Call-by-name is dual to call-by-value

Philip Wadler
Avaya Labs
wadler@avaya.com

Related work

- Filinski (1989)
  Symmetric λ-calculus
- Griffin (1990)
  Computational interpretation of classical logic
- Parigot (1992)
  $\lambda\mu$-calculus
- Barbanera and Berardi (1996)
  Symmetric λ-calculus
- Selinger (1999)
  Control categories and duality
- Curien and Herbelin (2000)
  The Duality of Computation

Gerhard Gentzen (1909-1945)

Gentzen 1934: Natural Deduction

&–*I*	&–*E*	∨–*I*	∨–*E*
$\dfrac{\mathfrak{A} \quad \mathfrak{B}}{\mathfrak{A} \,\&\, \mathfrak{B}}$	$\dfrac{\mathfrak{A} \,\&\, \mathfrak{B}}{\mathfrak{A}} \quad \dfrac{\mathfrak{A} \,\&\, \mathfrak{B}}{\mathfrak{B}}$	$\dfrac{\mathfrak{A}}{\mathfrak{A} \lor \mathfrak{B}} \quad \dfrac{\mathfrak{B}}{\mathfrak{A} \lor \mathfrak{B}}$	$\dfrac{\mathfrak{A} \lor \mathfrak{B} \quad \begin{matrix}[\mathfrak{A}] \\ \mathfrak{C}\end{matrix} \quad \begin{matrix}[\mathfrak{B}] \\ \mathfrak{C}\end{matrix}}{\mathfrak{C}}$
∀–*I*	∀–*E*	∃–*I*	∃–*E*
$\dfrac{\mathfrak{F}\mathfrak{a}}{\forall \mathfrak{x}\, \mathfrak{F}\mathfrak{x}}$	$\dfrac{\forall \mathfrak{x}\, \mathfrak{F}\mathfrak{x}}{\mathfrak{F}\mathfrak{a}}$	$\dfrac{\mathfrak{F}\mathfrak{a}}{\exists \mathfrak{x}\, \mathfrak{F}\mathfrak{x}}$	$\dfrac{\exists \mathfrak{x}\, \mathfrak{F}\mathfrak{x} \quad \begin{matrix}[\mathfrak{F}\mathfrak{a}] \\ \mathfrak{C}\end{matrix}}{\mathfrak{C}}$
⊃–*I*	⊃–*E*	¬–*I*	¬–*E*
$\dfrac{\begin{matrix}[\mathfrak{A}] \\ \mathfrak{B}\end{matrix}}{\mathfrak{A} \supset \mathfrak{B}}$	$\dfrac{\mathfrak{A} \quad \mathfrak{A} \supset \mathfrak{B}}{\mathfrak{B}}$	$\dfrac{\begin{matrix}[\mathfrak{A}] \\ \curlywedge\end{matrix}}{\neg \mathfrak{A}}$	$\dfrac{\mathfrak{A} \quad \neg \mathfrak{A}}{\curlywedge} \quad \dfrac{\curlywedge}{\mathfrak{D}}\ .$

Gentzen 1934: Sequent Calculus

$$\&\text{–}IS\colon\ \frac{\Gamma \to \Theta, \mathfrak{A} \qquad \Gamma \to \Theta, \mathfrak{B}}{\Gamma \to \Theta, \mathfrak{A} \,\&\, \mathfrak{B}},$$

$$\&\text{–}IA\colon\ \frac{\mathfrak{A}, \Gamma \to \Theta}{\mathfrak{A} \,\&\, \mathfrak{B}, \Gamma \to \Theta} \qquad \frac{\mathfrak{B}, \Gamma \to \Theta}{\mathfrak{A} \,\&\, \mathfrak{B}, \Gamma \to \Theta},$$

$$\vee\text{–}IA\colon\ \frac{\mathfrak{A}, \Gamma \to \Theta \qquad \mathfrak{B}, \Gamma \to \Theta}{\mathfrak{A} \vee \mathfrak{B}, \Gamma \to \Theta},$$

$$\vee\text{–}IS\colon\ \frac{\Gamma \to \Theta, \mathfrak{A}}{\Gamma \to \Theta, \mathfrak{A} \vee \mathfrak{B}} \qquad \frac{\Gamma \to \Theta, \mathfrak{B}}{\Gamma \to \Theta, \mathfrak{A} \vee \mathfrak{B}},$$

$$\forall\text{–}IS\colon\ \frac{\Gamma \to \Theta, \mathfrak{F}\mathfrak{a}}{\Gamma \to \Theta, \forall \mathfrak{x}\, \mathfrak{F}\mathfrak{x}},$$

$$\exists\text{–}IA\colon\ \frac{\mathfrak{F}\mathfrak{a}, \Gamma \to \Theta}{\exists \mathfrak{x}\, \mathfrak{F}\mathfrak{x}, \Gamma \to \Theta}.$$

