The time/space continuum: Continuous-time and continuous-space process algebras

Stephen Gilmore
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PASTA Workshop, Edinburgh, 7th September 2005
This talk is about PEPA.
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PEPA with CMTC semantics — a continuous-time process algebra
PEPA with ODE semantics — a continuous-space process algebra
This talk is about PEPA.

PEPA with CMTC semantics — a continuous-time process algebra
PEPA with ODE semantics — a continuous-space process algebra

Are they different?
A process is a **deterministic process** if knowledge of its values up to and including time $t$ allows us to **unambiguously** predict its value at any infinitesimally later time $t + dt$. 
A set of ordinary differential equations defines a memoryless deterministic process.

\[
X(t + dt) = X(t) + f(X(t), t)dt
\]

\[
\frac{dX}{dt} = f(X, t)
\]
A process is a **stochastic process** if knowledge of its values up to and including time $t$ allows us to **probabilistically** predict its value at any infinitesimally later time $t + dt$. 

**Background: Stochastic processes**
A process is a **stochastic process** if knowledge of its values up to and including time $t$ allows us to **probabilistically** predict its value at any infinitesimally later time $t + dt$.

Stochastic processes subsume deterministic processes.
A continuous-time Markov chain is a memoryless stochastic process.

\[ \Pr(X(t_{n+1}) = x_{n+1} \mid X(t_n) = x_n, \ldots, X(t_1) = x_1) = \Pr(X(t_{n+1}) = x_{n+1} \mid X(t_n) = x_n) \]
Background: Same mean, different standard deviations
Background: Same standard deviations, different mean
Two classes of PEPA models can be used to generate ODEs.

**High/Low models**¹: High and low concentrations of components are modelled, to indicate increase or decrease in quantity.

**Direct style**²: Models encode the behaviour of the system directly without the use of high and low labels.

This talk: models in direct style.


Outline

1 Quantitative modelling with CTMCs and ODEs
   - Modelling with quantified process algebras
   - Analysis based on Continuous-time Markov Chains
   - Analysis based on Ordinary Differential Equations

2 Performance modelling with process algebras
   - Performance Evaluation Process Algebra
   - PEPA model of jobs and servers
   - Analysis of the model

3 Comparing performance measures
   - Computed with continuous time
   - Computed with continuous space
   - Comparison of computed measures

4 Commentary and comparison
1. Quantitative modelling with CTMCs and ODEs
   - Modelling with quantified process algebras
   - Analysis based on Continuous-time Markov Chains
   - Analysis based on Ordinary Differential Equations

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   - Performance Evaluation Process Algebra
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4. Commentary and comparison
Modelling with quantified process algebras

Tiny example

\[ P_1 \overset{\text{def}}{=} (\text{start, } r).P_2 \quad P_2 \overset{\text{def}}{=} (\text{run, } r).P_3 \quad P_3 \overset{\text{def}}{=} (\text{stop, } r).P_1 \]

System \overset{\text{def}}{=} (P_1 \parallel P_1)
Modelling with quantified process algebras

Tiny example

\[
P_1 \overset{\text{def}}{=} (\text{start}, r).P_2 \quad P_2 \overset{\text{def}}{=} (\text{run}, r).P_3 \quad P_3 \overset{\text{def}}{=} (\text{stop}, r).P_1
\]

System \overset{\text{def}}{=} (P_1 \parallel P_1)

This example defines a system with nine reachable states:

1. \( P_1 \parallel P_1 \)
2. \( P_1 \parallel P_2 \)
3. \( P_1 \parallel P_3 \)
4. \( P_2 \parallel P_1 \)
5. \( P_2 \parallel P_2 \)
6. \( P_2 \parallel P_3 \)
7. \( P_3 \parallel P_1 \)
8. \( P_3 \parallel P_2 \)
9. \( P_3 \parallel P_3 \)

The transitions between states have quantified duration \( r \) which can be evaluated against a CTMC or ODE interpretation.
Analysis based on Continuous-time Markov Chains

Tiny example

\[ P_1 \overset{\text{def}}{=} (\text{start, } r).P_2 \quad P_2 \overset{\text{def}}{=} (\text{run, } r).P_3 \quad P_3 \overset{\text{def}}{=} (\text{stop, } r).P_1 \]
\[ \text{System} \overset{\text{def}}{=} (P_1 \parallel P_1) \]

Using transient analysis we can evaluate the probability of each state with respect to time. For \( t = 0 \):

\[
\begin{array}{cccc}
1 & 1.0000 & 4 & 0.0000 \\
2 & 0.0000 & 5 & 0.0000 \\
3 & 0.0000 & 6 & 0.0000 \\
4 & 0.0000 & 7 & 0.0000 \\
5 & 0.0000 & 8 & 0.0000 \\
6 & 0.0000 & 9 & 0.0000 \\
\end{array}
\]
Quantitative modelling with CTMCs and ODEs
Performance modelling with process algebras
Comparing performance measures
Commentary and comparison

Analysis based on Continuous-time Markov Chains

Tiny example

\[ P_1 \overset{\text{def}}{=} (\text{start}, r).P_2 \quad P_2 \overset{\text{def}}{=} (\text{run}, r).P_3 \quad P_3 \overset{\text{def}}{=} (\text{stop}, r).P_1 \]

System \overset{\text{def}}{=} (P_1 \parallel P_1)

Using transient analysis we can evaluate the probability of each state with respect to time. For \( t = 1 \):

\[
\begin{align*}
1 & \quad 0.1642 & 4 & \quad 0.1567 & 7 & \quad 0.0842 \\
2 & \quad 0.1567 & 5 & \quad 0.1496 & 8 & \quad 0.0804 \\
3 & \quad 0.0842 & 6 & \quad 0.0804 & 9 & \quad 0.0432
\end{align*}
\]
Tiny example

\[ P_1 \overset{\text{def}}{=} (\text{start}, r).P_2 \quad P_2 \overset{\text{def}}{=} (\text{run}, r).P_3 \quad P_3 \overset{\text{def}}{=} (\text{stop}, r).P_1 \]

System \overset{\text{def}}{=} (P_1 \parallel P_1)

Using transient analysis we can evaluate the probability of each state with respect to time. For \( t = 2 \):

1. 0.1056
2. 0.1159
3. 0.1034
4. 0.1159
5. 0.1272
6. 0.1135
7. 0.1034
8. 0.1135
9. 0.1012
Analysis based on Continuous-time Markov Chains

Tiny example

\[
P_1 \overset{\text{def}}{=} (\text{start}, r).P_2 \quad P_2 \overset{\text{def}}{=} (\text{run}, r).P_3 \quad P_3 \overset{\text{def}}{=} (\text{stop}, r).P_1
\]

