Continuous approximation of PEPA models and Petri nets

Vashti Galpin
Laboratory for Foundations of Computer Science
School of Informatics
University of Edinburgh

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Outline

Motivation

PEPA

Continuous Petri nets

Transformations

Conclusions
Motivation

- large systems and state space explosion
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- large systems and state space explosion
  - use approximation to avoid
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  - use approximation to avoid
- PEPA, continuous approximation using ODEs [Hillston 2005]
  - many identical components
  - equations for $dN(D, \tau)/d\tau$
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- PEPA, continuous approximation using ODEs [Hillston 2005]
  - many identical components
  - equations for $dN(D, \tau)/d\tau$
- timed continuous Petri nets [Alla & David, Recalde & Silva]
  - transitions have rates
  - markings take values from positive reals
  - large numbers of clients and servers
  - equations for $dm(p, \tau)/d\tau$
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- how do these compare?
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  - large numbers of clients and servers
  - equations for $dm(p, \tau)/d\tau$
- how do these compare?
- what are the server semantics of PEPA?
PEPA

- Performance Evaluation Process Algebra [Hillston 1996]
  - stochastic, action durations from exponential distribution
  - syntax, structured operational semantics
  - continuous time Markov chain (CTMC) to describe dynamic behaviour
### PEPA

- **Performance Evaluation Process Algebra** [Hillston 1996]
  - stochastic, action durations from exponential distribution
  - syntax, structured operational semantics
  - continuous time Markov chain (CTMC) to describe dynamic behaviour

- restricted PEPA syntax
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- restricted PEPA syntax
  - sequential component $S ::= (\alpha, r).S | S + S | C_s$
  - sequential constant $C_s \overset{\text{def}}{=} S$
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**PEPA**

- Performance Evaluation Process Algebra [Hillston 1996]
  - stochastic, action durations from exponential distribution
  - syntax, structured operational semantics
  - continuous time Markov chain (CTMC) to describe dynamic behaviour

- restricted PEPA syntax
  - sequential component \( S ::= (\alpha, r).S \mid S + S \mid C_s \)
  - sequential constant \( C_s \triangleq S \)
  - parallel cooperation with multiway synchronisation

\[
C_1[n_1] \overset{l_1}{\parallel} C_2[n_2] \overset{l_2}{\parallel} \ldots \overset{l_{m-1}}{\parallel} C_m[n_m]
\]

- \( C_i \)'s do not synchronise, \( C_i \)'s and \( C_j \)'s must synchronise
PEPA

- Performance Evaluation Process Algebra [Hillston 1996]
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- restricted PEPA syntax
  - sequential component $S ::= (\alpha, r).S \mid S + S \mid C_s$
  - sequential constant $C_s \overset{\text{def}}{=} S$
  - parallel cooperation with multiway synchronisation
    
    $$C_1[n_1] \Join_{l_1} C_2[n_2] \Join_{l_2} \ldots \Join_{l_{m-1}} C_m[n_m]$$
  - $C_i$’s do not synchronise, $C_i$’s and $C_j$’s must synchronise
  - identical rates for shared activities
ODE semantics of PEPA

- many identical sequential components
ODE semantics of PEPA

- many identical sequential components
- each sequential component may have a number of derivatives

\[ A \overset{\text{def}}{=} (a_1, r_1).B + (a_2, r_2).C \quad B \overset{\text{def}}{=} (b, s).A \quad C \overset{\text{def}}{=} (c, t).A \]
ODE semantics of PEPA

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- express states in numerical vector form \((n_1, \ldots, n_m)\)
ODE semantics of PEPA

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\]

- express states in numerical vector form \((n_1, \ldots, n_m)\)
- number of copies of each component/derivative
ODE semantics of PEPA

- many identical sequential components
- each sequential component may have a number of derivatives

\[
A \overset{\text{def}}{=} (a_1, r_1) \cdot B + (a_2, r_2) \cdot C \quad B \overset{\text{def}}{=} (b, s) \cdot A \quad C \overset{\text{def}}{=} (c, t) \cdot A
\]

- express states in numerical vector form \((n_1, \ldots, n_m)\)
- number of copies of each component/derivative
- transitions update the vector

\[
\begin{align*}
A \xrightarrow{(a_1, r_1)} B \\
(N(A), N(B), N(C)) \xrightarrow{(a_1, r_1)} (N(A) - 1, N(B) + 1, N(C))
\end{align*}
\]
ODE semantics of PEPA (continued)

- continuous approximation of changes in numbers
- consider what actions lead to change in numbers

\[
\begin{align*}
C_1 & \xrightarrow{(\alpha,s)} D_1 & \xleftarrow{(\alpha,s)} E_1 \\
C_2 & \xrightarrow{D_1} & \xleftarrow{E_2}
\end{align*}
\]
ODE semantics of PEPA (continued)

