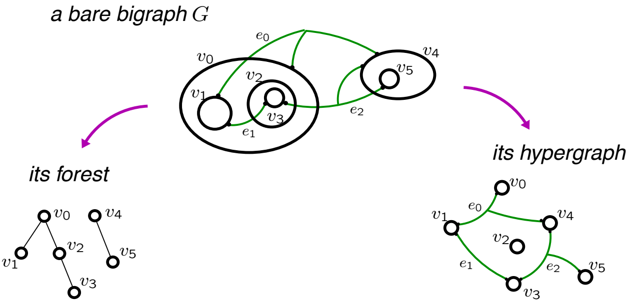
- We ask: what are bigraphs?
- To answer this, we aim at a standard algebraic account of bigraphs; we use a suitable linear kind of algebra.
- This gives a universal characterisation of bigraphs in terms of their algebraic structure.
- A possible external benefit could be that, as the account is standard, one can easily explore variations.
- A possible internal benefit is the provision of a (less) standard term language for bigraphs.
- Another possible internal benefit is a framework for discussing dynamics.

Robin's view

- I discussed some of these ideas with Robin when I was beginning to think about the problem.
- I wanted to avoid the partiality that Robin wished to deal with directly.
- He was not fond of that move, saying he had taken great care with his formalism. (I think the balance of structure and notational convenience was primary.)
- So he may well not have liked this work.
- He may also not have cared: I got the impression I would have needed to argue well to convince him that abstract universal characterisations of the category of bigraphs held any interest.
- I am sad that he is no longer with us, and that that such conversations can no longer be.

Example (bare) bigraph, place graph, and link graph (from Milner lectures)

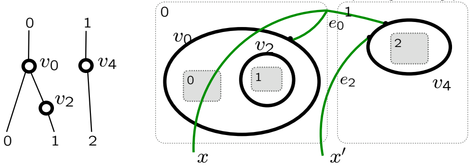
The bi-structure of bigraphs

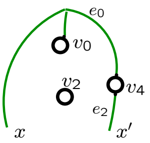


How to build bigraphs? *Give them interfaces . . .*

Example compositions (from Milner lectures)

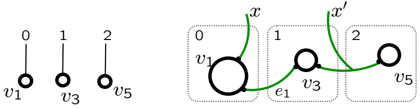
a contextual bigraph $H : \langle 3, \{x, x'\}\rangle \rightarrow \langle 2, \emptyset\rangle$

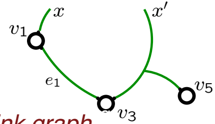




its place graph *its link graph*

a ground bigraph $F : \epsilon \rightarrow \langle 3, \{x, x'\}\rangle$

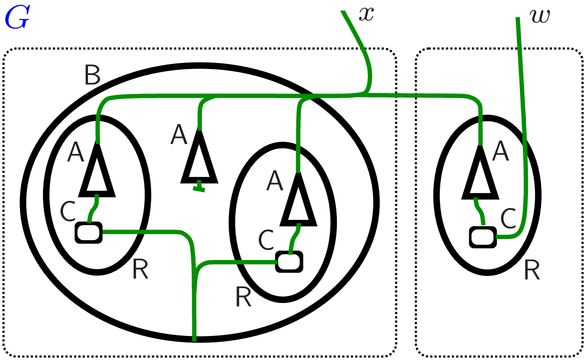




its place graph *its link graph*

An *interface* takes the form $\langle m, X\rangle$. The *origin* is $\epsilon \stackrel{\text{def}}{=} \langle 0, \emptyset\rangle$.

Example bigraph with controls (from Milner lectures)



Control Signature A:2 - an agent, B:1 - a building,
C:2 - a computer, R:0 - a room.

Outline

Definition of place graphs

- Signature A set $\mathcal{K}$ of controls, ranged over by K.
- Concrete Place Graph A tuple

$$F = \langle V, \mathrm{ctrl}, \mathrm{prnt} \rangle : m \to n$$

where:

  - V is the set of nodes
  - $\mathrm{ctrl} : V \to \mathcal{K}$ is the control map.
  - $\mathrm{prnt} : m \uplus V \to V \uplus n$ is the parent map, assumed *acyclic*. (We identify m with $\{0, \ldots, m-1\}$.)

- Abstract Place Graph An isomorphism class

$$[F]$$

of concrete place graphs.

Composition of abstract place graphs

$$I \xrightarrow{[\langle V,\mathrm{ctrl},\mathrm{prnt}\rangle]} m \xrightarrow{[\langle V',\mathrm{ctrl}',\mathrm{prnt}'\rangle]} n \quad = \quad I \xrightarrow{[\langle V \dot\cup V',\mathrm{ctrl} \dot\cup \mathrm{ctrl}',\mathrm{prnt}''\rangle]} m$$

where:

$$\mathrm{prnt}''(x) = \begin{cases} \mathrm{prnt}(x) & (x \in I \dot\cup V, \mathrm{prnt}(x) \notin m) \\ \mathrm{prnt}'(\mathrm{prnt}(x)) & (x \in I \dot\cup V, \mathrm{prnt}(x) \in m) \\ \mathrm{prnt}'(x) & (x \in V') \end{cases}$$

This gives a category $\mathbf{Place}_{\mathcal{K}}$.

Tensor of abstract place graphs

$$(l \xrightarrow{[\langle V,\mathrm{ctrl},\mathrm{prnt}\rangle]} m) \otimes (l' \xrightarrow{[\langle V',\mathrm{ctrl}',\mathrm{prnt}'\rangle]} m') \;=\; l{+}l' \xrightarrow{[\langle V \mathbin{\dot\cup} V',\mathrm{ctrl} \mathbin{\dot\cup} \mathrm{ctrl}',\mathrm{prnt}''\rangle]} m{+}m'$$

where:

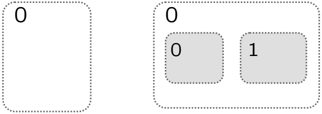
$$\mathrm{prnt}''(x) = \begin{cases} \mathrm{prnt}(x) & (x \in l \mathbin{\dot\cup} V) \\ \mathrm{prnt}(x-l)+l & (x \in \{l,\ldots,(l+l')-1\} \mathbin{\dot\cup} V') \end{cases}$$

- This makes $\mathbf{Place}_{\mathcal{K}}$ symmetric monoidal closed, with $l \otimes m = l + m$.
- But the tensor product is not a categorical product, and 0 is not the terminal object (eg, there are no place graphs $[F]:1 \to 0$).

