Equivalence of the Projected Forward Dynamics and the Dynamically Consistent Inverse Solution

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Task-based Constraints

We define a Task-based Constraint as

$$\Phi(q) = x(t), \quad (1)$$

where t is time, $x \in \mathbb{R}^m$ the task position, and $q \in \mathbb{R}^n$ the configuration position. Differentiating Eq. (1) twice leads to

$$A\ddot{q} = \ddot{x} - \dot{A}\dot{q}, \quad (2)$$

where $\ddot{x}$ and $\ddot{q}$ are the task and configuration accelerations, and $A \in \mathbb{R}^{m\times n}$ is the constraint Jacobian. Fig. 1 illustrates various Task-based Constraints and Fig. 2 categorizes it.

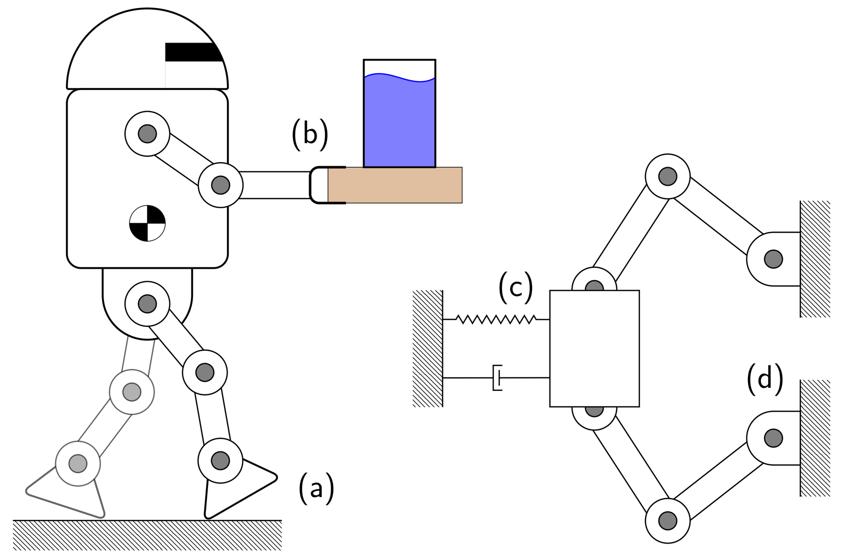


Fig. 1: Illustration of various Task-based Constraints, such as: physical constraints, motion tasks, and behaviours. Examples include: (a) using contacts for bipedal locomotion; (b) keeping the balance while holding a jar of water; (c) having a compliant behaviour while following a given trajectory; (d) and robots with closed kinematic loops.

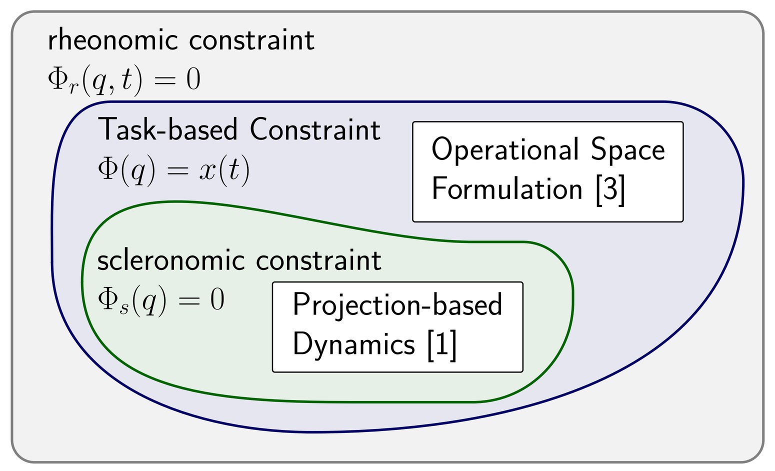


Fig. 2: Categorization regarding underlying equality constraint. Where a rheonomic constraing is a time dependent constraint, a scleronomic constraint is a time independent constraint, and a Task-based Constraint is a time dependent constraint with decoupled dependence on the configuration q and time t.