- continuous approximation of changes in numbers
- consider what actions lead to change in numbers

\[
\begin{align*}
\text{entry activity} \\
C_1 & \xrightarrow{(\alpha,s)} D_1 & (\alpha,s) & \xrightarrow{} E_1 \\
C_2 & \xrightarrow{} D_1 & \xrightarrow{} E_2
\end{align*}
\]
ODE semantics of PEPA (continued)

- continuous approximation of changes in numbers
- consider what actions lead to change in numbers

\[ \text{entry activity} \]

\[
\begin{align*}
C_1 &\xrightarrow{(\alpha,s)} D_1 & (\alpha,s) &\xrightarrow{} E_1 \\
C_2 &\xrightarrow{} D_1 & (\alpha,s) &\xrightarrow{\quad} E_2 \\
C_3 &\xrightarrow{(\alpha,s)} D_2 
\end{align*}
\]
ODE semantics of PEPA (continued)

- continuous approximation of changes in numbers
- consider what actions lead to change in numbers

\[ C_1 \xrightarrow{(\alpha,s)} D_1 \xleftarrow{(\alpha,s)} C_2 \]
\[ C_3 \rightarrow D_2 \rightarrow E_2 \xrightarrow{(\alpha,s)} E_1 \]

*entry activity*  \hspace{1cm}  *exit activity*  

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ODE semantics of PEPA (continued)

- continuous approximation of changes in numbers
- consider what actions lead to change in numbers

![Diagram of PEPA activity](image)

- **entry activity**
  - $C_1$ to $D_1$ with $(\alpha, s)$
  - $C_2$ to $D_1$ with $(\alpha, s)$
  - $C_3$ to $D_2$ with $(\alpha, s)$

- **exit activity**
  - $D_1$ to $E_1$ with $(\alpha, s)$
  - $D_1$ to $E_2$
  - $D_2$ to $E_3$ with $(\alpha, s)$
  - $D_3$ to $E_3$ with $(\alpha, s)$
Change in number of copies of component $D$

\[
\frac{dN(D, \tau)}{d\tau} = \sum_{(\alpha, r) \text{ entry activity}} r \times \min\{N(C, \tau) \mid C \xrightarrow{\alpha,r} \} - \sum_{(\alpha, r) \text{ exit activity}} r \times \min\{N(C, \tau) \mid C \xrightarrow{\alpha,r} \}
\]
Change in number of copies of component $D$

\[
\frac{dN(D, \tau)}{d\tau} = \sum_{(\alpha, r)} \text{activity} \times \min\{N(C, \tau) \mid C \xrightarrow{\alpha,r} \}
\]

\[
- \sum_{(\alpha, r)} \text{activity} \times \min\{N(C, \tau) \mid C \xrightarrow{\alpha,r} \}
\]

- can express ODEs as activity graph and activity matrix
Activity graph

- graph nodes are components and activities
Activity graph

- graph nodes are components and activities
- edges are added
Activity graph

- graph nodes are components and activities
- edges are added
  - from a component to an exit activity for that component
Activity graph

- graph nodes are components and activities
- edges are added
  - from a component to an exit activity for that component
  - from an entry activity for a component to that component
Activity graph

- graph nodes are components and activities

- edges are added
  - from a component to an exit activity for that component
  - from an entry activity for a component to that component

- $C \xrightarrow{(\alpha, r)} C'$

Motivation  PEPA  Continuous Petri nets  Transformations  Conclusions

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Timed continuous Petri nets

- places $P$, transitions $T$, disjoint
Timed continuous Petri nets

- places $P$, transitions $T$, disjoint
- ordinary Petri nets

\[
Pre : P \times T \rightarrow \{0, 1\} \quad Post : P \times T \rightarrow \{0, 1\}
\]
Timed continuous Petri nets

- places $P$, transitions $T$, disjoint
- ordinary Petri nets

$$Pre : P \times T \rightarrow \{0, 1\} \quad Post : P \times T \rightarrow \{0, 1\}$$

- presets and postsets

$$Pre(p, t) = 1 \iff p \in \cdot t \quad t \in p^*$$
$$Post(p, t) = 1 \iff p \in t^* \quad t \in \cdot p$$
Timed continuous Petri nets

- places $P$, transitions $T$, disjoint
- ordinary Petri nets

\[ Pre : P \times T \rightarrow \{0, 1\} \quad Post : P \times T \rightarrow \{0, 1\} \]

- presets and postsets

\[ Pre(p, t) = 1 \iff p \in ^*t \quad t \in p^* \]
\[ Post(p, t) = 1 \iff p \in t^* \quad t \in ^*p \]

- cost matrix $C = Post - Pre$
Timed continuous Petri nets

- places $P$, transitions $T$, disjoint

- ordinary Petri nets

\[ Pre : P \times T \rightarrow \{0, 1\} \quad Post : P \times T \rightarrow \{0, 1\} \]