System \overset{\text{def}}{=} (P_1 \parallel P_1)

Using transient analysis we can evaluate the probability of each state with respect to time. For \( t = 3 \):

\[
\begin{align*}
1 & : \ 0.1082 \\
2 & : \ 0.1106 \\
3 & : \ 0.1100 \\
4 & : \ 0.1106 \\
5 & : \ 0.1132 \\
6 & : \ 0.1125 \\
7 & : \ 0.1100 \\
8 & : \ 0.1125 \\
9 & : \ 0.1119
\end{align*}
\]
### Tiny example

\[
P_1 \overset{\text{def}}{=} (\text{start}, r).P_2 \quad P_2 \overset{\text{def}}{=} (\text{run}, r).P_3 \quad P_3 \overset{\text{def}}{=} (\text{stop}, r).P_1
\]

System \overset{\text{def}}{=} (P_1 \parallel P_1)

Using transient analysis we can evaluate the probability of each state with respect to time. For \( t = 4 \):

<table>
<thead>
<tr>
<th>State</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.1106</td>
</tr>
<tr>
<td>2</td>
<td>0.1108</td>
</tr>
<tr>
<td>3</td>
<td>0.1111</td>
</tr>
<tr>
<td>4</td>
<td>0.1108</td>
</tr>
<tr>
<td>5</td>
<td>0.1110</td>
</tr>
<tr>
<td>6</td>
<td>0.1113</td>
</tr>
<tr>
<td>7</td>
<td>0.1111</td>
</tr>
<tr>
<td>8</td>
<td>0.1113</td>
</tr>
<tr>
<td>9</td>
<td>0.1116</td>
</tr>
</tbody>
</table>

Stephen Gilmore. LFCS, University of Edinburgh.
Analysis based on Continuous-time Markov Chains

Tiny example

\[ P_1 \overset{\text{def}}{=} (\text{start}, r).P_2 \quad P_2 \overset{\text{def}}{=} (\text{run}, r).P_3 \quad P_3 \overset{\text{def}}{=} (\text{stop}, r).P_1 \]

\[ \text{System} \overset{\text{def}}{=} (P_1 \parallel P_1) \]

Using transient analysis we can evaluate the probability of each state with respect to time. For \( t = 5 \):

<table>
<thead>
<tr>
<th>State</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.1111</td>
</tr>
<tr>
<td>2</td>
<td>0.1110</td>
</tr>
<tr>
<td>3</td>
<td>0.1111</td>
</tr>
<tr>
<td>4</td>
<td>0.1110</td>
</tr>
<tr>
<td>5</td>
<td>0.1110</td>
</tr>
<tr>
<td>6</td>
<td>0.1111</td>
</tr>
<tr>
<td>7</td>
<td>0.1111</td>
</tr>
<tr>
<td>8</td>
<td>0.1111</td>
</tr>
<tr>
<td>9</td>
<td>0.1111</td>
</tr>
</tbody>
</table>
Quantitative modelling with CTMCs and ODEs
Performance modelling with process algebras
Comparing performance measures
Commentary and comparison

## Analysis based on Continuous-time Markov Chains

### Tiny example

\[
P_1 \overset{\text{def}}{=} (\text{start}, r).P_2 \quad P_2 \overset{\text{def}}{=} (\text{run}, r).P_3 \quad P_3 \overset{\text{def}}{=} (\text{stop}, r).P_1
\]

System \(\overset{\text{def}}{=} (P_1 \parallel P_1)\)

Using transient analysis we can evaluate the probability of each state with respect to time. For \(t = 6\):

<table>
<thead>
<tr>
<th></th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.1111</td>
</tr>
<tr>
<td>2</td>
<td>0.1111</td>
</tr>
<tr>
<td>3</td>
<td>0.1111</td>
</tr>
<tr>
<td>4</td>
<td>0.1111</td>
</tr>
<tr>
<td>5</td>
<td>0.1110</td>
</tr>
<tr>
<td>6</td>
<td>0.1111</td>
</tr>
<tr>
<td>7</td>
<td>0.1111</td>
</tr>
<tr>
<td>8</td>
<td>0.1111</td>
</tr>
<tr>
<td>9</td>
<td>0.1111</td>
</tr>
</tbody>
</table>
Analysis based on Continuous-time Markov Chains

Tiny example

\[ P_1 \overset{\text{def}}{=} (\text{start}, r).P_2 \quad P_2 \overset{\text{def}}{=} (\text{run}, r).P_3 \quad P_3 \overset{\text{def}}{=} (\text{stop}, r).P_1 \]

System \overset{\text{def}}{=} (P_1 \parallel P_1)

Using transient analysis we can evaluate the probability of each state with respect to time. For \( t = 7 \):

<table>
<thead>
<tr>
<th>State</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.1111</td>
</tr>
<tr>
<td>2</td>
<td>0.1111</td>
</tr>
<tr>
<td>3</td>
<td>0.1111</td>
</tr>
<tr>
<td>4</td>
<td>0.1111</td>
</tr>
<tr>
<td>5</td>
<td>0.1111</td>
</tr>
<tr>
<td>6</td>
<td>0.1111</td>
</tr>
<tr>
<td>7</td>
<td>0.1111</td>
</tr>
<tr>
<td>8</td>
<td>0.1111</td>
</tr>
<tr>
<td>9</td>
<td>0.1111</td>
</tr>
</tbody>
</table>
Analysis based on Ordinary Differential Equations

Tiny example

\[ P_1 \overset{\text{def}}{=} (\text{start}, r).P_2 \quad P_2 \overset{\text{def}}{=} (\text{run}, r).P_3 \quad P_3 \overset{\text{def}}{=} (\text{stop}, r).P_1 \]

System \overset{\text{def}}{=} (P_1 \parallel P_1)

Using the ordinary differential equation semantics we can compute the expected number of each type of component.

For \( t = 0 \):

\[ \begin{align*}
P_1 &\quad 2.0000 \\
P_2 &\quad 0.0000 \\
P_3 &\quad 0.0000
\end{align*} \]
Analysis based on Ordinary Differential Equations

Tiny example

\[
P_1 \overset{\text{def}}{=} (\text{start}, r).P_2 \quad P_2 \overset{\text{def}}{=} (\text{run}, r).P_3 \quad P_3 \overset{\text{def}}{=} (\text{stop}, r).P_1
\]

System \overset{\text{def}}{=} (P_1 \parallel P_1)

Using the ordinary differential equation semantics we can compute the expected number of each type of component.

