Equivalences for hybrid systems

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Introduction

- hybrid systems

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  - discrete behaviour

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  - well known
  - graphical rather than textual
  - not very compositional

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  - continuous behaviour, expressed as ODEs
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  - graphical rather than textual
  - not very compositional
- process algebras for hybrid systems
  - compositional language
  - semantic equivalences

Introduction (cont.)

- other process algebras

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  - ACP_{hs}^{srt} – Bergstra and Middelburg
  - HyPA – Cuijpers and Reniers
  - hybrid χ – van Beek *et al*
  - ϕ-calculus – Rounds and Song

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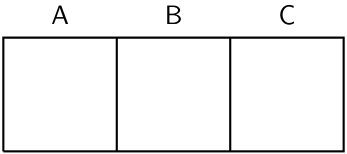
- HYPE
  - more fine-grained approach, individual additive flows
  - influence of continuous semantics of PEPA

Heater example



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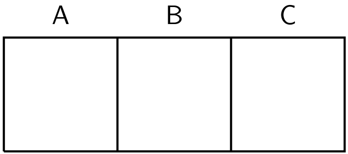

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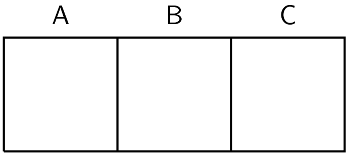

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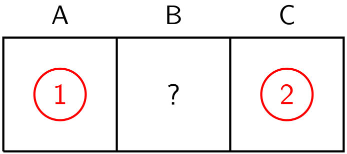

- three adjacent rooms
- fan heaters can be placed in each room
  - full effect on room
  - reduced effect on adjacent room
  - no effect on non-adjacent room
- heaters can be switched on and off, max of 25°C
- how does the temperature in Room B change if there is one heater in Room A and one in Room C?

HYPE syntax

events: $\underline{\mathsf{a}} \in \mathcal{E}$

activities: $\alpha(\vec{X}) = (\iota, r, I(\vec{X})) \subseteq \mathcal{A} \times \mathbb{R}^{+} \times \mathit{IN}$

HYPE syntax

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subcomponents: $S ::= \underline{a} : \alpha.C_s \mid S + S \qquad \underline{a} \in \mathcal{E}, \alpha \in \mathcal{A}$

components: $P ::= C(\vec{X}) \mid P \underset{L}{\bowtie} P \qquad L \subseteq \mathcal{E}$

uncontrolled system: $\Sigma ::= C(\vec{V}) \mid \Sigma \underset{L}{\bowtie} \Sigma \qquad L \subseteq \mathcal{E}$

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controller: $M ::= \underline{\text{a}}.M \mid 0 \mid M + M \qquad \underline{\text{a}} \in \mathcal{E}$

$Con ::= M \mid Con \underset{L}{\bowtie} Con \qquad L \subseteq \mathcal{E}$

controlled system: $ConSys ::= \Sigma \underset{L}{\bowtie} \underline{\text{init}}.Con \qquad L \subseteq \mathcal{E}$

HYPE syntax (cont.)

- HYPE model: $(ConSys, \mathcal{V}, \mathcal{X}, IN, IT, \mathcal{E}, \mathcal{A}, \mathrm{ec}, \mathrm{iv}, EC, ID)$

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  - event conditions: $\mathrm{ec} : \mathcal{E} \rightarrow \mathit{EC}$
  - influences and variables: $\mathrm{iv} : \mathit{IN} \rightarrow \mathcal{V}$
  - influence type $I(\vec{X})$ with $[\![I(\vec{X})]\!] = f(\vec{X})$

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  - subcomponents

$$\mathcal{C}_s(\vec{X}) \stackrel{def}{=} \underline{a}_1 : \alpha_1.\mathcal{C}_s(\vec{X}) + \ldots + \underline{a}_n : \alpha_n.\mathcal{C}_s(\vec{X}) \quad \underline{a}_i \neq \underline{a}_j$$

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  - synchronisation on shared events

Heater example (cont.)