Some structure in place graphs

We have:

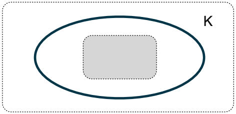
- A commutative monoid



$$1 : 0 \to 1 \qquad join : 2 \to 1$$

on 1,

- with unary functions



$K : 1 \to 1$

Example diagram: Associativity of *join*

$$
\begin{tikzcd}
(1 \otimes 1) \otimes 1 \arrow[r, "join \otimes \mathrm{id}_1"] \arrow[dd, "\mathrm{id}_3"'] & 1 \otimes 1 \arrow[dr, "join"] & \\
& & 1 \\
1 \otimes (1 \otimes 1) \arrow[r, "\mathrm{id}_1 \otimes join"'] & 1 \otimes 1 \arrow[ur, "join"'] &
\end{tikzcd}
$$

A (cartesian) equational theory

- Signature
  - $+ : 2$ and $0 : 0$
  - $K : 1 \qquad (K \in \mathcal{K})$
- Axioms Ax: $+$ and 0 form a commutative monoid, ie:

$$
\begin{array}{rcl}
(x + y) + x & = & x + (y + z) \\
x + y & = & y + x \\
x + 0 & = & x
\end{array}
$$

- Proof equivalence classes

$$[t] =_{\mathrm{def}} \{u \mid \mathrm{Ax} \vdash u = t\}$$

and write $[\langle t_0, \ldots, t_{n-1} \rangle]$ for $\langle [t_0], \ldots, [t_{n-1}] \rangle$.

A corresponding category $\mathbf{CMon}^c_{\mathcal{K}}$

- Objects The natural numbers $\mathbb{N}$
- Morphisms

$$[\langle t_0, \ldots, t_{n-1} \rangle] : m \longrightarrow n$$

where $\mathrm{FV}(t_i) \subseteq \{z_0, \ldots, z_{m-1}\}$, for $i = 0, m-1$.
(We assume a fixed infinite sequence $z_0, z_1, \ldots$ of distinct variables.)

- Composition

$$l \xrightarrow{[\langle u_0,\ldots,u_{m-1}\rangle]} m \xrightarrow{[\langle t_0,\ldots,t_{n-1}\rangle]} n \quad = \quad l \xrightarrow{[\langle t_0,\ldots,t_{n-1}\rangle[u_0/z_0,\ldots,u_{m-1}/z_{m-1}]]} m$$

- Identity

$$m \xrightarrow{[\langle z_0,\ldots,z_{m-1}\rangle]} m$$

Lawvere Theories

These are structures:

$$\mathbb{N}^{\mathrm{op}} \xrightarrow{I} \mathbf{L}$$

where

- $\mathbb{N}$ is the category of all natural numbers and maps between them,
- $\mathbf{L}$ is a small category with finite products, and
- I is a strict finite product preserving identity-on-objects functor

Example Lawvere Theory: $\mathbb{N}^{\mathrm{op}} \xrightarrow{I} \mathbf{CMon}^{c}_{\mathcal{K}}$ where

$$I(f : n \to m) = [\langle z_{f(0)}, \ldots, z_{f(m-1)} \rangle]$$

When I is obvious, we may omit it.

Maps of Lawvere theories

A map from $\mathbb{N}^{\mathrm{op}} \xrightarrow{I} \mathbf{L}$ to $\mathbb{N}^{\mathrm{op}} \xrightarrow{I'} \mathbf{L}'$, is just a functor

$$\mathcal{F} : \mathbf{L} \to \mathbf{L}'$$

such that the following diagram commutes:

$$\begin{tikzcd}
\mathbb{N}^{\mathrm{op}} \arrow[d, "I"'] \arrow[dr, "I'"] & \\
\mathbf{L} \arrow[r, "\mathcal{F}"] & \mathbf{L}'
\end{tikzcd}$$

It is necessarily the identity on objects and strictly finite product preserving.

Remark

$$\mathbf{Law} \simeq \mathbf{EqTh}$$

Characterisation of $\mathbf{CMon}^c_{\mathcal{K}}$

Define:

1. a commutative monoid $\langle +, 0 \rangle$ on 1 , where:

$$+ =_{\text{def}} \langle [z_0 + z_1] \rangle : 2 \to 1 \qquad 0 =_{\text{def}} \langle [0] \rangle : 0 \to 1$$

2. unary morphisms $K : 1 \to 1$ over P (for $K \in \mathcal{K}$) where:

$$K =_{\text{def}} \langle [K(z_0)] \rangle : 1 \to 1$$

Theorem

$\mathbf{CMon}^c_{\mathcal{K}}$ *is the free Lawvere theory* $\mathbf{L}$ *with*

1. *a specified commutative monoid* $\langle +_{\mathbf{L}}, 0_{\mathbf{L}} \rangle$ *on* 1*, and*
2. *specified unary morphisms* $K_{\mathbf{L}} : \mathrm{P}_{\mathbf{L}} \longrightarrow \mathrm{P}_{\mathbf{L}}$ *on* 1*.*

Linear equational logic

- Signature Σ: operation symbols $\mathrm{op}:n$ as usual.
- Linear Tems t: as usual, but restricted so that no variable appears twice.
- Linear Equations $t = u$ as usual, but with the same variables occurring on both sides
- Example As above with $+:2, 0:0$, and $K:1$ (for $K \in \mathcal{K}$), and the commutative monoid axioms.

$$
\begin{array}{ccc}
(x+y)+x & = & x+(y+z) \\
x+y & = & y+x \\
x+0 & = & x
\end{array}
$$

Linear equational logic: Inference Rules

Equality is an equivalence relation

$$t = t \qquad \frac{t = u \quad u = v}{t = v} \qquad \frac{t = u}{u = t}$$

Congruence

$$\frac{t_i = u_i \quad (i = 0, n-1)}{f(t_0, \ldots, t_{n-1}) = f(u_0, \ldots, u_{n-1})}$$

provided the terms in the conclusion are linear.