For \( t = 1 \):

\[
\begin{align*}
P_1 & \quad 0.8121 \\
P_2 & \quad 0.7734 \\
P_3 & \quad 0.4144
\end{align*}
\]
Analysis based on Ordinary Differential Equations

**Tiny example**

\[
P_1 \overset{\text{def}}{=} (\text{start}, r) . P_2 \quad P_2 \overset{\text{def}}{=} (\text{run}, r) . P_3 \quad P_3 \overset{\text{def}}{=} (\text{stop}, r) . P_1
\]

\[
\text{System} \overset{\text{def}}{=} (P_1 \parallel P_1)
\]

Using the ordinary differential equation semantics we can compute the expected number of each type of component.

For \( t = 2 \):

\[
\begin{align*}
P_1 & \quad 0.6490 \\
P_2 & \quad 0.7051 \\
P_3 & \quad 0.6457
\end{align*}
\]
Analysis based on Ordinary Differential Equations

Tiny example

\[ P_1 \overset{\text{def}}{=} (\text{start}, r).P_2 \quad P_2 \overset{\text{def}}{=} (\text{run}, r).P_3 \quad P_3 \overset{\text{def}}{=} (\text{stop}, r).P_1 \]
\[ \text{System} \overset{\text{def}}{=} (P_1 \parallel P_1) \]

Using the ordinary differential equation semantics we can compute the expected number of each type of component.

For \( t = 3 \):

\[ P_1 \quad 0.6587 \]
\[ P_2 \quad 0.6719 \]
\[ P_3 \quad 0.6692 \]
Analysis based on Ordinary Differential Equations

Tiny example

\[
P_1 \overset{\text{def}}{=} (\text{start}, r).P_2 \quad P_2 \overset{\text{def}}{=} (\text{run}, r).P_3 \quad P_3 \overset{\text{def}}{=} (\text{stop}, r).P_1
\]

System \overset{\text{def}}{=} (P_1 \parallel P_1)

Using the ordinary differential equation semantics we can compute the expected number of each type of component.

For \( t = 4 \):

\[
\begin{align*}
P_1 & \quad 0.6648 \\
P_2 & \quad 0.6665 \\
P_3 & \quad 0.6685
\end{align*}
\]
### Analysis based on Ordinary Differential Equations

#### Tiny example

<table>
<thead>
<tr>
<th>$P_1$</th>
<th>$P_2$</th>
<th>$P_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$= (start, r).P_2$</td>
<td>$= (run, r).P_3$</td>
<td>$= (stop, r).P_1$</td>
</tr>
</tbody>
</table>

System $= (P_1 \parallel P_1)$

Using the ordinary differential equation semantics we can compute the expected number of each type of component.

For $t = 5$:

<table>
<thead>
<tr>
<th>$P_1$</th>
<th>$P_2$</th>
<th>$P_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.6666</td>
<td>0.6663</td>
<td>0.6669</td>
</tr>
</tbody>
</table>
Analysis based on Ordinary Differential Equations

Tiny example

\[ P_1 \overset{\text{def}}{=} (\text{start}, r).P_2 \]
\[ P_2 \overset{\text{def}}{=} (\text{run}, r).P_3 \]
\[ P_3 \overset{\text{def}}{=} (\text{stop}, r).P_1 \]
\[ \text{System} \overset{\text{def}}{=} (P_1 \parallel P_1) \]

Using the ordinary differential equation semantics we can compute the expected number of each type of component.

For \( t = 6 \):

\[ P_1 \quad 0.6666 \]
\[ P_2 \quad 0.6666 \]
\[ P_3 \quad 0.6666 \]
Tiny example

\[ P_1 \overset{\text{def}}{=} (\text{start, } r).P_2 \quad P_2 \overset{\text{def}}{=} (\text{run, } r).P_3 \quad P_3 \overset{\text{def}}{=} (\text{stop, } r).P_1 \]

System \overset{\text{def}}{=} (P_1 \parallel P_1)

Using the ordinary differential equation semantics we can compute the expected number of each type of component.

For \( t = 7 \):  
\[
\begin{align*}
P_1 & \quad 0.6666 \\
P_2 & \quad 0.6666 \\
P_3 & \quad 0.6666
\end{align*}
\]
Analysis based on Ordinary Differential Equations

Slightly larger example

\[ P_1 \overset{\text{def}}{=} (\text{start}, r).P_2 \quad P_2 \overset{\text{def}}{=} (\text{run}, r).P_3 \quad P_3 \overset{\text{def}}{=} (\text{stop}, r).P_1 \]

System \overset{\text{def}}{=} (P_1 \parallel P_1 \parallel P_1)

A slightly larger example with a third copy of the process also initiated in state \( P_1 \).

For \( t = 0 \):

\[
\begin{align*}
P_1 & \quad 3.0000 \\
P_2 & \quad 0.0000 \\
P_3 & \quad 0.0000 \\
\end{align*}
\]
Analysis based on Ordinary Differential Equations

Slightly larger example

\[ P_1 \overset{\text{def}}{=} (\text{start}, r).P_2 \quad P_2 \overset{\text{def}}{=} (\text{run}, r).P_3 \quad P_3 \overset{\text{def}}{=} (\text{stop}, r).P_1 \]

System \overset{\text{def}}{=} (P_1 | P_1 | P_1)

A slightly larger example with a third copy of the process also initiated in state \( P_1 \).

For \( t = 1 \):

\[
\begin{align*}
P_1 &\quad 1.1782 \\
P_2 &\quad 1.1628 \\
P_3 &\quad 0.6590
\end{align*}
\]
Analysis based on Ordinary Differential Equations

Slightly larger example

\[
P_1 \overset{\text{def}}{=} (\text{start}, r).P_2 \quad P_2 \overset{\text{def}}{=} (\text{run}, r).P_3 \quad P_3 \overset{\text{def}}{=} (\text{stop}, r).P_1
\]

System \overset{\text{def}}{=} (P_1 \parallel P_1 \parallel P_1)

A slightly larger example with a third copy of the process also initiated in state \( P_1 \).

For \( t = 2 \):

\[
\begin{align*}
P_1 & \quad 0.9766 \\
P_2 & \quad 1.0754 \\
P_3 & \quad 0.9479
\end{align*}
\]
Analysis based on Ordinary Differential Equations

Slightly larger example

\[ P_1 \overset{\text{def}}{=} (\text{start}, r).P_2 \quad P_2 \overset{\text{def}}{=} (\text{run}, r).P_3 \quad P_3 \overset{\text{def}}{=} (\text{stop}, r).P_1 \]

\[ \text{System} \overset{\text{def}}{=} (P_1 \parallel P_1 \parallel P_1) \]

A slightly larger example with a third copy of the process also initiated in state \( P_1 \).