Operational Space Formulation

The Dynamically Consistent Inverse of a Jacobian A is the matrix G that satisfies the condition

$$AM^{-1}\left(I_n - A^\top G^\top\right)\tau_\star = 0, \quad (3)$$

valid for $G = \overline{A} \triangleq M^{-1}A^\top(AM^{-1}A^\top)^\dagger$, where $A^\dagger$ is the pseudo-inverse of A.

Control Decomposition

$$\tau = \underbrace{A^\top f}_{\tau_t} + \underbrace{\overline{P}^\top \tau_\star}_{\tau_{\mathcal{N}}}, \quad (4)$$

where $\overline{P} = I_n - \overline{A}A$.

Equivalence

Analytical dynamics solution equivalence:

$$\ddot{q} = M^{-1}A^\top\left(\boxed{M_x\ddot{x} + h_x - f}\right) + M^{-1}(\tau - h)$$
$$= \underbrace{\overline{A}(\ddot{x} - \dot{A}\dot{q})}_{\ddot{q}_t} + \underbrace{\overline{P}M^{-1}(\tau - h)}_{\ddot{q}_{\mathcal{N}}}$$

Multiple Task-based Constraints

By stacking two constraints as $A = \left[A_1^\top \ A_2^\top\right]^\top$:

$$M_x = \begin{bmatrix} M_1 & -\overline{A}_1^\top A_2^\top M_2 \\ -\overline{A}_2^\top A_1^\top M_1 & M_2 \end{bmatrix}, \quad (5)$$

with

$$M_1 \triangleq \left(A_1\overline{P}_2M^{-1}A_1^\top\right)^\dagger$$
$$M_2 \triangleq \left(A_2\overline{P}_1M^{-1}A_2^\top\right)^\dagger,$$

and the dynamically consistent inverse

$$\overline{A}^\top = \begin{bmatrix} M_1A_1\overline{P}_2M^{-1} \\ M_2A_2\overline{P}_1M^{-1} \end{bmatrix} \triangleq \begin{bmatrix} A_1^{\#^\top} \\ A_2^{\#^\top} \end{bmatrix}, \quad (6)$$

where we define $A_1^{\#^\top}$ and $A_2^{\#^\top}$ as the partial dynamically consistent inverses. By partitioning $f = \left[f_1^\top \ f_2^\top\right]^\top$, and making $\lambda_2 = 0$, $\ddot{x}_1 = 0$, and $R = I_n$, we get

$$f_2 = M_2(\ddot{x}_2 - \dot{A}_2\dot{q}) + \overline{A}_2A_1^\top M_1\dot{A}_1\dot{q} + A_2^{\#^\top}h$$
$$= M_2[\ddot{x}_2 + A_2M_{c1}^{-1}P_1h$$
$$- (\dot{A}_2 - A_2M_{c1}^{-1}A_1^\dagger\dot{A}_1)\dot{q}]$$

which correspond to the operational space controllers with rigid constraints proposed by [2, 4].