Substitution

$$\frac{t = u}{t[v_0/y_0, \ldots, v_{n-1}/y_{n-1}] = u[v_0/y_0, \ldots, v_{n-1}/y_{n-1}]}$$

provided the terms in the conclusion are linear.

A corresponding category $\mathbf{CMon}^{I}_{\mathcal{K}}$

- Objects The natural numbers $\mathbb{N}$
- Morphisms

$$[\langle t_0, \ldots, t_{n-1} \rangle] : m \longrightarrow n$$

  where
  - $\{z_0, \ldots, z_{m-1}\} = \bigcup_{i=0}^{n-1} \mathrm{FV}(t_i)$, and
  - The $\mathrm{FV}(t_i)$ are mutually disjoint
- Composition

$$l \xrightarrow{[\langle u_0,\ldots,u_{m-1}\rangle]} m \xrightarrow{[\langle t_0,\ldots,t_{n-1}\rangle]} n \;=\; l \xrightarrow{[\langle t_0,\ldots,t_{n-1}\rangle[u_0/z_0,\ldots,u_{m-1}/z_{m-1}]]} m$$

- Identity

$$m \xrightarrow{[\langle z_0,\ldots,z_{m-1}\rangle]} m$$

Symmetric Monoidal Lawvere Theories (aka PROPs)

These are structures:

$$\mathbb{B}^{\mathrm{op}} \xrightarrow{I} \mathbf{L}$$

where

- $\mathbb{B}$ is the category of all natural numbers and bijections over them,
- $\mathbf{L}$ is a small symmetric monoidal category, and
- I is a strict symmetric monoidal identity-on-objects functor

Example symmetric monoidal Lawvere theory: $\mathbb{B}^{\mathrm{op}} \xrightarrow{I} \mathbf{CMon}^{I}_{\mathcal{K}}$ where

$$I(f : n \cong n) = [\langle z_{f(0)}, \ldots, z_{f(n-1)} \rangle]$$

Much as before, morphisms of symmetric monoidal theories are functors making the evident triangle commute.

Remark Presumably (a slight surprise):

$$\mathbf{LinEqTh} \simeq \mathbf{Operad}$$

Characterisation of $\mathbf{CMon}^I_{\mathcal{K}}$

Define:

1. a commutative monoid $\langle +, 0 \rangle$ on 1 , where:

$$+ = \langle [z_0 + z_1] \rangle \qquad 0 = \langle [0] \rangle$$

2. unary morphisms $K : 1 \to 1$ over P (for $K \in \mathcal{K}$) where:

$$K = \langle [K(z_0)] \rangle$$

Theorem

$\mathbf{CMon}^I_{\mathcal{K}}$ *is the free symmetric monoidal Lawvere theory* $\mathbf{L}$ *with*

1. *a specified commutative monoid* $\langle +_{\mathbf{L}}, 0_{\mathbf{L}} \rangle$ *on* 1*, and*
2. *specified unary morphisms* $K_{\mathbf{L}} : 1 \longrightarrow 1$ *on* 1*.*

The same thing in normal form

Multilevel Multiset Terms

- Every finite multiset $\{a_0, \ldots, a_{n-1}\}$ $(n \geq 0)$ of atomic multilevel multiset terms is a multilevel multiset term, provided that no variable appears in more than one a_i.

Atomic Multilevel Multiset Terms

- Every variable x is an atomic multilevel multiset term.
- If t is a multilevel multiset term and $K \in \mathcal{K}$ then $K(t)$ is an atomic multilevel multiset term.

The corresponding category

- Objects The natural numbers $\mathbb{N}$
- Morphisms

$$\langle t_0, \ldots, t_{n-1} \rangle : m \longrightarrow n$$

where $\{z_0, \ldots, z_{m-1}\} = \dot\bigcup_{i=0}^{n-1} \mathrm{FV}(t_i)$.

- Composition

$$l \xrightarrow{\langle u_0,\ldots,u_{m-1}\rangle} m \xrightarrow{\langle t_0,\ldots,t_{n-1}\rangle} n \quad = \quad \xrightarrow{\langle t_0,\ldots,t_{n-1}\rangle[u_0/z_0,\ldots,u_{m-1}/z_{m-1}]} m$$

- Identity

$$m \xrightarrow{\langle z_0,\ldots,z_{m-1}\rangle} m$$

Normal forms of place graphs

- Every abstract place graph $[F]: m \to 1$ can be written essentially uniquely as a permutation of a join of *atomic* place graphs:

$$join_a \circ ((K_1 \circ [F_0]) \otimes \ldots \otimes (K_{a-1} \circ [F_{a-1}]) \otimes join_b) \circ \alpha$$

where $[F_i]: m_i \to 1$, with $m = \sum_{i=0}^{a-1} m_i$.

- Every abstract place graph $[F]: m \to n$ can be written uniquely as a permutation of a tensor of unary place graphs:

$$([F_0] \otimes \ldots \otimes [F_{n-1}]) \circ \alpha$$

where $[F_i]: m_i \to 1$, with $m = \sum_{i=0}^{a-1} m_i$.