- presets and postsets

\[
Pre(p, t) = 1 \iff p \in t^* \quad t \in p^* \\
Post(p, t) = 1 \iff p \in t^* \quad t \in t^* 
\]

- cost matrix $C = Post - Pre$

- firing rates $\lambda : T \rightarrow (0, \infty)$
Dynamic behaviour

- marking $M : P \times Time \rightarrow [0, \infty)$
Dynamic behaviour

- marking \( M : P \times Time \rightarrow [0, \infty) \)

- \( t \) is enabled at \( \tau \) if places preceding \( t \) have nonzero marking
Dynamic behaviour

- marking $M : P \times Time \rightarrow [0, \infty)$
- $t$ is enabled at $\tau$ if places preceding $t$ have nonzero marking
- enabling degree of $t$: minimum value of markings at places preceding $t$,

$$enab(t, \tau) = \min_{p \in \bullet t} \{m(p, \tau)\}$$
Dynamic behaviour

- marking \( M : P \times Time \rightarrow [0, \infty) \)
- \( t \) is enabled at \( \tau \) if places preceding \( t \) have nonzero marking
- enabling degree of \( t \): minimum value of markings at places preceding \( t \),
  \[
  \text{enab}(t, \tau) = \min_{p \in \bullet t} \{ m(p, \tau) \}
  \]
- infinite server semantics
  - assume many clients and many servers
  - \( t \) fires \( \text{enab}(t, \tau) \)
Change in marking at place \( p \)
Change in marking at place $p$

- fundamental equation for Petri nets

$$m(\cdot, \tau + \delta \tau) = m(\cdot, \tau) + C \cdot \sigma(\tau)$$

- change in marking of place $p$

$$\frac{dm(p, \tau)}{d\tau} = \sum_{j=1}^{n} C(p, t_j) \cdot \lambda(t_j) \cdot \min_{p' \in t_j} \{ m(p', \tau) \}$$

$$= \sum_{t \in p} \lambda(t) \cdot \min_{p' \in t} \{ m(p', \tau) \}$$

$$- \sum_{t \in p} \lambda(t) \cdot \min_{p' \in t} \{ m(p', \tau) \}$$
PEPA model to continuous Petri net

- translate a PEPA model into a timed continuous Petri net
- example – clients and servers

\[
\begin{align*}
C \ &\overset{\text{def}}{=} \ (\text{serv}_1, s_1).C' + (\text{serv}_2, s_2).C' \\
C' \ &\overset{\text{def}}{=} \ (\text{do}, d).C \\
S_1 \ &\overset{\text{def}}{=} \ (\text{serv}_1, s_1).S'_1 \\
S'_1 \ &\overset{\text{def}}{=} \ (\text{reset}_1, r_1).S_1 \\
S_2 \ &\overset{\text{def}}{=} \ (\text{serv}_2, s_2).S'_2 \\
S'_2 \ &\overset{\text{def}}{=} \ (\text{reset}_2, r_2).S_2 \\
\text{Sys} \ &\overset{\text{def}}{=} \ (C[100] \ \text{\&} \ \{\text{serv}_1, \text{serv}_2\}) \ (S_1[50] \ || \ S_2[50])
\end{align*}
\]
**Continuous approximation of PEPA models and Petri nets**

**ODEs**

\[
\frac{d N(C, \tau)}{d \tau} = \begin{cases} 
+ d \cdot N(C', \tau) & - s_1 \cdot \min(N(C, \tau), N(S_1, \tau)) \\
& - s_2 \cdot \min(N(C, \tau), N(S_2, \tau))
\end{cases}
\]

\[
\frac{d N(C', \tau)}{d \tau} = \begin{cases} 
- d \cdot N(C', \tau) & + s_1 \cdot \min(N(C, \tau), N(S_1, \tau)) \\
& + s_2 \cdot \min(N(C, \tau), N(S_2, \tau))
\end{cases}
\]

**Conclusions**
**ODEs**

\[
\frac{dN(C, \tau)}{d\tau} = +d \cdot N(C', \tau) - s_1 \cdot \min(N(C, \tau), N(S_1, \tau)) \\
\quad - s_2 \cdot \min(N(C, \tau), N(S_2, \tau))
\]

\[
\frac{dN(C', \tau)}{d\tau} = -d \cdot N(C', \tau) + s_1 \cdot \min(N(C, \tau), N(S_1, \tau)) \\
\quad + s_2 \cdot \min(N(C, \tau), N(S_2, \tau))
\]

\[
\frac{dN(S_i, \tau)}{d\tau} = r_i \cdot N(S'_i) - s_i \cdot \min(N(C, \tau), N(S_i, \tau)) \quad i = 1, 2
\]

\[
\frac{dN(S'_i, \tau)}{d\tau} = s_i \cdot \min(N(C, \tau), N(S_i, \tau)) - r_i \cdot N(S'_i) \quad i = 1, 2
\]
Activity graph