- room: $Room_x(T) \stackrel{def}{=} \underline{\text{init}}:(t_{0,x}, -1, linear(T)).Room_x(T)$
  - $t_{0,x}$ represents influence of cooling on Room y

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- fan i in Room x affecting Room y:
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$$
\begin{aligned}
Fan_{i,x,y} \stackrel{def}{=} \; & \underline{\text{init}}:(t_{i,y}, 0, const).Fan_{i,x,y} + \\
& \underline{\text{off}}_i:(t_{i,y}, 0, const).Fan_{i,x,y} + \\
& \underline{\text{on}}_i:(t_{i,y}, r_i, const_{\psi(x,y)}).Fan_{i,x,y}
\end{aligned}
$$

where

$$
\psi(x,y) = \begin{cases} in & \text{if } x = y \\ adj & \text{if } x \text{ and } y \text{ are adjacent} \\ far & \text{otherwise} \end{cases}
$$

Heater example (cont.)

- uncontrolled system:

$$Sys \stackrel{def}{=} (Fan_{1,A,B} \underset{\{\underline{init}\}}{\bowtie} Fan_{2,C,B}) \underset{\{\underline{init}\}}{\bowtie} Room_B(T_B)$$

Heater example (cont.)

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- controller:

$$Con \stackrel{def}{=} Con_1 \underset{\emptyset}{\bowtie} Con_2 \qquad Con_i \stackrel{def}{=} \underline{\text{on}}_i.\underline{\text{off}}_i.Con_i$$

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$$MF \stackrel{def}{=} Sys \underset{M}{\bowtie} \underline{\mathrm{init}}.Con \qquad M = \{\underline{\mathrm{init}}, \underline{\mathrm{on}}_1, \underline{\mathrm{off}}_1, \underline{\mathrm{on}}_2, \underline{\mathrm{off}}_2\}$$

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- MF is well-defined

Heater example (cont.)

- $(MF, \{T_B\}, \{T\}, IN, IT, \mathcal{E}, \mathcal{A}, \mathrm{ec}, \mathrm{iv}, EC, ID)$

$$
\begin{array}{rcl}
IN & = & \{t_{0,B}, t_{1,B}, t_{2,B}\} \\
IT & = & \{const, const_{in}, const_{adj}, const_{far}, linear(T)\} \\
\mathcal{E} & = & \{\underline{\text{init}}, \underline{\text{on}}_1, \underline{\text{off}}_1, \underline{\text{on}}_2, \underline{\text{off}}_2\}
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$$\begin{aligned} \mathrm{ec}(\underline{\text{init}}) &= (true, (T'_B = T_0)) \\ \mathrm{ec}(\underline{\text{on}}_i) &= (\bot, (T'_B = T_B)) \\ \mathrm{ec}(\underline{\text{off}}_i) &= ((T_B = 25), (T'_B = T_B)) \end{aligned}$$

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$$\mathrm{iv}(t_{i,B}) = T_B$$

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\end{aligned}
$$

$$
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$$

$$
\begin{aligned}
[\![const]\!] &= 0 \qquad [\![linear(X)]\!] = X \\
[\![const_{in}]\!] &= 1 \qquad [\![const_{adj}]\!] = 0.5 \qquad [\![const_{far}]\!] = 0
\end{aligned}
$$

Operational semantics

- state: $\sigma : \mathit{IN} \rightarrow (\mathbb{R}^+ \times \mathit{IT})$

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- configuration: $\big\langle \mathit{ConSys}, \sigma \big\rangle$
- labelled transition system: $(\mathcal{F}, \mathcal{E}, \rightarrow \subseteq \mathcal{F} \times \mathcal{E} \times \mathcal{F})$
- updating function: $\sigma[\iota \mapsto (r, I)]$

$$\sigma[\iota \mapsto (r, I)](x) = \begin{cases} (r, I) & \text{if } x = \iota \\ \sigma(x) & \text{otherwise} \end{cases}$$

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- change identifying function: $\Gamma : \mathcal{S} \times \mathcal{S} \times \mathcal{S} \rightarrow \mathcal{S}$

$$(\Gamma(\sigma, \tau, \tau'))(\iota) = \begin{cases} \tau(\iota) & \text{if } \sigma(\iota) = \tau'(\iota) \\ \tau'(\iota) & \text{if } \sigma(\iota) = \tau(\iota) \\ \text{undefined} & \text{otherwise} \end{cases}$$

Operational semantics (cont.)