For \( t = 3 \):

\begin{align*}
P_1 & \quad 0.9838 \\
P_2 & \quad 1.0142 \\
P_3 & \quad 1.0020
\end{align*}
A slightly larger example with a third copy of the process also initiated in state $P_1$.

For $t = 4$:

- $P_1 \approx 0.9981$
- $P_2 \approx 0.9995$
- $P_3 \approx 1.0023$
A slightly larger example with a third copy of the process also initiated in state $P_1$.

For $t = 5$:

- $P_1 \approx 1.0001$
- $P_2 \approx 0.9996$
- $P_3 \approx 1.0003$
Slightly larger example

\[ P_1 \overset{\text{def}}{=} (\text{start}, r).P_2 \quad P_2 \overset{\text{def}}{=} (\text{run}, r).P_3 \quad P_3 \overset{\text{def}}{=} (\text{stop}, r).P_1 \]

\[ \text{System} \overset{\text{def}}{=} (P_1 \parallel P_1 \parallel P_1) \]

A slightly larger example with a third copy of the process also initiated in state \( P_1 \).

For \( t = 6 \):

\[ \begin{align*}
P_1 & \quad 1.0001 \\
P_2 & \quad 0.9999 \\
P_3 & \quad 1.0000
\end{align*} \]
A slightly larger example with a third copy of the process also initiated in state $P_1$.

For $t = 7$:

- $P_1 = 1.0000$
- $P_2 = 0.9999$
- $P_3 = 0.9999$
### Analysis based on Ordinary Differential Equations

#### Slightly larger example

<table>
<thead>
<tr>
<th></th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_1$</td>
<td>$\text{def} = (\text{start, r}).P_2$</td>
</tr>
<tr>
<td>$P_2$</td>
<td>$\text{def} = (\text{run, r}).P_3$</td>
</tr>
<tr>
<td>$P_3$</td>
<td>$\text{def} = (\text{stop, r}).P_1$</td>
</tr>
<tr>
<td>System</td>
<td>$\text{def} = (P_1 \parallel P_1 \parallel P_1)$</td>
</tr>
</tbody>
</table>

A slightly larger example with a third copy of the process also initiated in state $P_1$.

For $t = 8$:

<table>
<thead>
<tr>
<th>$P_1$</th>
<th>1.0000</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_2$</td>
<td>1.0000</td>
</tr>
<tr>
<td>$P_3$</td>
<td>1.0000</td>
</tr>
</tbody>
</table>
An ODE specifies how the value of some continuous variable varies over continuous time. For example, the temperature in a container may be modelled by an ODE describing how the temperature will change dependent on the current temperature and pressure. The pressure can be similarly modelled and the equations together form a system of ODEs describing the state of the container.
What just happened?

In a PEPA model the state at any current time is the local derivative or state of each component of the model. When we have large numbers of repeated components it can make sense to represent each component type as a continuous variable, and the state of the model as a whole as the set of such variables. The evolution of each such variable can then be described by an ODE.
The PEPA definitions of the component specify the activities which can increase or decrease the number of components exhibited in the current state. The cooperations show when the number of instances of another component will have an influence on the evolution of this component.
Isn’t this just the Chapman-Kolmogorov equations?

It is possible to perform transient analysis of a continuous-time Markov chain by solving the Chapman-Kolmogorov differential equations:

$$\frac{d\pi(t)}{dt} = \pi(t)Q$$

[Stewart, 1994]
Isn’t this just the Chapman-Kolmogorov equations?

It is possible to perform transient analysis of a continuous-time Markov chain by solving the Chapman-Kolmogorov differential equations:

\[ \frac{d\pi(t)}{dt} = \pi(t)Q \]

[Stewart, 1994]

That’s not what we’re doing. We go directly to ODEs.
The bottleneck for Markovian modelling of systems is the size of the solution vector, which is bounded by the product of the state-space sizes of the processes which are composed in parallel ("state-space explosion").
The bottleneck for Markovian modelling of systems is the size of the solution vector, which is bounded by the product of the state-space sizes of the processes which are composed in parallel (“state-space explosion”).

The size of the solution vector for the system of ODEs may be exponentially smaller.
Outline

1. Quantitative modelling with CTMCs and ODEs
   - Modelling with quantified process algebras
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   - Analysis based on Ordinary Differential Equations

2. Performance modelling with process algebras
   - Performance Evaluation Process Algebra
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4. Commentary and comparison

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Performance Evaluation Process Algebra

PEPA components perform activities either independently or in co-operation with other components.
Performance Evaluation Process Algebra

PEPA components perform activities either independently or in co-operation with other components.

The rate at which an activity is performed is quantified by some component in each co-operation. The symbol $\top$ indicates that the rate value is quantified elsewhere (not in this component).
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$$(\alpha, r).P$$ Prefix

$P_1 + P_2$ Choice

$P_1 \lhd P_2$ Co-operation

$P/L$ Hiding

$X$ Variable
$P_1 \parallel P_2$ is a derived form for $P_1 \lozenge P_2$. 

Because we are interested in transient behaviour we use the deadlocked process $\text{Stop}$.

When working with large numbers of jobs and servers, we write $P[n]$ to denote an array of $n$ copies of $P$ executing in parallel.

$P[5] \equiv (P \parallel P \parallel P \parallel P \parallel P)$
$P_1 \parallel P_2$ is a derived form for $P_1 \beth_\emptyset P_2$.

Because we are interested in transient behaviour we use the deadlocked process *Stop*.
$P_1 \parallel P_2$ is a derived form for $P_1 \boxplus_\emptyset P_2$.

Because we are interested in transient behaviour we use the deadlocked process $Stop$.

When working with large numbers of jobs and servers, we write $P[n]$ to denote an array of $n$ copies of $P$ executing in parallel.
Derived forms and additional syntax

\[ P_1 \parallel P_2 \] is a derived form for \[ P_1 \uplus P_2. \]

Because we are interested in transient behaviour we use the deadlocked process \textit{Stop}.

When working with large numbers of jobs and servers, we write \[ P[n] \] to denote an array of \( n \) copies of \( P \) executing in parallel.

\[ P[5] \equiv (P \parallel P \parallel P \parallel P \parallel P) \]
Consider jobs with a number of ordered stages. (Here three.)
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Jobs must be loaded onto a node before execution. Stage 1 must be completed before Stage 2 and Stage 2 before Stage 3. After Stage 3 the job is cleared by being unloaded from the node, and is then finished.
Modelling jobs and nodes

Consider jobs with a number of ordered stages. (Here three.)