- activities and components

- reset$_1$
  - $S_1$
  - serv$_1$
    - $S_1'$
  - $C$
    - do
      - $C'$
  - serv$_2$
  - $S_2$
  - reset$_2$
    - $S_2'$
Petri net

- activities become transitions and components become places
Petri net (continued)

- activities become transitions and components become places
Petri net (continued)

- activities become transitions and components become places
- rate of transition is rate of activity
Petri net (continued)

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- rate of transition is rate of activity
- \( \text{Post}(p, t) = 1 \iff t \in \bullet p \iff t \text{ is an entry activity of } p \)
Petri net (continued)

- activities become transitions and components become places
- rate of transition is rate of activity
- \( \text{Post}(p, t) = 1 \iff t \in \bullet p \iff t \) is an entry activity of \( p \)
- \( \text{Pre}(p, t) = 1 \iff t \in p^\bullet \iff t \) is an exit activity of \( p \)
Petri net (continued)

- activities become transitions and components become places
- rate of transition is rate of activity
- $Post(p, t) = 1 \Leftrightarrow t \in \bullet p \Leftrightarrow t$ is an entry activity of $p$
- $Pre(p, t) = 1 \Leftrightarrow t \in p^\bullet \Leftrightarrow t$ is an exit activity of $p$
- a marking value of $x$ at $p$ is the same as $x$ copies of $p$

$$m(p, \tau) = N(p, \tau)$$
Petri net (continued)

- activities become transitions and components become places
- rate of transition is rate of activity
- $Post(p, t) = 1 \iff t \in \bullet p \iff t$ is an entry activity of $p$
- $Pre(p, t) = 1 \iff t \in p^\bullet \iff t$ is an exit activity of $p$
- a marking value of $x$ at $p$ is the same as $x$ copies of $p$
  \[ m(p, \tau) = N(p, \tau) \]
- initial marking $m(p, 0) = N(p, 0)$ for each $p$
Comparison of ODEs

\[
\frac{dm(p, \tau)}{d\tau} = \sum_{t \in \bullet p} \lambda(t) \cdot \min_{p' \in \bullet t} \{ m(p', \tau) \} - \sum_{t \in p'} \lambda(t) \cdot \min_{p' \in \bullet t} \{ m(p', \tau) \}
\]

\[
\frac{dN(D, \tau)}{d\tau} = \sum_{(\alpha, r)} r \cdot \min \{ N(C, \tau) \mid C \xrightarrow{(\alpha, r)} \} - \sum_{(\alpha, r)} r \cdot \min \{ N(C, \tau) \mid C \xrightarrow{(\alpha, r)} \}
\]
Comparison of ODEs

\[
\frac{dm(p, \tau)}{d\tau} = \sum_{t \in \bullet p} \lambda(t) \cdot \min_{p' \in \bullet t} \{m(p', \tau)\} - \sum_{t \in p^\bullet} \lambda(t) \cdot \min_{p' \in \bullet t} \{m(p', \tau)\}
\]

\[
\frac{dN(D, \tau)}{d\tau} = \sum_{(\alpha, r) \text{ entry activity}} r \cdot \min\{N(C, \tau) \mid C \xrightarrow{\alpha,r} \} - \sum_{(\alpha, r) \text{ exit activity}} r \cdot \min\{N(C, \tau) \mid C \xrightarrow{\alpha,r} \}
\]
Comparison of ODEs

\[
\frac{dm(p, \tau)}{d\tau} = \sum_{t \in \bullet p} \lambda(t) \cdot \min_{p' \in \bullet t} \{ m(p', \tau) \} - \sum_{p' \in \bullet t} \min_{p' \in \bullet t} \{ m(p', \tau) \}
\]

\[
\frac{dN(D, \tau)}{d\tau} = \sum_{(\alpha, r)} r \cdot \min \{ N(C, \tau) \mid C \xrightarrow{(\alpha, r)} \} - \sum_{(\alpha, r)} r \cdot \min \{ N(C, \tau) \mid C \xrightarrow{(\alpha, r)} \}
\]
Comparison of ODEs

\[
\frac{dm(p, \tau)}{d\tau} = \sum_{t \in \bullet p} \lambda(t) \cdot \min_{p' \in \bullet t} \{m(p', \tau)\} - \sum_{t \in p'} \lambda(t) \cdot \min_{p' \in \bullet t} \{m(p', \tau)\}
\]

\[
\frac{dN(D, \tau)}{d\tau} = \sum\{r \cdot \min\{N(C, \tau) \mid C \xrightarrow{\alpha,r} \}\} - \sum\{r \cdot \min\{N(C, \tau) \mid C \xrightarrow{\alpha,r} \}\}
\]
Comparison of ODEs