From unary place graphs to normal forms of terms

We inductively define a map $\mathcal{G}_1$ sending unary place graphs $[F]: m \to 1$ to multiset multilevel terms $\mathcal{G}_1([F])$ with free variables $\{z_0, \ldots, z_{m-1}\}$, by:

$$\mathcal{G}_1(join_a \circ ((K_1 \circ [F_0]) \otimes \ldots \otimes (K_{a-1} \circ [F_{l-1}]) \otimes join_b) \circ \alpha) =$$

$$\sum_{i=0}^{a-1} K_i(\mathcal{G}_1([F_i])[z_{\alpha^{-1}(0)}/z_0, \ldots, z_{\alpha^{-1}(m_i-1)}/z_{m_i-1}]$$
$$+ \sum_{j=0}^{b-1} z_{\alpha^{-1}(j)}$$

From place graphs to tuples of normal forms of terms

We can then define a map $\mathcal{G}$ sending place graphs $[F] : m \to n$ to n tuples of terms with free variables $\{z_0, \ldots, z_{m-1}\}$, occurring disjointly:

$$\begin{aligned}&\mathcal{G}(([F_0] \otimes \ldots \otimes [F_{n-1}]) \circ \alpha)\\ &= \langle \mathcal{G}([F_0])[z_{\alpha^{-1}(\mathbf{m_0})}/z_{\mathbf{m_0}}], \ldots, \mathcal{G}([F_{n-1}])[z_{\alpha^{-1}(\mathbf{m_{n-1}})}/z_{\mathbf{m_{n-1}}}]\rangle\end{aligned}$$

Proposition

We have a faithful (ie, locally 1-1) morphism of sm Lawvere theories:

$$\mathcal{G} : \mathbf{Place}_{\mathcal{K}} \longrightarrow \mathbf{CMon}^{I}_{\mathcal{K}}$$

Identification of place graphs

Theorem

***Place**$_{\mathcal{K}}$ is the free symmetric monoidal Lawvere theory **L** with*

1. *a specified commutative monoid $\langle +_{\mathbf{L}}, 0_{\mathbf{L}} \rangle$ on 1, and*
2. *specified unary morphisms $K_{\mathbf{L}} : 1 \longrightarrow 1$ on 1, for $K \in \mathcal{K}$.*

Proof.

By the freeness of $\mathbf{CMon}^{l}_{\mathcal{K}}$ there is a suitable morphism $\mathcal{F} : \mathbf{CMon}^{l}_{\mathcal{K}} \longrightarrow \mathbf{Place}_{\mathcal{K}}$. Composing with $\mathcal{G}$ and using the proposition we see that $\mathcal{G}\mathcal{F} : \mathbf{CMon}^{l}_{\mathcal{K}} \longrightarrow \mathbf{CMon}^{l}_{\mathcal{K}}$ is also a suitable morphism.
So, by the freeness of $\mathbf{CMon}^{l}_{\mathcal{K}}$, we have: $\mathcal{G}\mathcal{F} = \mathrm{id}$. So $\mathcal{G}\mathcal{F}\mathcal{G} = \mathcal{G}$.
So, $\mathcal{F}\mathcal{G} = \mathrm{id}$ as $\mathcal{G}$ is faithful. So $\mathcal{F}$ and $\mathcal{G}$ are mutually inverse. $\square$

Summary

- Up to isomorphism, the category $\mathbf{Place}_{\mathcal{K}}$ of place graphs with signature $\mathcal{K}$ is given by a standard term model construction.
- This identifies it as the linear equational theory of a commutative monoid with unary function symbols K, for $K \in \mathcal{K}$.

Outline

Definition of link graphs

- Signature A set $\mathcal{K}$ of controls K with natural number arities, written $K : k$.
- Concrete Link Graph A tuple

$$F = \langle V, E, \mathrm{ctrl}, \mathrm{link} \rangle : X \to Y$$

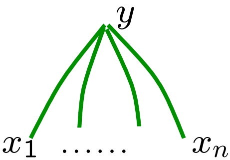
where:

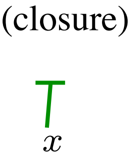
  - V, E are, respectively, the sets of nodes and edges.
  - $X, Y \subseteq_{\mathrm{fin}} \mathcal{X}$, the set of names, are the inner and outer faces.
  - $\mathrm{ctrl} : V \to \mathcal{K}$ is the control map.
  - $\mathrm{link} : X \dot{\cup} P \to E \dot{\cup} Y$ is the link map, assumed to cover E, where the set P of ports is:

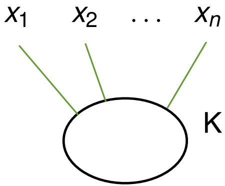
$$P =_{\mathrm{def}} \{ \langle v, i \rangle \mid \mathrm{ctrl}(v) : k, i < k \}$$

- Abstract Link Graph An isomorphism class $[F]$ of concrete link graphs.

Some elementary link graphs







$y/X : X \to \{y\}$

Elementary Substitution

$/x : x \to \epsilon$

Elementary Closure

$K_{x_1,\ldots,x_n} : \epsilon \to X$

Atom

Composition of abstract link graphs

$$X \xrightarrow{[\langle V,E,\mathrm{ctrl},\mathrm{link}\rangle]} Y \xrightarrow{[\langle V',E',\mathrm{ctrl}',\mathrm{link}'\rangle']} Z \;=\; X \xrightarrow{[\langle V\dot\cup V',E\dot\cup E',\mathrm{ctrl}\dot\cup\mathrm{ctrl}',\mathrm{link}''\rangle^{-}]} Z$$

where

$$\mathrm{link}''(x) = \begin{cases} \mathrm{link}(x) & (x \in X \dot\cup P, \mathrm{link}(x) \notin Y) \\ \mathrm{link}'(\mathrm{link}(x)) & (x \in X \dot\cup P, \mathrm{link}(x) \in Y) \\ \mathrm{link}'(x) & (x \in P') \end{cases}$$

and where $\langle \ldots \rangle^{-}$ is $\langle \ldots \rangle^{-}$, less any uncovered edges.
This gives a category.

Tensor of abstract link graphs

$$(X \xrightarrow{[\langle V, E, \mathrm{ctrl}, \mathrm{link}\rangle]} Y) \otimes (X' \xrightarrow{[\langle V', E', \mathrm{ctrl}', \mathrm{link}'\rangle']} Y')$$

$$= \quad X \dot\cup X' \xrightarrow{[\langle V \dot\cup V', E \dot\cup E', \mathrm{ctrl} \dot\cup \mathrm{ctrl}', \mathrm{link}' \dot\cup \mathrm{link}'\rangle]} Y \dot\cup Y'$$

... but this only gives a partial symmetric monoidal category.