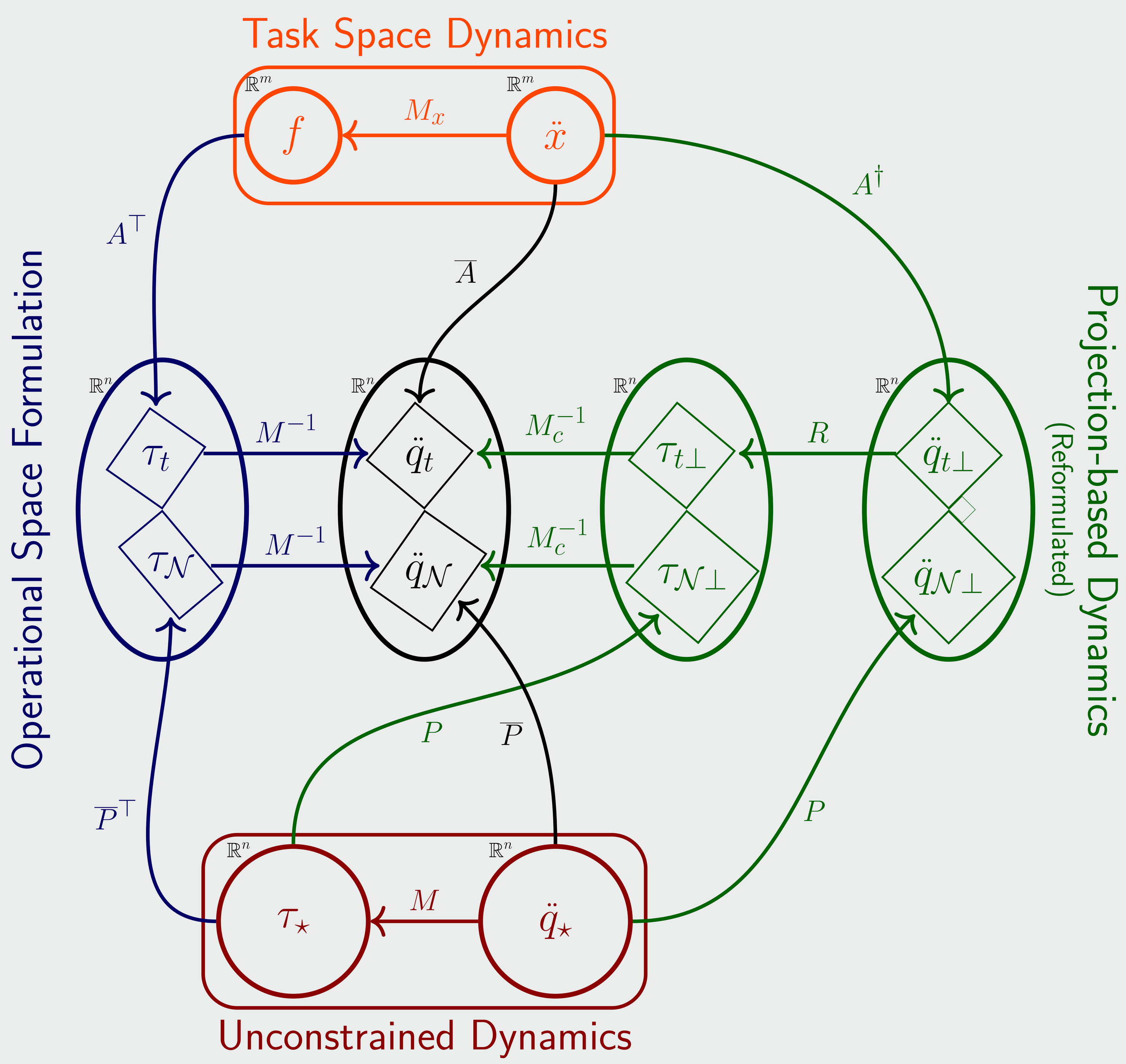
Task Space Dynamics

$$M_x\ddot{x} + h_x - \lambda = f, \quad (7)$$

where

$$M_x \triangleq \left(AM^{-1}A^\top\right)^\dagger = \overline{A}^\top M\overline{A} \quad (8)$$

is the task space inertia matrix, and with $h_x \triangleq \overline{A}^\top h - M_x\dot{A}\dot{q}$ and $f \triangleq \overline{A}^\top\tau_\star$.



Unconstrained Dynamics

The equation of motion of an unconstrained system in the configuration space is

$$M(q_\star)\ddot{q}_\star + h(q_\star, \dot{q}_\star) = \tau_\star \quad (9)$$

where $h \in \mathbb{R}^n$ contains the Coriolis, centrifugal, and gravitational contributions, $M(q_\star)$ is the unconstrained inertia matrix, $\tau_\star \in \mathbb{R}^n$ is the generalized force vector in the configuration space, and $q_\star, \dot{q}_\star, \ddot{q}_\star \in \mathbb{R}^n$ are, respectively, the unconstrained generalized position, velocity, and acceleration. We can compute the forward dynamics by simply inverting M as