\[
\frac{dm(p, \tau)}{d\tau} = \sum_{t \in \cdot p} \lambda(t) \cdot \min_{p' \in \cdot t} \{m(p', \tau)\} - \sum_{t \in p^\bullet} \lambda(t) \cdot \min_{p' \in \cdot t} \{m(p', \tau)\}
\]

\[
\frac{dN(D, \tau)}{d\tau} = \sum_{(\alpha, r) \text{ entry activity}} r \cdot \min \{N(C, \tau) \mid C \xrightarrow{(\alpha, r)} \} - \sum_{(\alpha, r) \text{ exit activity}} r \cdot \min \{N(C, \tau) \mid C \xrightarrow{(\alpha, r)} \}
\]
Comparison of ODEs

\[
\frac{dm(p, \tau)}{d\tau} = \sum_{t \in \bullet p} \lambda(t) \cdot \min_{p' \in \bullet t} \{ m(p', \tau) \} - \sum_{t \in p} \lambda(t) \cdot \min_{p' \in \bullet t} \{ m(p', \tau) \}
\]

\[
\frac{dN(D, \tau)}{d\tau} = \sum_{(\alpha, r)} r \cdot \min\{ N(C, \tau) \mid C \overset{(\alpha, r)}{\rightarrow} \} - \sum_{(\alpha, r)} r \cdot \min\{ N(C, \tau) \mid C \overset{(\alpha, r)}{\rightarrow} \}
\]
Comparison of ODEs

\[
\frac{dm(p, \tau)}{d\tau} = \sum_{t \in \bullet p} \lambda(t) \cdot \min_{p' \in \bullet t} \{m(p', \tau)\} - \sum_{t \in p'} \lambda(t) \cdot \min_{p' \in \bullet t} \{m(p', \tau)\}
\]

\[
\frac{dN(D, \tau)}{d\tau} = \sum (\alpha, r) \min \{N(C, \tau) \mid C \xrightarrow{(\alpha, r)} \} - \sum (\alpha, r) \min \{N(C, \tau) \mid C \xrightarrow{(\alpha, r)} \}
\]

▶ both approaches give the same equations
Continuous Petri net to PEPA model

- transformation of *bounded* net to PEPA model
Continuous Petri net to PEPA model

- transformation of *bounded* net to PEPA model
- concept of implicit/complementary place: marking of $p$ does not change enabling degree of any $t$

\[
m(p, τ) \geq \min_{p' \in \cdot t \setminus \{p\}} \{m(p', τ)\} = enab(t, τ)
\]
Continuous Petri net to PEPA model

- transformation of *bounded* net to PEPA model

- concept of implicit/complementary place: marking of $p$ does not change enabling degree of any $t$

  $$m(p, \tau) \geq \min_{p' \in \cdot t \setminus \{p\}} \{m(p', \tau)\} = enab(t, \tau)$$

- addition of complementary places: for each $p$, add new $\bar{p}$

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Continuous Petri net to PEPA model

▶ transformation of *bounded* net to PEPA model

▶ concept of implicit/complementary place: marking of \( p \) does not change enabling degree of any \( t \)

\[
m(p, \tau) \geq \min_{p' \in \bullet t \setminus \{p\}} \{m(p', \tau)\} = \text{enab}(t, \tau)
\]

▶ addition of complementary places: for each \( p \), add new \( \bar{p} \)
  ▶ for every arc from \( p \) to any \( t \), add an arc from \( t \) to \( \bar{p} \)
  ▶ for every arc from \( t \) to any \( p \), add an arc from \( \bar{p} \) to \( t \)
Continuous Petri net to PEPA model

- transformation of *bounded* net to PEPA model

- concept of implicit/complementary place: marking of $p$ does not change enabling degree of any $t$

$$m(p, \tau) \geq \min_{p' \in \cdot t \setminus \{p\}} \{m(p', \tau)\} = enab(t, \tau)$$

- addition of complementary places: for each $p$, add new $\overline{p}$
  - for every arc from $p$ to any $t$, add an arc from $t$ to $\overline{p}$
  - for every arc from $t$ to any $p$, add an arc from $\overline{p}$ to $t$
  - $m(\overline{p}, 0) = b(p) - m(p, 0)$ where $b(p)$ is bound for $p$
Continuous Petri net to PEPA model

- transformation of *bounded* net to PEPA model
- concept of implicit/complementary place: marking of $p$ does not change enabling degree of any $t$

\[ m(p, \tau) \geq \min_{p' \in \cdot t\setminus \{p\}} \{ m(p', \tau) \} = \text{enab}(t, \tau) \]

- addition of complementary places: for each $p$, add new $\overline{p}$
  - for every arc from $p$ to any $t$, add an arc from $t$ to $\overline{p}$
  - for every arc from $t$ to any $p$, add an arc from $\overline{p}$ to $t$
  - $m(\overline{p}, 0) = b(p) - m(p, 0)$ where $b(p)$ is bound for $p$