Prefix with influence:

$$\frac{}{\langle \underline{a}:(\iota, r, I).E, \sigma \rangle \xrightarrow{\underline{a}} \langle E, \sigma[\iota \mapsto (r, I)] \rangle}$$

Prefix without influence:

$$\frac{}{\langle \underline{a}.E, \sigma \rangle \xrightarrow{\underline{a}} \langle E, \sigma \rangle}$$

Choice:

$$\frac{\langle E, \sigma \rangle \xrightarrow{\underline{a}} \langle E', \sigma' \rangle}{\langle E + F, \sigma \rangle \xrightarrow{\underline{a}} \langle E', \sigma' \rangle} \qquad \frac{\langle F, \sigma \rangle \xrightarrow{\underline{a}} \langle F', \sigma' \rangle}{\langle E + F, \sigma \rangle \xrightarrow{\underline{a}} \langle F', \sigma' \rangle}$$

Constant:

$$\frac{\langle E, \sigma \rangle \xrightarrow{\underline{a}} \langle E', \sigma' \rangle}{\langle A, \sigma \rangle \xrightarrow{\underline{a}} \langle E', \sigma' \rangle} (A \stackrel{def}{=} E)$$

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Operational semantics (cont.)

Parallel without synchronisation:

$$\frac{\langle E, \sigma \rangle \xrightarrow{\underline{a}} \langle E', \sigma' \rangle}{\langle E \underset{M}{\bowtie} F, \sigma \rangle \xrightarrow{\underline{a}} \langle E' \underset{M}{\bowtie} F, \sigma' \rangle} \quad \underline{a} \notin M$$

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Parallel with synchronisation:

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$\underline{a} \in M, \Gamma$ defined

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Heater example (cont.)

- transition derivation

$$\frac{\langle F_{1,A,B}, \tau\rangle \xrightarrow{\underline{\text{init}}} \langle F_{1,A,B}, \tau_1\rangle \qquad \langle F_{2,C,B}, \tau\rangle \xrightarrow{\underline{\text{init}}} \langle F_{2,C,B}, \tau_2\rangle}{\langle F_{1,A,B} \underset{\underline{\text{init}}}{\bowtie} F_{2,C,B}, \tau\rangle \xrightarrow{\underline{\text{init}}} \langle F_{1,A,B} \underset{\underline{\text{init}}}{\bowtie} F_{2,C,B}, \tau_3\rangle}$$

$$\begin{aligned}
\tau &= \{t_{0,B} \mapsto *, t_{1,B} \mapsto *, t_{2,B} \mapsto *\} \\
\tau_1 &= \tau[t_{1,B} \mapsto (0,c)] = \{t_{0,B} \mapsto *, t_{1,B} \mapsto (0,c), t_{2,B} \mapsto *\} \\
\tau_2 &= \tau[t_{2,B} \mapsto (0,c)] = \{t_{0,B} \mapsto *, t_{1,B} \mapsto *, t_{2,B} \mapsto (0,c)\} \\
\tau_3 &= \Gamma(\tau, \tau_1, \tau_2) = \{t_{0,B} \mapsto *, t_{1,B} \mapsto (0,c), t_{2,B} \mapsto (0,c)\}
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Heater example (cont.)