Jobs must be loaded onto a node before execution. Stage 1 must be completed before Stage 2 and Stage 2 before Stage 3. After Stage 3 the job is cleared by being unloaded from the node, and is then finished.

Here the number of compute jobs is larger than the number of nodes available to execute them. Nodes specify the rate at which jobs are completed.
PEPA model of jobs and nodes

Jobs

\[
\begin{align*}
Job & \overset{\text{def}}{=} (load, \top).Job1 \\
Job1 & \overset{\text{def}}{=} (stage1, \top).Job2 \\
Job2 & \overset{\text{def}}{=} (stage2, \top).Job3 \\
Job3 & \overset{\text{def}}{=} (stage3, \top).Clearing \\
Clearing & \overset{\text{def}}{=} (unload, \top).Finished \\
Finished & \overset{\text{def}}{=} \text{Stop}
\end{align*}
\]
PEPA model of jobs and nodes

\[
\begin{align*}
\text{NodeIdle} & \overset{\text{def}}{=} (\text{load}, r_0).\text{Node1} \\
\text{Node1} & \overset{\text{def}}{=} (\text{stage1}, r_1).\text{Node2} \\
\text{Node2} & \overset{\text{def}}{=} (\text{stage2}, r_2).\text{Node3} \\
\text{Node3} & \overset{\text{def}}{=} (\text{stage3}, r_3).\text{Node4} \\
\text{Node4} & \overset{\text{def}}{=} (\text{unload}, r_0).\text{NodeIdle}
\end{align*}
\]
PEPA model of jobs and nodes

System

\[ \text{Node} \text{idle}[100] \parallel L \rightarrow \text{Job}[1000] \]

where \( L \) is \{ \text{load}, \text{stage1}, \text{stage2}, \text{stage3}, \text{unload} \}.  

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Continuous-time and continuous-space process algebras
Analysis of the model proceeds by choosing particular values for the rates. The values below are chosen to make the analysis easy to follow.

<table>
<thead>
<tr>
<th>Rate</th>
<th>Value</th>
<th>Interpretation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_0$</td>
<td>1</td>
<td>(Un)loading takes one time unit</td>
</tr>
<tr>
<td>$r_1$</td>
<td>0.1</td>
<td>Stage 1 takes ten time units</td>
</tr>
<tr>
<td>$r_2$</td>
<td>0.05</td>
<td>Stage 2 takes twenty time units</td>
</tr>
<tr>
<td>$r_3$</td>
<td>0.025</td>
<td>Stage 3 takes forty time units</td>
</tr>
</tbody>
</table>
Analysis of the model: Nodes

![Graph showing analysis of nodes in the model](image-url)
Analysis of the model: Jobs
We take the modelling decision to ignore the potential failures which could occur during the very brief stages of loading and unloading jobs.
A failure/repair model

We take the modelling decision to ignore the potential failures which could occur during the very brief stages of loading and unloading jobs.

We model a failure and repair cycle taking a job back to re-execute the present stage (rather than restart the execution of the job from the beginning).
Nodes

\[
\begin{align*}
    NodeIdle & \overset{\text{def}}{=} (load, r_0).Node1 \\
    Node1 & \overset{\text{def}}{=} (stage1, r_1).Node2 + (\text{fail1}, r_4).NodeFailed1 \\
    Node2 & \overset{\text{def}}{=} (stage2, r_2).Node3 + (\text{fail2}, r_4).NodeFailed2 \\
    Node3 & \overset{\text{def}}{=} (stage3, r_3).Node4 + (\text{fail3}, r_4).NodeFailed3 \\
    Node4 & \overset{\text{def}}{=} (unload, r_0).NodeIdle \\
    NodeFailed1 & \overset{\text{def}}{=} (\text{repair1}, r_5).Node1 \\
    NodeFailed2 & \overset{\text{def}}{=} (\text{repair2}, r_5).Node2 \\
    NodeFailed3 & \overset{\text{def}}{=} (\text{repair3}, r_5).Node3
\end{align*}
\]
With regard to the rates of failure of jobs, we estimate that one in ten jobs may fail during stage 3 (and so one in 20 during stage 2 and one in 40 during stage 1) and that the cost of repairs is relatively high, perhaps requiring a reboot of the failed node.

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<tr>
<td>$r_4$</td>
<td>0.0025</td>
<td>On average 1 in 10 stage 3 jobs will fail</td>
</tr>
<tr>
<td>$r_5$</td>
<td>0.0025</td>
<td>Repairing may require the reboot of a node</td>
</tr>
</tbody>
</table>
Analysis of the failure/repair model: Nodes
Analysis of the failure/repair model: Jobs
Outline

1. Quantitative modelling with CTMCs and ODEs
   - Modelling with quantified process algebras
   - Analysis based on Continuous-time Markov Chains
   - Analysis based on Ordinary Differential Equations

2. Performance modelling with process algebras
   - Performance Evaluation Process Algebra
   - PEPA model of jobs and servers
   - Analysis of the model

3. Comparing performance measures
   - Computed with continuous time
   - Computed with continuous space
   - Comparison of computed measures

4. Commentary and comparison
Computing performance measures: CTMCs

Queue example

\[ Q_0 \overset{\text{def}}{=} (\text{arrive}, \lambda).Q_1 \]
\[ Q_i \overset{\text{def}}{=} (\text{arrive}, \lambda).Q_{i+1} + (\text{serve}, \mu).Q_{i-1} \quad (0 < i < 8) \]
\[ Q_8 \overset{\text{def}}{=} (\text{serve}, \mu).Q_7 \]

A queue with arrivals at rate \( \lambda \), service at rate \( \mu \) and capacity 8 (thus \( 0 \leq \text{len} < 9 \)).
Computing performance measures: CTMCs

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A queue with arrivals at rate \( \lambda \), service at rate \( \mu \) and capacity 8 (thus \( 0 \leq \text{len} < 9 \)). For \( \lambda = 1 \), \( \mu = 4 \) steady-state is:

\[
\begin{array}{cccc}
0 & 0.7500 & 3 & 0.0117 \\
1 & 0.1875 & 4 & 0.0029 \\
2 & 0.0468 & 5 & 0.0007 \\
6 & 0.0000 & 7 & 0.0000 \\
8 & 0.0000 & & \\
\end{array}
\]
Computing performance measures: CTMCs