- ODEs remain unchanged by addition of complementary places

\[ \min_{p' \in \cdot t} \{ m(p', \tau) \} = \min_{p' \in \cdot t \cap P} \{ m(p', \tau) \} \]
Complementation

- example

![Complementation Diagram]

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Complementation

▶ example
Construction of PEPA model

- use algorithm by Hillston, Recalde, Ribaudo and Silva [2001]
- converts stochastic Petri net to PEPA model
Construction of PEPA model

- use algorithm by Hillston, Recalde, Ribaudo and Silva [2001]
- converts stochastic Petri net to PEPA model
  - convert each pair $p$ and $\overline{p}$ into a sequential component
    \[
    C_p \equiv \sum_{t \in p} (t, \lambda(t)).C_{\overline{p}} \quad C_{\overline{p}} \equiv \sum_{t \in \overline{p}} (t, \lambda(t)).C_p
    \]
Construction of PEPA model

- use algorithm by Hillston, Recalde, Ribaudo and Silva [2001]
- converts stochastic Petri net to PEPA model
  - convert each pair $p$ and $\overline{p}$ into a sequential component
    \[ C_p \overset{\text{def}}{=} \sum_{t \in p^\bullet} (t, \lambda(t)).C_{\overline{p}} \]
    \[ C_{\overline{p}} \overset{\text{def}}{=} \sum_{t \in \bullet \cdot p} (t, \lambda(t)).C_p \]
  - for each $p$ define model component
    \[ M_p \overset{\text{def}}{=} C_p \ || \ . \ || C_p \ || C_{\overline{p}} \ || \ . \ || C_{\overline{p}} \]
    with $m(p, 0)$ copies of $C_p$ and $m(\overline{p}, 0)$ copies of $C_{\overline{p}}$. 
Construction of PEPA model

- use algorithm by Hillston, Recalde, Ribaudo and Silva [2001]
- converts stochastic Petri net to PEPA model
  - convert each pair $p$ and $\overline{p}$ into a sequential component

\[
C_p \overset{\text{def}}{=} \sum_{t \in p^\bullet} (t, \lambda(t)).C_{\overline{p}} \quad C_{\overline{p}} \overset{\text{def}}{=} \sum_{t \in \overline{p}^\bullet} (t, \lambda(t)).C_p
\]

- for each $p$ define model component

\[
M_p \overset{\text{def}}{=} C_p \parallel \ldots \parallel C_p \parallel C_{\overline{p}} \parallel \ldots \parallel C_{\overline{p}}
\]

with $m(p, 0)$ copies of $C_p$ and $m(\overline{p}, 0)$ copies of $C_{\overline{p}}$.
- recursively build up synchronisation sets and system equation
Construction of PEPA model (cont.)

example

\[ C_{p_1} \overset{\text{def}}{=} (\text{prepare}, p).C_{\overline{p}_1} \quad C_{\overline{p}_1} \overset{\text{def}}{=} (\text{finish}, f).C_{p_1} \]
Construction of PEPA model (cont.)

Example

\[
\begin{align*}
C_{p_1} & \overset{\text{def}}{=} (\text{prepare}, p).C_{\overline{p}_1} & C_{\overline{p}_1} & \overset{\text{def}}{=} (\text{finish}, f).C_{p_1} \\
C_{p_2} & \overset{\text{def}}{=} (\text{serve}, s).C_{\overline{p}_2} & C_{\overline{p}_2} & \overset{\text{def}}{=} (\text{prepare}, p).C_{p_2}
\end{align*}
\]
Construction of PEPA model (cont.)

▶ example

\[ C_{p_1} \overset{\text{def}}{=} (\text{prepare}, p).C_{p_1} \quad C_{p_1} \overset{\text{def}}{=} (\text{finish}, f).C_{p_1} \]
\[ C_{p_2} \overset{\text{def}}{=} (\text{serve}, s).C_{p_2} \quad C_{p_2} \overset{\text{def}}{=} (\text{prepare}, p).C_{p_2} \]
\[ C_{p_3} \overset{\text{def}}{=} (\text{finish}, f).C_{p_3} \quad C_{p_3} \overset{\text{def}}{=} (\text{serve}, s).C_{p_3} \]
\[ C_{p_4} \overset{\text{def}}{=} (\text{serve}, s).C_{p_4} \quad C_{p_4} \overset{\text{def}}{=} (\text{reset}, r).C_{p_4} + (\text{repair}, e).C_{p_4} \]
\[ C_{p_5} \overset{\text{def}}{=} (\text{reset}, r).C_{p_5} + (\text{fail}, a).C_{p_5} \quad C_{p_5} \overset{\text{def}}{=} (\text{serve}, s).C_{p_5} \]
\[ C_{p_6} \overset{\text{def}}{=} (\text{repair}, e).C_{p_6} \quad C_{p_6} \overset{\text{def}}{=} (\text{fail}, a).C_{p_6} \]
Construction of PEPA model (cont.)