A sm category of link graphs

Using the above partial smc, we define a total smc $\mathbf{Link}_{\mathcal{K}}$:

- Objects the natural numbers $\mathbb{N}$
- Morphisms

$$l \xrightarrow{[F]} m$$

for $F:\{n_0, \ldots, n_{l-1}\} \to \{n_0, \ldots, n_{m-1}\}$

- Composition (as above)
- Tensor

$$(l \xrightarrow{[F]} m) \otimes (l' \xrightarrow{[F']} m') \quad = \quad l{+}l' \xrightarrow{[F]\otimes(\sigma_{0,m,m'} \circ [F'] \circ \sigma_{l,0,l'})} m{+}m'$$

where $\sigma_{k,l,m} = [n_l/n_k, \ldots, n_{l+m-1}/n_{k+m-1}]$

Some structure in the category $\mathbf{Link}_{\mathcal{K}}$ of link graphs

We have:

- A commutative monoid $\langle n_0/\{n_0, n_1\}, n_0/\epsilon\rangle$ on 1
- whose zero $n_0/\epsilon : 0 \to 1$ has a left inverse $/n_0 : 1 \to 0$, ie:

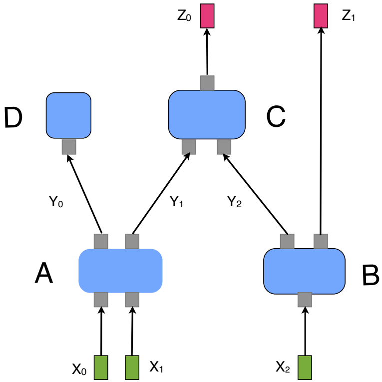
$$\epsilon \xrightarrow{n_0/\epsilon} 1 \xrightarrow{/n_0} 0 \;=\; 0 \xrightarrow{\mathrm{id}} 0$$

- and morphisms

$$K_{n_0,\ldots,n_{k-1}} : 0 \to k$$

for $K : k \in \mathcal{K}$.

Symmetric monoidal equational logic: Example term



$$\{A(x_0, x_1; y_0, y_1), B(x_2; y_2, z_1), D(y_0;), C(y_1, y_2; z_0)\}$$

A dag

Corresponding term

Symmetric monoidal equational logic (CCS style)

- Signature Σ of operation symbols op with arities and co-arities: $\mathrm{op}: m \rightarrow n$
- Atomic terms These are either
  - Wires
    $$a = y/x$$
    when:
    $$\mathrm{IV}(a) =_{\mathrm{def}} \{x\} \qquad \mathrm{OV}(a) =_{\mathrm{def}} \{y\}$$
    or
  - Boxes
    $$a = \mathrm{op}(x_0, \ldots, x_{m-1}; y_0, \ldots, y_{n-1})$$
    for $\mathrm{op}: m \rightarrow n$, where no two of $x_0, \ldots, x_{m-1}, y_0, \ldots, y_{n-1}$ are the same, when:
    $$\mathrm{IV}(a) =_{\mathrm{def}} \{x_0, \ldots, x_{m-1}\} \qquad \mathrm{OV}(a) =_{\mathrm{def}} \{y_0, \ldots, y_{n-1}\}$$

Terms

- Terms are acyclic multisets of atomic terms:

$$t =_{\text{def}} \{a_0, \ldots, a_{n-1}\} : \mathrm{IV}(t) \longrightarrow \mathrm{OV}(t)$$

where no two atomic terms a_i have a common input or output variable, and when:

$$\mathrm{IV}(t) =_{\text{def}} \bigcup \mathrm{IV}(a_i) \backslash \bigcup \mathrm{OV}(a_i) \qquad \mathrm{OV}(t) =_{\text{def}} \bigcup \mathrm{OV}(a_i) \backslash \bigcup \mathrm{IV}(a_i)$$

The term t is said to be acyclic if this graph is:

$$\{\langle x, y\rangle \mid \exists i.\, x \in \mathrm{IV}(a_i) \wedge y \in \mathrm{OV}(a_i)\}$$

More on terms

- Free variables

$$\mathrm{FV}(t) =_{\text{def}} \mathrm{IV}(t) \cup \mathrm{OV}(t)$$

- Bound Variables

$$\mathrm{BV}(t) =_{\text{def}} \mathrm{IV}(t) \cap \mathrm{OV}(t)$$

  We identify terms up to α-equivalence; acyclicity is invariant under α-equivalence.

- Substitution

$$t[y/x] \qquad (x \in \mathrm{FV}(t), y \notin \mathrm{FV}(t))$$

Equational reasoning

Equations

$$t = u$$

provided $\mathrm{IV}(t) = \mathrm{IV}(u)$ and $\mathrm{OV}(t) = \mathrm{OV}(u)$.

Rules Those of an equivalence relation, plus:

- Congruence

$$\frac{t = u}{t, a = u, a}$$

provided $\mathrm{IV}(t) \cap \mathrm{IV}(a) = \mathrm{OV}(t) \cap \mathrm{OV}(a) = \varnothing$.

- Rewiring

$$t, y/x = t[y/x] \qquad (x \in \mathrm{OV}(t))$$

$$t, y/x = t[x/y] \qquad (y \in \mathrm{IV}(t))$$

Corresponding category **L**

- Objects The natural numbers $\mathbb{N}$
- Morphisms These are equivalence classes of terms:

$$[t] : m \to n$$

where $\mathrm{IV}(t) = \{z_0, \ldots, z_{m-1}\}$ and $\mathrm{OV}(t) = \{z'_0, \ldots, z'_{n-1}\}$. (We assume mutually disjoint fixed infinite sequences $z_0, z_1, \ldots$ and $z'_0, z'_1, \ldots$ and etc, of distinct variables.)