Queue example

\[ Q_0 \triangleq (\text{arrive}, \lambda).Q_1 \quad Q_i \triangleq (\text{arrive}, \lambda).Q_{i+1} + (\text{serve}, \mu).Q_{i-1} \]
\[ Q_8 \triangleq (\text{serve}, \mu).Q_7 \]

A queue with arrivals at rate $\lambda$, service at rate $\mu$ and capacity 8 (thus $0 \leq \text{len} < 9$). For $\lambda = 1, \mu = 2$ steady-state is:

0 0.5009 3 0.0626 6 0.0078
1 0.2504 4 0.0313 7 0.0039
2 0.1252 5 0.0156 8 0.0019

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Computing performance measures: CTMCs

Queue example

\[ Q_0 \overset{\text{def}}{=} (\text{arrive}, \lambda).Q_1 \]
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\[ 0 < i < 8 \]

A queue with arrivals at rate \( \lambda \), service at rate \( \mu \) and capacity 8 (thus \( 0 \leq \text{len} < 9 \)). For \( \lambda = 1, \mu = 1 \) steady-state is:

\[ \begin{array}{cccc}
0 & 0.1111 & 3 & 0.1111 \\
1 & 0.1111 & 4 & 0.1111 \\
2 & 0.1111 & 5 & 0.1111 \\
\end{array} \]
Computing performance measures: CTMCs

Queue example

\[
Q_0 \overset{\text{def}}{=} (\text{arrive}, \lambda).Q_1 \\
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Q_8 \overset{\text{def}}{=} (\text{serve}, \mu).Q_7
\]

(0 < i < 8)

A queue with arrivals at rate \(\lambda\), service at rate \(\mu\) and capacity 8 (thus \(0 \leq \text{len} < 9\)). For \(\lambda = 2\), \(\mu = 1\) steady-state is:

\[
\begin{align*}
0 & \quad 0.0019 \\
1 & \quad 0.0039 \\
2 & \quad 0.0078 \\
3 & \quad 0.0156 \\
4 & \quad 0.0313 \\
5 & \quad 0.0626 \\
6 & \quad 0.1252 \\
7 & \quad 0.2504 \\
8 & \quad 0.5009 \\
\end{align*}
\]
Queue example

\[ Q_0 \overset{\text{def}}{=} (\text{arrive}, \lambda).Q_1 \]
\[ Q_i \overset{\text{def}}{=} (\text{arrive}, \lambda).Q_{i+1} + (\text{serve}, \mu).Q_{i-1} \]
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(0 < i < 8)

A queue with arrivals at rate \( \lambda \), service at rate \( \mu \) and capacity 8 (thus 0 \( \leq \) len < 9). For \( \lambda = 4 \), \( \mu = 1 \) steady-state is:

\[
\begin{array}{cccc}
0 & 0.0000 & 3 & 0.0007 \\
1 & 0.0000 & 4 & 0.0029 \\
2 & 0.0000 & 5 & 0.0117 \\
& & 6 & 0.0468 \\
& & 7 & 0.1875 \\
& & 8 & 0.7500 \\
\end{array}
\]

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Continuous-time and continuous-space process algebras
Calculating average queue length: CTMCs

To calculate the average queue length, weight the probability of a state by the number of customers in the queue at that point.

\[
a = \sum_{i=0}^{8} i \pi(i)
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Queues and differential equations

CTMC:

ODEs:
### Queues and differential equations

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<td><img src="image1.png" alt="Diagram" /></td>
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Queues and differential equations

CTMC: 

ODEs:
Queues and differential equations

CTMC: ...
ODEs: ...

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Computing performance measures: ODEs

\[ \lambda = 1 \]
\[ \mu = 4 \]
Computing performance measures: ODEs

\[ \lambda = 1 \]
\[ \mu = 2 \]
Computing performance measures: ODEs

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Computing performance measures: ODEs

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Computing performance measures: ODEs

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Calculating average queue length: ODEs

To calculate the average queue length, weight the fraction of queues of a given length by the number of customers in the queue.

\[ a = \sum_{i=0}^{8} i \frac{[Q_i]}{90} \]
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<th>Av. queue length (CTMCs at equilibrium)</th>
<th>Av. queue length (ODEs at $t = 50$)</th>
<th>Difference</th>
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<td>$-3.2 \times 10^{-7}$</td>
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</tbody>
</table>
Small queue example: CTMCs

\[
\begin{align*}
Q_0 & \overset{\text{def}}{=} (\text{arrive}, \lambda).Q_1 \\
Q_1 & \overset{\text{def}}{=} (\text{arrive}, \lambda).Q_2 + (\text{serve}, \mu).Q_0 \\
Q_2 & \overset{\text{def}}{=} (\text{serve}, \mu).Q_1
\end{align*}
\]
Small queue example: CTMCs

\[
\begin{align*}
Q_0 & \overset{\text{def}}{=} (\text{arrive, } \lambda).Q_1 \\
Q_1 & \overset{\text{def}}{=} (\text{arrive, } \lambda).Q_2 + (\text{serve, } \mu).Q_0 \\
Q_2 & \overset{\text{def}}{=} (\text{serve, } \mu).Q_1 \\
Q & = \\
& = \begin{bmatrix}
-\lambda & \lambda & 0 \\
\mu & -\lambda - \mu & \lambda \\
0 & \mu & -\mu
\end{bmatrix}
\end{align*}
\]
Small queue example: CTMCs

\[ Q_0 \overset{\text{def}}{=} (\text{arrive}, \lambda).Q_1 \]
\[ Q_1 \overset{\text{def}}{=} (\text{arrive}, \lambda).Q_2 + (\text{serve}, \mu).Q_0 \]
\[ Q_2 \overset{\text{def}}{=} (\text{serve}, \mu).Q_1 \]

\[
Q = \begin{bmatrix}
-\lambda & \lambda & 0 \\
\mu & -\lambda - \mu & \lambda \\
0 & \mu & -\mu
\end{bmatrix}
\]

\[
\pi Q = 0
\]
Small queue example: CTMCs

\[ Q_0 \overset{\text{def}}{=} (\text{arrive}, \lambda).Q_1 \]
\[ Q_1 \overset{\text{def}}{=} (\text{arrive}, \lambda).Q_2 + (\text{serve}, \mu).Q_0 \]
\[ Q_2 \overset{\text{def}}{=} (\text{serve}, \mu).Q_1 \]

\[ Q = \begin{bmatrix}
-\lambda & \lambda & 0 \\
\mu & -\lambda - \mu & \lambda \\
0 & \mu & -\mu
\end{bmatrix} \]