example

\[
\begin{align*}
C_{p1} & \overset{\text{def}}{=} (\text{prepare}, p).C_{\bar{p}1} & C_{\bar{p}1} & \overset{\text{def}}{=} (\text{finish}, f).C_{p1} \\
C_{p2} & \overset{\text{def}}{=} (\text{serve}, s).C_{\bar{p}2} & C_{\bar{p}2} & \overset{\text{def}}{=} (\text{prepare}, p).C_{p2} \\
C_{p3} & \overset{\text{def}}{=} (\text{finish}, f).C_{\bar{p}3} & C_{\bar{p}3} & \overset{\text{def}}{=} (\text{serve}, s).C_{p3} \\
C_{p4} & \overset{\text{def}}{=} (\text{serve}, s).C_{\bar{p}4} & C_{\bar{p}4} & \overset{\text{def}}{=} (\text{reset}, r).C_{p4} + (\text{repair}, e).C_{p4} \\
C_{p5} & \overset{\text{def}}{=} (\text{reset}, r).C_{\bar{p}5} + (\text{fail}, a).C_{\bar{p}5} & C_{\bar{p}5} & \overset{\text{def}}{=} (\text{serve}, s).C_{p5} \\
C_{p6} & \overset{\text{def}}{=} (\text{repair}, e).C_{\bar{p}6} & C_{\bar{p}6} & \overset{\text{def}}{=} (\text{fail}, a).C_{p6}
\end{align*}
\]

\[
\mathcal{M} \overset{\text{def}}{=} C_{p1}[200] \{\text{prepare, finish}\} C_{\bar{p}2}[200] \{\text{serve}\} C_{\bar{p}3}[200] \{\text{serve}\} C_{p4}[100] \{\text{serve, reset, repair}\} C_{\bar{p}5}[100] \{\text{fail}\} C_{\bar{p}6}[100]
\]
ODE extraction

- view PEPA model as high/low concentration, $C_p$ and $C_{\bar{p}}$
ODE extraction

- view PEPA model as high/low concentration, $C_p$ and $C_{\overline{p}}$
- construct a high/low activity graph
ODE extraction

- view PEPA model as high/low concentration, $C_p$ and $C_{\overline{p}}$

- construct a high/low activity graph
  - node for each component $C_p$
  - node for each activity $\alpha$
ODE extraction

- view PEPA model as high/low concentration, $C_p$ and $C_{\overline{p}}$

- construct a high/low activity graph
  - node for each component $C_p$
  - node for each activity $\alpha$
  - edge from component to activity if exit activity
  - edge from activity to component if entry activity
ODE extraction

- view PEPA model as high/low concentration, $C_p$ and $C_{\overline{p}}$
- construct a high/low activity graph
  - node for each component $C_p$
  - node for each activity $\alpha$
  - edge from component to activity if exit activity
  - edge from activity to component if entry activity
- extract ODEs
Comparison of ODEs

\[
\frac{dN(C_p, \tau)}{d\tau} = \sum_{(t, \lambda(t))} \lambda(t) \cdot \min\{N(C_{p'}, \tau)|C_{p'}(t, \lambda(t))\} - \sum_{(t, \lambda(t))} \lambda(t) \cdot \min\{N(C_{p'}, \tau)|C_{p'}(t, \lambda(t))\}
\]

\[
\frac{dm(p, \tau)}{d\tau} = \sum_{t \in \bullet p} \lambda(t) \cdot \min_{p' \in \bullet t \cap P} \{m(p', \tau)\} - \sum_{t \in \bullet p} \lambda(t) \cdot \min_{p' \in \bullet t \cap P} \{m(p', \tau)\}
\]
Comparison of ODEs

\[
\frac{dN(C_p, \tau)}{d\tau} = \sum_{(t, \lambda(t))} \lambda(t) \cdot \min\{N(C_{p'}, \tau) | C_{p'}(t, \lambda(t)) \} - \sum_{(t, \lambda(t))} \lambda(t) \cdot \min\{N(C_{p'}, \tau) | C_{p'}(t, \lambda(t)) \}
\]

\[
\frac{dm(p, \tau)}{d\tau} = \sum_{t \in \cdot p} \lambda(t) \cdot \min_{p' \in \cdot t \cap P} \{m(p', \tau)\} - \sum_{t \in \cdot p} \lambda(t) \cdot \min_{p' \in \cdot t \cap P} \{m(p', \tau)\}
\]
Comparison of ODEs

\[
\frac{dN(C_p, \tau)}{d\tau} = \sum \lambda(t). \min \{N(C_{p'}, \tau) | C_{p'} \xrightarrow{(t, \lambda(t))} \} - \sum \lambda(t). \min \{N(C_{p'}, \tau) | C_{p'} \xrightarrow{(t, \lambda(t))} \}
\]