- transition derivation

$$\frac{\langle F_{1,A,B}, \tau\rangle \xrightarrow{\text{init}} \langle F_{1,A,B}, \tau_1\rangle \qquad \langle F_{2,C,B}, \tau\rangle \xrightarrow{\text{init}} \langle F_{2,C,B}, \tau_2\rangle}{\langle F_{1,A,B} \underset{\underline{\text{init}}}{\bowtie} F_{2,C,B}, \tau\rangle \xrightarrow{\text{init}} \langle F_{1,A,B} \underset{\underline{\text{init}}}{\bowtie} F_{2,C,B}, \tau_3\rangle}$$

$$\begin{aligned}
\tau &= \{t_{0,B} \mapsto *, t_{1,B} \mapsto *, t_{2,B} \mapsto *\} \\
\tau_1 &= \tau[t_{1,B} \mapsto (0,c)] = \{t_{0,B} \mapsto *, t_{1,B} \mapsto (0,c), t_{2,B} \mapsto *\} \\
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Heater example (cont.)

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Heater example (cont.)

- transition derivation

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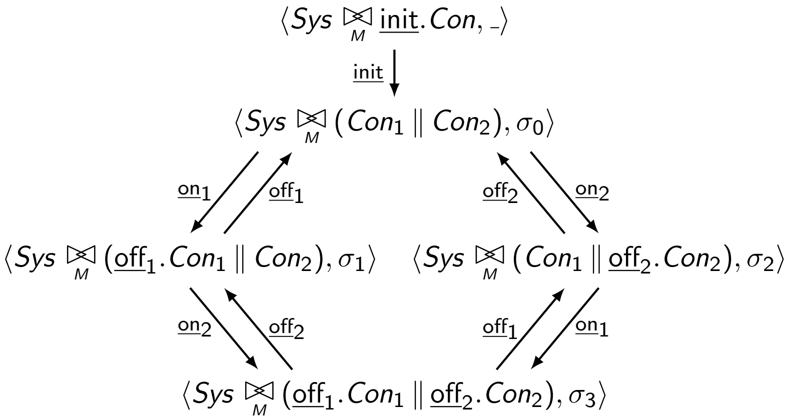
\end{aligned}$$

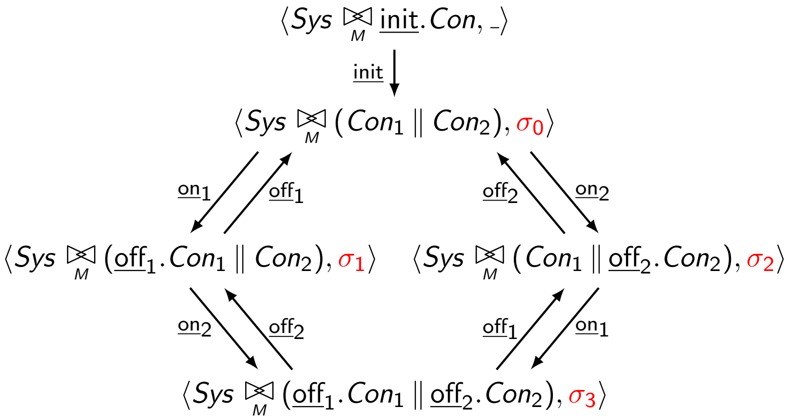
Heater example (cont.)

- labelled transition system



Heater example (cont.)

- labelled transition system



Heater example (cont.)

- for MF, there are four states

$$
\begin{aligned}
\sigma_0 &= \{t_{0,B} \mapsto (-1, \mathit{linear}(T_B)), t_{1,B} \mapsto (0, \mathit{const}), \\
&\qquad t_{2,B} \mapsto (0, \mathit{const})\} \\
\sigma_1 &= \{t_{0,B} \mapsto (-1, \mathit{linear}(T_B)), t_{1,B} \mapsto (r_1, \mathit{const}_{adj}), \\
&\qquad t_{2,B} \mapsto (0, \mathit{const})\} \\
\sigma_2 &= \{t_{0,B} \mapsto (-1, \mathit{linear}(T_B)), t_{1,B} \mapsto (0, \mathit{const}), \\
&\qquad t_{2,B} \mapsto (r_2, \mathit{const}_{adj})\} \\
\sigma_3 &= \{t_{0,B} \mapsto (-1, \mathit{linear}(T_B)), t_{1,B} \mapsto (r_1, \mathit{const}_{adj}), \\
&\qquad t_{2,B} \mapsto (r_2, \mathit{const}_{adj})\}
\end{aligned}
$$