\[ \pi Q = 0 \]
\[ \sum \pi = 1 \]
Small queue example: CTMCs

\[
Q_0 \overset{\text{def}}{=} (\text{arrive}, \lambda).Q_1 \quad Q_1 \overset{\text{def}}{=} (\text{arrive}, \lambda).Q_2 + (\text{serve}, \mu).Q_0 \\
Q_2 \overset{\text{def}}{=} (\text{serve}, \mu).Q_1
\]

\[
Q = \begin{bmatrix}
-\lambda & \lambda & 0 \\
\mu & -\lambda - \mu & \lambda \\
0 & \mu & -\mu
\end{bmatrix}
\]

\[
\pi Q = 0 \quad \sum \pi = 1
\]

\[
\pi = \begin{bmatrix}
\frac{\mu^2}{\lambda^2 + \mu \lambda + \mu^2}, & \frac{\mu \lambda}{\lambda^2 + \mu \lambda + \mu^2}, & \frac{\lambda^2}{\lambda^2 + \mu \lambda + \mu^2}
\end{bmatrix}
\]
Small queue example: ODEs

\[
\begin{align*}
Q_0 & \overset{\text{def}}{=} (\text{arrive}, \lambda) \cdot Q_1 \\
Q_1 & \overset{\text{def}}{=} (\text{arrive}, \lambda) \cdot Q_2 + (\text{serve}, \mu) \cdot Q_0 \\
Q_2 & \overset{\text{def}}{=} (\text{serve}, \mu) \cdot Q_1
\end{align*}
\]
Small queue example: ODEs

\[ \begin{align*}
Q_0 & \overset{\text{def}}{=} (\text{arrive}, \lambda).Q_1 \\
Q_2 & \overset{\text{def}}{=} (\text{serve}, \mu).Q_1 \\
Q_1 & \overset{\text{def}}{=} (\text{arrive}, \lambda).Q_2 + (\text{serve}, \mu).Q_0
\end{align*} \]

\[ \frac{dQ_0}{dt} = -\lambda Q_0 + \mu Q_1 \]
Small queue example: ODEs

\[
\begin{align*}
Q_0 & \overset{\text{def}}{=} (\text{arrive}, \lambda).Q_1 \\
Q_1 & \overset{\text{def}}{=} (\text{arrive}, \lambda).Q_2 + (\text{serve}, \mu).Q_0 \\
Q_2 & \overset{\text{def}}{=} (\text{serve}, \mu).Q_1
\end{align*}
\]

\[
\begin{align*}
\frac{dQ_0}{dt} &= -\lambda Q_0 + \mu Q_1 \\
\frac{dQ_1}{dt} &= \lambda Q_0 - \lambda Q_1 - \mu Q_1 + \mu Q_2
\end{align*}
\]
Small queue example: ODEs

\[
Q_0 \overset{\text{def}}{=} (\text{arrive}, \lambda).Q_1 \quad Q_1 \overset{\text{def}}{=} (\text{arrive}, \lambda).Q_2 + (\text{serve}, \mu).Q_0 \\
Q_2 \overset{\text{def}}{=} (\text{serve}, \mu).Q_1
\]

\[
\frac{dQ_0}{dt} = -\lambda Q_0 + \mu Q_1 \\
\frac{dQ_1}{dt} = \lambda Q_0 - \lambda Q_1 - \mu Q_1 + \mu Q_2 \\
\frac{dQ_2}{dt} = \lambda Q_1 - \mu Q_2
\]
Small queue example: ODEs (stationary points)

\[ Q_0 \overset{\text{def}}{=} (\text{arrive}, \lambda).Q_1 \]
\[ Q_1 \overset{\text{def}}{=} (\text{arrive}, \lambda).Q_2 + (\text{serve}, \mu).Q_0 \]
\[ Q_2 \overset{\text{def}}{=} (\text{serve}, \mu).Q_1 \]

\[ 0 = -\lambda Q_0 + \mu Q_1 \]
\[ 0 = \lambda Q_0 - \lambda Q_1 - \mu Q_1 + \mu Q_2 \]
\[ 0 = \lambda Q_1 - \mu Q_2 \]
Small queue example

\[
Q_0 \overset{\text{def}}{=} (\text{arrive}, \lambda).Q_1 \\
Q_1 \overset{\text{def}}{=} (\text{arrive}, \lambda).Q_2 + (\text{serve}, \mu).Q_0 \\
Q_2 \overset{\text{def}}{=} (\text{serve}, \mu).Q_1
\]

\[
0 = \begin{bmatrix}
Q_0 & Q_1 & Q_2
\end{bmatrix}
\begin{bmatrix}
-\lambda & \lambda & 0 \\
\mu & -\lambda - \mu & \lambda \\
0 & \mu & -\mu
\end{bmatrix}
\]
Small queue example: ODEs (and CTMC solution)

Small queue example

\[
Q_0 \overset{\text{def}}{=} (\text{arrive}, \lambda).Q_1 \\
Q_1 \overset{\text{def}}{=} (\text{arrive}, \lambda).Q_2 + (\text{serve}, \mu).Q_0 \\
Q_2 \overset{\text{def}}{=} (\text{serve}, \mu).Q_1
\]

\[
p = [Q_0 \quad \frac{\lambda}{\mu} Q_0 \quad \frac{\lambda^2}{\mu^2} Q_0]
\]
Small queue example: ODEs (and CTMC solution)

\[ \begin{align*}
Q_0 & \overset{\text{def}}{=} (\text{arrive, } \lambda).Q_1 & Q_1 & \overset{\text{def}}{=} (\text{arrive, } \lambda).Q_2 + (\text{serve, } \mu).Q_0 \\
Q_2 & \overset{\text{def}}{=} (\text{serve, } \mu).Q_1
\end{align*} \]

\[ p = \begin{bmatrix}
Q_0 & \frac{\lambda}{\mu}Q_0 & \frac{\lambda^2}{\mu^2}Q_0
\end{bmatrix} \]

\[ \pi = \begin{bmatrix}
\frac{\mu^2}{\lambda^2 + \mu \lambda + \mu^2}, & \frac{\mu \lambda}{\lambda^2 + \mu \lambda + \mu^2}, & \frac{\lambda^2}{\lambda^2 + \mu \lambda + \mu^2}
\end{bmatrix} \]
We found that, for a sequential PEPA component, the differential equations are recording the same information as found in the infinitesimal generator matrix of the Markov chain.
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The stationary points of the system of ODEs for an initial value of 1 make up the stationary probability distribution of the CTMC.
Isn’t this just the Chapman-Kolmogorov equations?