$$\ddot{q}_\star = M^{-1}(\tau_\star - h). \quad (10)$$

Projection-based Dynamics

Reformulation

By pre-multiplying the configuration dynamics with P, obtaining

$$PM\ddot{q} = P(\tau - h), \quad (11)$$

and Eq. (2) with $A^\dagger$, obtaining

$$(I_n - P)\ddot{q} = A^\dagger(\ddot{x} - \dot{A}\dot{q}), \quad (12)$$

and combining them both in different ways, we get

$$M_c\,\ddot{q} = P(\tau - h) + C_c\,(\ddot{x} - \dot{A}\dot{q}) \quad (13)$$

$$M_c^{(1)} = PM + (I - P) \qquad C_c^{(1)} = -A^\dagger$$
$$M_c^{(2)} = M + PM + (PM)^\top \qquad C_c^{(2)} = -MA^\dagger$$
$$M_c^{(3)} = PMP + (I - P)M(I - P) \qquad C_c^{(3)} = -(I - 2P)MA^\dagger$$

$$M_c \triangleq PM + R(I - P) \qquad C_c \triangleq -RA^\dagger$$

Equivalence

Analytical dynamics solution equivalence:

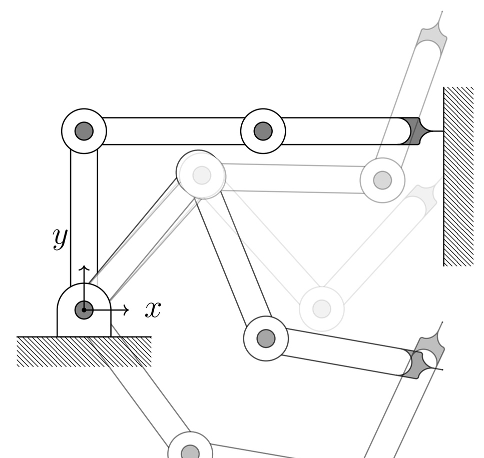
$$\ddot{q} = \boxed{M_c^{-1}RA^\dagger}(\ddot{x} - \dot{A}\dot{q}) + \boxed{M_c^{-1}P}(\tau - h)$$
$$= \boxed{\overline{A}}(\ddot{x} - \dot{A}\dot{q}) + \boxed{\overline{P}M^{-1}}(\tau - h)$$

Condition Number Minimization

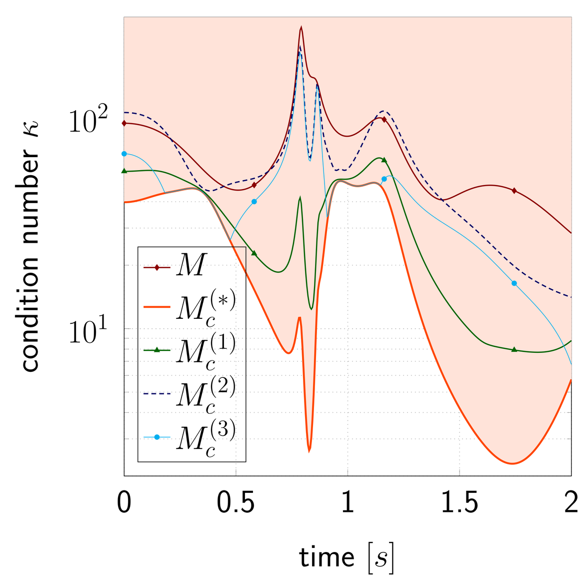
The $R^{(*)}$ that minimizes $\kappa(M_c)$, where $\kappa(.)$ represents the condition number, is given by

$$R^{(*)} = \mu I_n - PM, \quad (14)$$

for some $\mu \in \mathbb{R}$ such that $\{\sigma_{min}(PMP) \neq 0\} \leq \mu \leq \sigma_{max}(PMP)$, where $\sigma(.)$ represents singular values.



(a) Five configuration samples.



(b) Time evolution of the condition number.

Fig. 3: Free fall (i.e. $\tau = 0$) simulation of a two dimensional serial robot arm with three links and with the end-effector constrained to a vertical slider.

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