Hybrid semantics

- extract ODEs from each state σ in the lts of CS

$$CS_\sigma = \left\{ \text{ODE for variable } V \;\middle|\; V \in \mathcal{V} \right\} \text{ where}$$

$$\frac{dV}{dt} = \sum \{ r[\![I(\vec{W})]\!] \mid \mathrm{iv}(\iota) = V \text{ and } \sigma(\iota) = (r, I(\vec{W})) \}$$

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  - determine from σ its rate and influence type

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  - multiply its rate and influence function together

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  - for any influence name associated with V
  - determine from σ its rate and influence type
  - multiply its rate and influence function together
  - sum these over all associated influence names

Heater example (cont.)

- state σ_0 occurs when both fans are off

$$MF_{\sigma_0} = \left\{ \frac{dT_B}{dt} = -T_B \right\}$$

Heater example (cont.)

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$$MF_{\sigma_2} = \left\{ \frac{dT_B}{dt} = -T_B + 0.5r_2 \right\}$$

Heater example (cont.)

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$$MF_{\sigma_2} = \left\{ \frac{dT_B}{dt} = -T_B + 0.5r_2 \right\}$$

- state σ_3 occurs when both fans are on

$$MF_{\sigma_3} = \left\{ \frac{dT_B}{dt} = -T_B + 0.5(r_1 + r_2) \right\}$$

Heater example (cont.)

- translation from HYPE system to hybrid automaton

Heater example (cont.)

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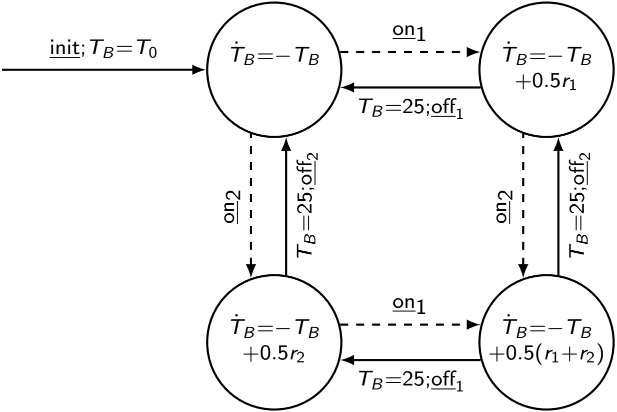

Equivalence semantics

- system bisimulation: relation B if for all $(P, Q) \in B$ whenever
  1. $\langle P, \sigma \rangle \xrightarrow{\text{a}} \langle P', \sigma' \rangle$, there exists $\langle Q', \sigma' \rangle$ with $\langle Q, \sigma \rangle \xrightarrow{\text{a}} \langle Q', \sigma' \rangle$ and $(P', Q') \in B$.
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- *Theorem 2*: if Σ_1 and Σ_2 have the same prefixes then $\Sigma_1 \underset{L}{\bowtie} \underline{\mathrm{init}}.Con \sim_s \Sigma_2 \underset{L}{\bowtie} \underline{\mathrm{init}}.Con$, assuming well-defined systems

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- *Theorem 3*: if $P \sim_s Q$ then $P_\sigma = Q_\sigma$ for all σ, assuming well-defined systems

Heater example (cont.)

- Consider two fans in Room C and none in Room A

$$Sys' \stackrel{def}{=} (Fan_{1,C,B} \underset{\{\underline{\text{init}}\}}{\bowtie} Fan_{2,C,B}) \underset{\{\underline{\text{init}}\}}{\bowtie} Room_B(T_B)$$

$$MF' \stackrel{def}{=} Sys' \underset{M}{\bowtie} \underline{\text{init}}.Con \qquad M = \{\underline{\text{init}}, \underline{\text{on}}_1, \underline{\text{off}}_1, \underline{\text{on}}_2, \underline{\text{off}}_2\}$$

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- Sys and Sys' have the same prefixes

$$\bigcup_{i=1,2} \{\underline{\text{init}} : (t_{i,y}, 0, const), \underline{\text{on}}_i : (t_{i,y}, r_i, const_{adj}), \underline{\text{off}}_i : (t_{i,y}, 0, const)\}$$

$$\cup \ \{\underline{\text{init}} : (t_{0,x}, -1, linear(T))\}$$

Heater example (cont.)