Now that we have discovered that we have a copy of a generator matrix in the ODEs aren’t we just back to

\[
\frac{d\pi(t)}{dt} = \pi(t)Q
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\[
\frac{d\pi(t)}{dt} = \pi(t)Q
\]

Only if the system is a single sequential component. For even only two parallel queues, the generator matrix is much larger than the system of ODEs.
Generator matrix for two parallel queues

\[
Q = \begin{bmatrix}
-2\lambda & \lambda & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\mu & -2\lambda - \mu & 0 & \lambda & \lambda & 0 & 0 & 0 & 0 & 0 \\
\mu & 0 & -2\lambda - \mu & 0 & \lambda & 0 & 0 & 0 & 0 & \lambda \\
0 & \mu & 0 & -\lambda - \mu & 0 & \lambda & 0 & 0 & 0 & 0 \\
0 & \mu & \mu & 0 & -2\lambda - 2\mu & \lambda & 0 & \lambda & 0 & 0 \\
0 & 0 & 0 & \mu & \mu & -\lambda - 2\mu & \lambda & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \mu & 0 & \mu & -2\mu & \mu & 0 \\
0 & 0 & 0 & 0 & \mu & 0 & \lambda & -\lambda - 2\mu & \mu & 0 \\
0 & 0 & \mu & 0 & 0 & 0 & 0 & \lambda & -\lambda - \mu & 0 \\
0 & 0 & 0 & \mu & 0 & 0 & 0 & 0 & \lambda & -\lambda - \mu \\
\end{bmatrix}
\]
Steady-state for two parallel queues

\[
\pi = \begin{bmatrix}
\frac{\mu^4}{2\mu\lambda^3 + 3\mu^2\lambda^2 + 2\mu^3\lambda + \lambda^4 + \mu^4}, \\
\frac{\mu^3\lambda}{2\mu\lambda^3 + 3\mu^2\lambda^2 + 2\mu^3\lambda + \lambda^4 + \mu^4}, \\
\frac{\mu^3\lambda}{2\mu\lambda^3 + 3\mu^2\lambda^2 + 2\mu^3\lambda + \lambda^4 + \mu^4}, \\
\frac{\mu^2\lambda^2}{2\mu\lambda^3 + 3\mu^2\lambda^2 + 2\mu^3\lambda + \lambda^4 + \mu^4}, \\
\frac{\mu^2\lambda^2}{2\mu\lambda^3 + 3\mu^2\lambda^2 + 2\mu^3\lambda + \lambda^4 + \mu^4}, \\
\frac{\mu\lambda^3}{2\mu\lambda^3 + 3\mu^2\lambda^2 + 2\mu^3\lambda + \lambda^4 + \mu^4}, \\
\frac{\lambda^4}{2\mu\lambda^3 + 3\mu^2\lambda^2 + 2\mu^3\lambda + \lambda^4 + \mu^4}, \\
\frac{\mu\lambda^3}{2\mu\lambda^3 + 3\mu^2\lambda^2 + 2\mu^3\lambda + \lambda^4 + \mu^4}, \\
\frac{\mu\lambda^3}{2\mu\lambda^3 + 3\mu^2\lambda^2 + 2\mu^3\lambda + \lambda^4 + \mu^4}, \\
\frac{\mu^2\lambda^2}{2\mu\lambda^3 + 3\mu^2\lambda^2 + 2\mu^3\lambda + \lambda^4 + \mu^4}
\end{bmatrix}
\]
Quantitative modelling with CTMCs and ODEs
- Modelling with quantified process algebras
- Analysis based on Continuous-time Markov Chains
- Analysis based on Ordinary Differential Equations

Performance modelling with process algebras
- Performance Evaluation Process Algebra
- PEPA model of jobs and servers
- Analysis of the model

Comparing performance measures
- Computed with continuous time
- Computed with continuous space
- Comparison of computed measures

Commentary and comparison

Stephen Gilmore. LFCS, University of Edinburgh. Continuous-time and continuous-space process algebras
Previous performance modelling with PEPA used continuous-time Markov chains (CTMCs). These admit steady-state and transient analysis (by solving the CTMC).
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- Steady-state is cheaper but less informative. Transient is more informative but more expensive.
- Major drawback: state-space explosion. Generating the state-space is slow. Solving the CTMC is slow.
- In practice effective only to systems of size $10^6$ states, even when using clever storage representations.
Mapping PEPA to ODEs admits *course-of-values* analysis by solving the ODE (akin to transient analysis).
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Major benefit: ODE solving is effective in practice, leaning towards suitability for interactive experimentation. Good for modellers, gaining more insights into the system behaviour.

Effective for systems of size $10^{10^6}$ states and beyond.
Models in the PEPA stochastic process algebra are concise, and in direct style they generate a system of ODEs the number of which is linear in the number of distinct component types in the PEPA model.
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Thus there is no hidden cost in the use of the high-level language but there are many advantages.
Discussion: process algebras and ODEs

- Models in the PEPA stochastic process algebra are concise, and in direct style they generate a system of ODEs the number of which is linear in the number of distinct component types in the PEPA model.

- Thus there is no hidden cost in the use of the high-level language but there are many advantages.
  - PEPA models can be checked for freedom from deadlock, satisfaction of logical properties, or compared using relations such as bisimulation.
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Thus there is no hidden cost in the use of the high-level language but there are many advantages.

- PEPA models can be checked for freedom from deadlock, satisfaction of logical properties, or compared using relations such as bisimulation.
- As a compositional modelling language PEPA components can be re-used in other models, promoting best practice.
Commentary and comparison

- **Analysis capabilities**
  - Numerical integration, course-of-values analysis
  - Interested in the solution of *initial value problems*
  - Interested in finding *stationary points*
  - Verification at process algebra level (freedom from deadlock)
Commentary and comparison

- **Analysis capabilities**
  - Numerical integration, course-of-values analysis
  - Interested in the solution of *initial value problems*
  - Interested in finding *stationary points*
  - Verification at process algebra level (freedom from deadlock)

- **Relationship to other analysis methods**
  - For sequential components, we can understand the relationship between the CTMC and ODE solution via the (same) generator matrix.
  - No such relationship exists for stochastic simulation and Markov chains.
In the process algebra world, algebras with an interleaving semantics are termed **false concurrency**. PEPA [Hillston 1994] was the first timed process algebra to have an interleaving semantics allowing it to generate a CTMC. The interleaving semantics gives rise to the **state-space explosion** problem.
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PEPA [Hillston 2005] is the first timed process algebra to have a true concurrency semantics via the mapping to ODEs. The true concurrency semantics avoids the state-space explosion problem.