- Composition

$$l \xrightarrow{[t]} m \xrightarrow{[u]} n \quad = \quad l \xrightarrow{[t[z''_0/z'_0, \ldots, z''_{m-1}/z'_{m-1}], u[z''_0/z_0, \ldots, z''_{m-1}/z_{m-1}]]} n$$

Corresponding symmetric monoidal Lawvere theory

- Tensor of morphisms

$$(m \xrightarrow{[t]} n) \otimes (m' \xrightarrow{[u]} n') = (m+m') \xrightarrow{[t,u[z_m/z_0,\ldots,z_{m+m'-1}/z_{m'-1}][z'_n/z'_0,\ldots,z'_{n+n'-1}/z'_{n'-1}]]} (n+n')$$

- The functor $I : \mathbb{B}^{\mathrm{op}} \to \mathbf{L}$ is given by:

$$I(f : n \cong n) = [z'_{f(0)}/z_0, \ldots, z'_{f(n-1)}/z_{n-1}]$$

Equational theory for link graphs

- Signature
  - $\| : 2 \to 1$, $\mathrm{NIL} : 0 \to 1$, $\mathrm{NIL}^{-1} : 1 \to 0$
  - $K : 0 \to k$ $(K : k)$.
- Axioms

$$\|(x, y; u), \|(u, z; v) = \|(y, z; u), \|(x, u; v)$$

$$\mathrm{NIL}(; u), \|(u, x; y) = y/x = \mathrm{NIL}(; u), \|(x, u; y)$$

$$\mathrm{NIL}(; x), \mathrm{NIL}^{-1}(x;) =$$

Note We are omitting the multiset brackets.

Abbreviatory conventions

- Two conventions For unary $\mathrm{op}:n \to 1$ (eg, $\|, \mathrm{NIL}$),

$$\mathrm{op}'(\ldots, \mathrm{op}(\ldots), \ldots; \ldots) \equiv_{\mathrm{def}} \mathrm{op}(\ldots; x), \mathrm{op}'(\ldots, x, \ldots; \ldots)$$

$$\mathrm{op}(\ldots)^{x} \equiv_{\mathrm{def}} \mathrm{op}(\ldots; x)$$

- Examples

$$\|(\|(x,y),z)^{v} = \|(x,\|(y,z))^{v}$$

$$\|(\mathrm{NIL}(),x)^{y} = y/x = \|(x,\mathrm{NIL}())^{y}$$

$$\mathrm{NIL}^{-1}(\mathrm{NIL}();) =$$

Normal forms: Atomic terms

- Atoms

$$K(y_0, \ldots, y_{k-1}) : \epsilon \to \{y_0, \ldots, y_{k-1}\} \qquad (K : k \in \mathcal{K})$$

- Elementary substitutions

$$y/x_0, \ldots, x_{n-1} : \{x_0, \ldots, x_{n-1}\} \to \{y\}$$

where the x_i and y are all distinct

Normal forms: terms

These are multisets of atomic terms

$$t = \{a_0, \ldots, a_{n-1}\}/X$$

"closed-off" by a finite set of variables, such that:

- no two atomic terms have a common input variable,
- no input variable of an elementary substitution is an output variable of any a_i.
- for every output variable of an a_i, there is exactly one elementary substitution with that as its output variable, and
- $X \subseteq \bigcup \mathrm{OV}(a_i)$.

We set:

$$\mathrm{IV}(t) =_{\mathrm{def}} \bigcup \mathrm{IV}(a_i) \qquad \mathrm{OV}(t) =_{\mathrm{def}} \bigcup \mathrm{OV}(a_i) \backslash X$$

From link graphs to normal forms

From

$$(V, E, \mathrm{ctrl}, \mathrm{link}: X \dot{\cup} P \to E \dot{\cup} Y): X \to Y$$

where $X = \{n_{i_0}, \ldots, n_{i_{|X|-1}}\}$ and $Y = \{n_{o_0}, \ldots, n_{o_{|Y|-1}}\}$ and $n_0, \ldots$ enumerates the set of names, obtain the term:

$$\begin{aligned}
&(\{K(\mathrm{link}(v,0), \ldots, \mathrm{link}(v, k-1)) \mid v \in V, K = \mathrm{ctrl}(v), K : k\} \\
&\quad + \quad \{n/\mathrm{link}^{-1}(n) \cap X \mid n \in Y\} \\
&\quad + \quad \{e/\mathrm{link}^{-1}(e) \cap X \mid e \in E\})\backslash E : X \to Y
\end{aligned}$$

From link graphs to normal forms

From

$$(V, E, \mathrm{ctrl}, \mathrm{link}: X \dot\cup P \to E \dot\cup Y): X \to Y$$

where $X = \{n_{i_0}, \ldots, n_{i_{|X|-1}}\}$ and $Y = \{n_{o_0}, \ldots, n_{o_{|Y|-1}}\}$ and $n_0, \ldots$ enumerates the names. obtain the term:

$$\begin{array}{ll} (\{K(\mathrm{var}(\mathrm{link}(v,0)), \ldots, \mathrm{var}(\mathrm{link}(v,k-1))) \mid v \in V, K = \mathrm{ctrl}(v), K:k\} & \\ + \quad \{\mathrm{var}(n)/\mathrm{var}(\mathrm{link}^{-1}(n) \cap X) \mid n \in Y\} & \\ + \quad \{\mathrm{var}(e)/\mathrm{var}(\mathrm{link}^{-1}(e) \cap X) \mid e \in E\})\backslash\mathrm{var}(E) : \mathrm{var}(X) \to \mathrm{var}(Y) & \end{array}$$

where:

$$\mathrm{var}(x) = \begin{cases} z_i & x = n_i \\ z_i'' & x = e_i \end{cases} \qquad \mathrm{var}(x) = \begin{cases} z_i' & x = n_i \\ z_i'' & x = e_i \end{cases}$$

Identification of link graphs

Theorem

$\mathbf{Link}_{\mathcal{K}}$ *is the free symmetric monoidal Lawvere theory* $\mathbf{L}$ *with*

1. *a specified commutative monoid* $\langle \|_{\mathbf{L}}, \mathrm{NIL}_{\mathbf{L}} \rangle$ *on* 1,
2. *a specified left inverse* $\mathrm{NIL}_{\mathbf{L}}^{-1}$ *of* $0_{\mathbf{L}}$, *and*
3. *specified* k*-ary morphisms* $K_{\mathbf{L}} : 0 \longrightarrow k$ *on* 1, *for* $K \in \mathcal{K}$.