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- by Theorem 2, $MF \sim_s MF'$

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- by Theorem 2, $MF \sim_s MF'$
- by Theorem 3, MF and MF' have the same ODEs

Bisimulations for ACP_{hs}^{srt}

- bisimulation: relation B if for all $(\langle P, \sigma \rangle, \langle Q, \sigma \rangle) \in B$ whenever
  1. $\langle P, \sigma \rangle \xrightarrow{\underline{a}} \langle P', \sigma' \rangle$, there exists $\langle Q', \sigma' \rangle$ with $\langle Q, \sigma \rangle \xrightarrow{\underline{a}} \langle Q', \sigma' \rangle$ and $(\langle P', \sigma' \rangle, \langle Q', \sigma' \rangle) \in B$.
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- bisimilar: $P \leftrightarroweq Q$ if $\langle P, \sigma \rangle \leftrightarroweq \langle Q, \sigma \rangle$ for all σ
- not a congruence for parallel operator of ACP_{hs}^{srt}

Bisimulations for ACP^{srt}_{hs} (cont.)

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  - $\langle P, \sigma\rangle \xrightarrow{a} \langle P', \sigma[u_1, \ldots, u_n]\rangle$, $\langle Q, \sigma\rangle \xrightarrow{a} \langle Q', \sigma[u_1, \ldots, u_n]\rangle$

More general bisimulation

- let $\equiv$ be an equivalence over states
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More general bisimulation (cont.)

- $\equiv$ preserves updating if

$$\sigma \equiv \tau \Rightarrow \sigma[\iota \mapsto (r, I)] \equiv \tau[\iota \mapsto (r, I)]$$

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- *Theorem 5*: if $\equiv$ preserves updating then $\sim_s^{\equiv}$ is a congruence for all operators
- what are interesting equivalences over states?

$$P \sim_s^{\equiv} Q \quad \overset{?}{\Rightarrow} \quad P_\sigma = Q_\sigma$$

More general bisimulation (cont.)

- $\sigma_1 \doteq \sigma_2$ if

  $$add(\sigma_1, V, f(\vec{X})) = add(\sigma_2, V, f(\vec{X})) \text{ for all } V,\ f(\vec{X}) \text{ where}$$

  $$add(\sigma, V, f(\vec{X})) =$$

  $$\sum\{r \mid \mathrm{iv}(\iota) = V, \sigma(\iota) = (r, I(\vec{X})), f(\vec{X}) = [\![I(\vec{X})]\!]\}$$

- $P \sim_s^{\doteq} Q \quad \Rightarrow \quad P_\sigma = Q_\sigma$
- $\doteq$ does not preserve updating
- solutions
  - redefine *add* to preserve updates
  - require iv to be injective
  - consider individual equivalences

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  - more modelling

Thank you

Hybrid automata

- $(V, E, \mathbf{X}, \mathcal{E}, \mathit{flow}, \mathit{init}, \mathit{inv}, \mathit{event}, \mathit{jump}, \mathit{reset}, \mathit{urgent})$
- $\mathbf{X} = \{X_1, \ldots, X_n\}$, $\dot{X}_j$, X'_j
- control graph: $G = (V, E)$
- (control) modes: $v \in V$
  - associated ODEs: $\dot{\mathbf{X}} = \mathit{flow}(v)$
  - initial conditions: $\mathit{init}(v)$
  - invariants: $\mathit{inv}(v)$
- (control) switches: $e \in E$
  - events: $\mathit{event}(e) \in \mathcal{E}$
  - predicate on $\mathbf{X}$: $\mathit{jump}(e)$
  - predicate on $\mathbf{X} \cup \mathbf{X}'$: $\mathit{reset}(e)$
  - boolean: $\mathit{urgent}(e)$