\[
\frac{dm(p, \tau)}{d\tau} = \sum_{t \in \bullet p} \lambda(t). \min_{p' \in \bullet t \cap P} \{m(p', \tau)\} - \sum_{t \in \bullet p} \lambda(t). \min_{p' \in \bullet t \cap P} \{m(p', \tau)\}
\]
Comparison of ODEs

\[
\frac{dN(C_p, \tau)}{d\tau} = \sum_{(t, \lambda(t))} \lambda(t). \min\{N(C_{p'}, \tau) | C_{p'}(t, \lambda(t))\} - \sum_{(t, \lambda(t))} \lambda(t). \min\{N(C_{p'}, \tau) | C_{p'}(t, \lambda(t))\}
\]

- entry activity
- exit activity

\[
\frac{dm(p, \tau)}{d\tau} = \sum_{t \in \cdot p} \lambda(t). \min_{p' \in \cdot t \cap P} \{m(p', \tau)\} - \sum_{t \in \cdot p} \lambda(t). \min_{p' \in \cdot t \cap P} \{m(p', \tau)\}
\]
Comparison of ODEs

\[
\frac{dN(C_p, \tau)}{d\tau} = \sum_{(t, \lambda(t))} \lambda(t). \min\{N(C_{p'}, \tau)|C_{p'}(t, \lambda(t))\} - \sum_{(t, \lambda(t))} \lambda(t). \min\{N(C_{p'}, \tau)|C_{p'}(t, \lambda(t))\}
\]

\[
\frac{dm(p, \tau)}{d\tau} = \sum_{t \in \bullet p} \lambda(t). \min_{p' \in t \cap P} \{m(p', \tau)\} - \sum_{t \in \bullet p} \lambda(t). \min_{p' \in t \cap P} \{m(p', \tau)\}
\]
Comparison of ODEs

\[
\frac{dN(C_p, \tau)}{d\tau} = \sum_{(t, \lambda(t))} \lambda(t) \cdot \min \{ N(C_{p'}, \tau) \mid C_{p'} \xrightarrow{(t, \lambda(t))} \} - \sum_{(t, \lambda(t))} \lambda(t) \cdot \min \{ N(C_{p'}, \tau) \mid C_{p'} \xrightarrow{(t, \lambda(t))} \}
\]

\[
\frac{dm(p, \tau)}{d\tau} = \sum_{t \in \bullet p} \lambda(t) \cdot \min_{p' \in t \cap P} \{ m(p', \tau) \} - \sum_{t \in \bullet p} \lambda(t) \cdot \min_{p' \in t \cap P} \{ m(p', \tau) \}
\]
Comparison of ODEs

\[ \frac{dN(C_p, \tau)}{d\tau} = \sum_{(t, \lambda(t))} \lambda(t) \cdot \min \{ N(C_{p'}, \tau) | C_{p'} \xrightarrow{(t, \lambda(t))} \} - \sum_{(t, \lambda(t))} \lambda(t) \cdot \min \{ N(C_{p'}, \tau) | C_{p'} \xrightarrow{(t, \lambda(t))} \} \]

entry activity

exit activity

\[ \frac{dm(p, \tau)}{d\tau} = \sum_{t \in \cdot p} \lambda(t) \cdot \min_{p' \in \cdot t \cap P} \{ m(p', \tau) \} - \sum_{t \in \cdot p} \lambda(t) \cdot \min_{p' \in \cdot t \cap P} \{ m(p', \tau) \} \]

▶ both approaches give the same equations
Further work and conclusions

- further work
  - different approaches to finite server semantics
  - robustness of ODEs
Further work and conclusions

▶ further work
  ▶ different approaches to finite server semantics
  ▶ robustness of ODEs

▶ PEPA $\rightarrow$ timed continuous Petri nets
Further work and conclusions

- **further work**
  - different approaches to finite server semantics
  - robustness of ODEs

- PEPA $\rightarrow$ timed continuous Petri nets
  - ODEs are identical
Further work and conclusions

- further work
  - different approaches to finite server semantics
  - robustness of ODEs

- PEPA $\rightarrow$ timed continuous Petri nets
  - ODEs are identical
- bounded timed continuous Petri nets $\rightarrow$ PEPA
Further work and conclusions

- further work
  - different approaches to finite server semantics
  - robustness of ODEs

- PEPA $\rightarrow$ timed continuous Petri nets
  - ODEs are identical

- bounded timed continuous Petri nets $\rightarrow$ PEPA
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Further work and conclusions

- further work
  - different approaches to finite server semantics
  - robustness of ODEs

- PEPA → timed continuous Petri nets
  - ODEs are identical

- bounded timed continuous Petri nets → PEPA
  - ODEs are identical

- ODE semantics of PEPA has infinite server semantics