Summary

- Up to isomorphism the category $\mathbf{Link}_{\mathcal{K}}$ of link graphs with signature $\mathcal{K}$ is given by a standard term model construction.
- This identifies it as the commutative monoidal equational theory of a commutative monoid, whose zero has a left inverse, and with k-ary constants for each k-ary control.

Outline

Definition of bigraphs

- Signature A set $\mathcal{K}$ of controls K with arities $K : k$.
- Concrete Bigraph A tuple

$$F = \langle V, E, \mathrm{ctrl}, \mathrm{prnt}, \mathrm{link} \rangle : \langle m, X \rangle \to \langle n, Y \rangle$$

where:

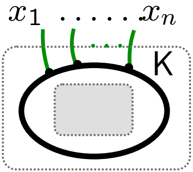
  - $$F_p =_{\mathrm{def}} \langle V, \mathrm{ctrl}, \mathrm{prnt} \rangle : m \to n$$

    is a concrete place graph, and
  - $$F_l =_{\mathrm{def}} \langle V, E, \mathrm{ctrl}, \mathrm{link} \rangle : X \to Y$$

    is a concrete link graph.
- Abstract Bigraph An isomorphism class $[F]$ of concrete bigraphs. And we set $[F]_p =_{\mathrm{def}} [F_p]$ and $[F]_l =_{\mathrm{def}} [F_l]$.

Example Bigraphs

discrete ion:



$$K_X : \langle 1, \epsilon \rangle \to \langle 1, X \rangle$$

plus:

- Every place graph $F : m \to n$ can be regarded as a bigraph $F : \langle m, \epsilon \rangle \to \langle n, \epsilon \rangle$.
  Examples: $1 : 0 \to 1$, $join : 2 \to 1$.
- Every link graph $F : X \to Y$ can be regarded as a bigraph $F : \langle 0, X \rangle \to \langle 0, Y \rangle$.
  Examples: Elementary substitutions $y/X : X \to \{y\}$ and closures $/x : x \to \epsilon$.

A partial symmetric monoidal category of bigraphs

- Objects Pairs $\langle m, X\rangle$
- Morphisms Abstract bigraphs

$$[F]:\langle m, X\rangle \to \langle n, Y\rangle$$

where $F:\langle m, X\rangle \to \langle n, Y\rangle$.

- Composition is uniquely specified by:

$$([G]\circ[F])_p = [G_p]\circ[F_p] \qquad ([G]\circ[F])_l = [G_l]\circ[F_l]$$

- Tensor Product

$$\langle m, X\rangle \otimes \langle n, Y\rangle = \langle m+n, X \dot{\cup} Y\rangle$$

and the (partial) tensor of morphisms is inherited from the place and link tensors, analogously to the case of composition.

A total symmetric monoidal category of bigraphs

The smc $\mathbf{Bigraph}_{\mathcal{K}}$ is given as follows:

- Objects Pairs $\langle m, n \rangle \in \mathbb{N}^2$
- Morphisms Abstract bigraphs

$$[F] : \langle m, n \rangle \to \langle m', n' \rangle$$

where $[F_p] : m \to n$ in $\mathbf{Place}_{\mathcal{K}}$ and $[F_l] : m' \to n'$ in $\mathbf{Link}_{\mathcal{K}}$.

- Composition Inherited as before.
- Tensor Product On objects:

$$\langle m, n \rangle \otimes \langle m', n' \rangle =_{\text{def}} \langle m + m', n + n' \rangle$$

and on morphisms defined as before, so it is total.

Some structure in $\mathbf{Bigraph}_{\mathcal{K}}$

- Distinguished place and link objects P and L where:

$$\mathrm{P} =_{\mathrm{def}} \langle 1, 0\rangle \qquad \mathrm{L} =_{\mathrm{def}} \langle 0, 1\rangle$$

- A commutative monoid $\langle 1, \mathit{join}\rangle$ on P.
- A commutative monoid $\langle n_0/\{n_0, n_1\}, n_0/\epsilon\rangle$ on L whose zero has a left inverse $/n_0$.
- For each control $K\!:\!k$ a morphism:

$$K_{n_0,\ldots,n_{k-1}} : \mathrm{P} \to \mathrm{P} \otimes \overbrace{\mathrm{L} \otimes \ldots \otimes \mathrm{L}}^{k \text{ times}}$$

Multisorted Symmetric Monoidal Lawvere Theories (aka coloured PROPs)

Assume a set S of sorts.
These are structures:

$$(\mathbb{B}^{\mathrm{S}})^{\mathrm{op}} \xrightarrow{I} \mathbf{L}$$

where

- $\mathbf{L}$ is a small symmetric monoidal category, and
- I is a strict symmetric monoidal identity-on-objects functor

Example With $\mathrm{S}_b =_{\mathrm{def}} \{\mathrm{p}, \mathrm{l}\}$, bigraphs form an S_b-sorted symmetric monoidal Lawvere theory

$$(\mathbb{B}^{\mathrm{S}_b})^{\mathrm{op}} \xrightarrow{I} \mathbf{Bigraphs}_{\mathcal{K}}$$

for an evident I (and suitably identifying $\mathbb{B}^{\mathrm{S}_b}$ and $\mathbb{B}^2$).