HYPE model to hybrid automaton

- modes V: set of reachable configurations
- edges E: transitions between configurations
- variables $\mathbf{X}$: variables $\mathcal{V}$
- if $v_j = \langle P_j, \sigma_j \rangle$ then

$$\mathit{flow}(v_j)[X_i] \;=\; \sum\{r[\![I(\vec{W})]\!] \mid iv(\iota) = X_i \text{ and } \sigma_j(\iota) = (r, I(\vec{W}))\}$$

- $\mathit{inv}(v) = \mathit{true}$
- let e be an edge associated with $\underline{a}$ and let $ec(\underline{a}) = (\mathit{act}_{\underline{a}}, \mathit{res}_{\underline{a}})$
  - $\mathit{event}(e) = \underline{a}$ and $\mathit{reset}(e) = \mathit{res}_{\underline{a}}$
  - if $\mathit{act}_{\underline{a}} \neq \bot$ then $\mathit{jump}(e) = \mathit{act}_{\underline{a}}$ and $\mathit{urgent}(e) = \mathit{true}$
    else $\mathit{jump}(e) = \mathit{true}$ and $\mathit{urgent}(e) = \mathit{false}$
- $\mathit{init}(v) = \begin{cases} \mathit{res}_{\underline{\mathrm{init}}} & \text{if } v = \langle P, \sigma \rangle \text{ with primes removed} \\ \mathit{false} & \mathit{otherwise} \end{cases}$

Heater example in ACP^{srt}_{hs}

- system with temperature limit
- $\theta \equiv (T_B^{\bullet} = {}^{\bullet}T_B) \quad \psi \equiv (T_B = 25)$

$$
\begin{aligned}
\text{Start} &\stackrel{def}{=} (T_B = T_0) \wedge\!\!\blacktriangle\ \text{Off12} \\
\text{Off12} &\stackrel{def}{=} (\dot{T}_B = -T_B) \curvearrowright\!\!\blacktriangledown\ \sigma^*_{\text{rel}}(\theta \ulcorner\!\!\blacktriangledown\ (on_1 \cdot \text{On1} + on_2 \cdot \text{On2})) \\
\text{On1} &\stackrel{def}{=} (T_B \leq 25 \wedge \dot{T}_B = -T_B + 0.5r_1) \\
&\quad \curvearrowright\!\!\blacktriangledown\ \sigma^*_{\text{rel}}((\theta \ulcorner\!\!\blacktriangledown\ on_2 \cdot \text{On12}) + (\psi :\longrightarrow (\theta \ulcorner\!\!\blacktriangledown\ off_1 \cdot \text{Off12}))) \\
\text{On2} &\stackrel{def}{=} (T_B \leq 25 \wedge \dot{T}_B = -T_B + 0.5r_2) \\
&\quad \curvearrowright\!\!\blacktriangledown\ \sigma^*_{\text{rel}}((\theta \ulcorner\!\!\blacktriangledown\ on_1 \cdot \text{On12}) + (\psi :\longrightarrow (\theta \ulcorner\!\!\blacktriangledown\ off_2 \cdot \text{Off12}))) \\
\text{On12} &\stackrel{def}{=} (T_B \leq 25 \wedge \dot{T}_B = -T_B + 0.5(r_1 + r_2)) \\
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$$\mathrm{On2} \stackrel{def}{=} (T_B \leq 25 \wedge \dot{T}_B = -T_B + 0.5r_2)$$
$$\curvearrowright\!\blacktriangledown\ \sigma^*_{\mathrm{rel}}((\theta \ulcorner\!\blacktriangledown\ on_1 \cdot \mathrm{On12}) + (\psi :\longrightarrow (\theta \ulcorner\!\blacktriangledown\ off_2 \cdot \mathrm{Off12})))$$

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