Multisorted symmetric monoidal equational logic

- Signature Σ of *sorted operation symbols* $\mathrm{op}:\mathbf{s} \to \mathbf{s}'$, for $\mathbf{s}, \mathbf{s}' \in \mathrm{S}_b^*$.
- Sorted variables x^s $(s \in \mathrm{S})$.
- Atomic terms are:
  - Wires
    $$y^s/x^s : \{x^s\} \to \{y^s\}$$
  - Boxes
    $$\mathrm{op}(x_0^{s_0}, \ldots, x_{m-1}^{s_{m-1}}; y_0^{s'_0}, \ldots, y_{n-1}^{s'_{n-1}}) : \{x_0^{s_0}, \ldots, x_{m-1}^{s_{m-1}}\} \to \{y_0^{s'_0}, \ldots, y_{n-1}^{s'_{n-1}}\}$$
    for $\mathrm{op}: s_0 \ldots s_{m-1} \to s'_0 \ldots s'_{n-1}$
- Terms are suitable multisets of atomic terms:
  $$t = \{a_0, \ldots, a_{n-1}\} : \mathrm{IV}(t) \longrightarrow \mathrm{OV}(t)$$

Input and output variables, and variable constraints as before.

Unary open place normal forms

- Variables Place variables: $p, q, \ldots$ Link variables $x, y, \ldots$
- Atomic terms
  - Molecules

$$\frac{t : \epsilon \rightarrow Y}{K(t; y_0, \ldots, y_{k-1}) : \epsilon \rightarrow Y \cup \{y_0, \ldots, y_{k-1}\}} \qquad (K : k \in \mathcal{K})$$

  - Place Variables

$$p : \epsilon \rightarrow \epsilon$$

- Terms Multisets of atomic terms

$$\frac{a_i : \epsilon \rightarrow Y_i \quad (i = 0, n-1)}{\{a_o, \ldots, a_{n-1}\} : \epsilon \rightarrow \bigcup Y_i}$$

with no place variable occurring twice.
Correspond to bigraphs of type $m \rightarrow \langle 1, Y \rangle$ with no edges.

Unary normal forms

These have the form:

$$\frac{t : \epsilon \to Y \qquad w_i : X_i \to Y_i \quad (i < n)}{\langle t, \{w_0, \ldots, w_{n-1}\}\rangle / X : \bigcup X_i \to (Y \cup \bigcup Y_i) \backslash X}$$

where the “wires" w_i are either:

- substitutions $y/x_0, \ldots, x_{n-1} : \{x_0, \ldots, x_{n-1}\} \to \{y\}$, or
- closures $/x : \{x\} \to \epsilon$

and such that:

- no two of the w_i have a common input link variable,
- no input link variable of an a_i is an output link variable of t or any a_i,
- for every $y \in \mathrm{OLV}(w_i) \cup \mathrm{OLV}(t)$, there is exactly one elementary substitution of type $X \to y$, and
- $X \subseteq \mathrm{OLV}(t) \cup \bigcup \mathrm{OLV}(a_i)$.

Correspond to bigraphs of type $\langle m, X\rangle \to \langle 1, Y\rangle$.

Characterisation of **Bigraphs**$_{\mathcal{K}}$

Theorem

Bigraphs$_{\mathcal{K}}$ *is the free* S_b*-sorted sm Lawvere theory* $(\mathbb{B}^{\mathrm{S}})^{\mathrm{op}} \xrightarrow{I} \mathbf{L}$ *with:*

1. *a specified commutative monoid on* $\mathrm{P}_{\mathbf{L}} =_{def} I(1,0)$,
2. *a specified commutative monoid on* $\mathrm{L}_{\mathbf{L}} =_{def} I(0,1)$ *with a left-inverse of its zero, and*
3. *a specified morphism*

$$K_{\mathbf{L}} : \mathrm{P} \rightarrow \mathrm{P} \otimes \overbrace{\mathrm{L} \otimes \ldots \otimes \mathrm{L}}^{k \text{ times}}$$

for each $K : k \in \mathcal{K}$.

Summary

- Up to isomorphism the category **Bigraph**$_{\mathcal{K}}$ of bigraphs with signature $\mathcal{K}$ is given by a standard term model construction.
- This identifies it as the multi-sorted commutative monoidal equational theory with:
  - two sorts P and L,
  - a commutative monoid on P,
  - a commutative monoid on L whose zero has a left inverse, and
  - morphisms

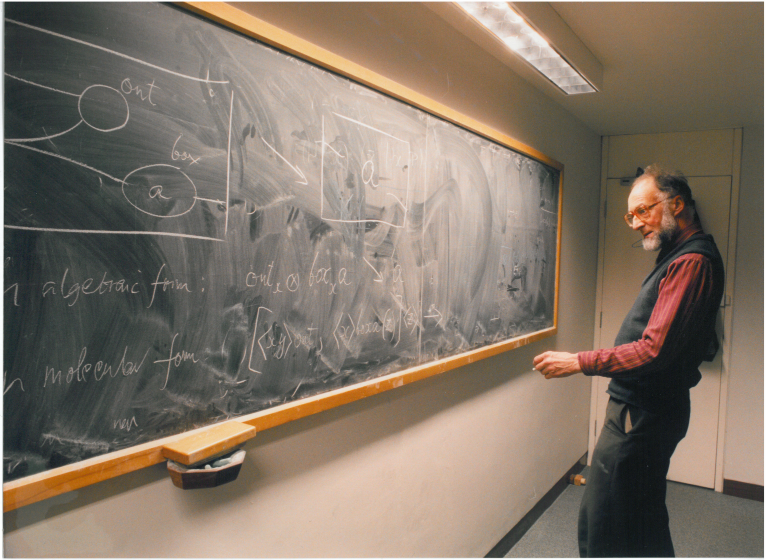
$$K : \mathrm{P} \to \mathrm{P} \otimes \overbrace{\mathrm{L} \otimes \ldots \otimes \mathrm{L}}^{k \text{ times}}$$

for $K : k \in \mathcal{K}$.

- *Finish present work.*
- Statics Relate to sorting and other kinds of bigraphs, eg. binding bigraphs (cf Garner, Hirschowitz, and Pardon).
- Dynamics Relate to term-rewriting. In sm equational logic, CCS style, a rule is just an oriented equation $t \Rightarrow t'$, as usual, and its application takes on a simple form:

  $$\frac{t \Rightarrow t'}{t, u \Rightarrow t', u}$$

  (cf Krivine, Milner, Troina).
- Externally For bio applications, consider replacing link graphs by κ-graphs, after Danos, Laneve.


out
box
a
algebraic form:
molecular form