Part 1

The dual calculus

Syntax and Judgements

Terms	M, N	$::=$	$x \mid \langle M, N\rangle \mid \langle M\rangle\mathrm{inl} \mid \langle N\rangle\mathrm{inr} \mid [K]\mathrm{not} \mid (C).\bar{x}$
Co-terms	K, L	$::=$	$\bar{x} \mid [K, L] \mid \mathrm{fst}[K] \mid \mathrm{snd}[L] \mid \mathrm{not}\langle M\rangle \mid x.(C)$
Commands	C, D	$::=$	$M \cdot K$

$$\Gamma \triangleright M : A; \Delta$$

$$\Gamma; K : A \triangleleft \Delta$$

$$\Gamma \triangleright C \triangleleft \Delta$$

Type rules

$$\frac{\Gamma \triangleright M : A;\ \Delta \qquad \Gamma \triangleright N : B;\ \Delta}{\Gamma \triangleright \langle M, N\rangle : A \mathbin{\&} B;\ \Delta}\ \text{\&-R}$$

$$\frac{\Gamma;\ K : A \triangleleft \Delta}{\Gamma;\ \mathrm{fst}[K] : A \mathbin{\&} B \triangleleft \Delta}\ \text{\&-L}_1 \qquad \frac{\Gamma;\ L : B \triangleleft \Delta}{\Gamma;\ \mathrm{snd}[L] : A \mathbin{\&} B \triangleleft \Delta}\ \text{\&-L}_2$$

$$\frac{\Gamma \triangleright M : A;\ \Delta}{\Gamma \triangleright \langle M\rangle\mathrm{inl} : A \vee B;\ \Delta}\ \vee\text{-R}_1 \qquad \frac{\Gamma \triangleright N : B;\ \Delta}{\Gamma \triangleright \langle N\rangle\mathrm{inr} : A \vee B;\ \Delta}\ \vee\text{-R}_2$$

$$\frac{\Gamma;\ K : A \triangleleft \Delta \qquad \Gamma;\ L : B \triangleleft \Delta}{\Gamma;\ [K, L] : A \vee B \triangleleft \Delta}\ \vee\text{-L}$$

Type rules, continued

$$\frac{}{\Gamma, x : A \triangleright x : A; \Delta}\ \text{Id-R} \qquad \frac{}{\Gamma; \bar{x} : A \triangleleft \bar{x} : A, \Delta}\ \text{Id-L}$$

$$\frac{\Gamma; K : A \triangleleft \Delta}{\Gamma \triangleright [K]\text{not} : \neg A; \Delta}\ \neg\text{-R} \qquad \frac{\Gamma \triangleright M : A; \Delta}{\Gamma; \text{not}\langle M \rangle : \neg A \triangleleft \Delta}\ \neg\text{-L}$$

$$\frac{\Gamma \triangleright C \triangleleft \bar{x} : A, \Delta}{\Gamma \triangleright (C).\bar{x} : A; \Delta}\ \mu \qquad \frac{\Gamma, x : A \triangleright C \triangleleft \Delta}{\Gamma; x.(C) : A \triangleleft \Delta}\ \lambda$$

$$\frac{\Gamma \triangleright M : A; \Delta \qquad \Gamma; K : A \triangleleft \Delta}{\Gamma \triangleright M \cdot K \triangleleft \Delta}\ \text{Cut}$$

Part 2

Call-by-value

Call-by-value reductions

$$\text{Values} \quad V, W \quad ::= \quad x \mid \langle V, W\rangle \mid \langle V\rangle\text{inl} \mid \langle W\rangle\text{inr} \mid [K]\text{not}$$

$$
\begin{array}{lll}
(\beta\&_1) & \langle V, W\rangle \cdot \text{fst}[K] \Rightarrow_v & V \cdot K \\
(\beta\&_2) & \langle V, W\rangle \cdot \text{snd}[L] \Rightarrow_v & W \cdot L \\
(\beta\vee_1) & \langle V\rangle\text{inl} \cdot [K, L] \Rightarrow_v & V \cdot K \\
(\beta\vee_2) & \langle W\rangle\text{inr} \cdot [K, L] \Rightarrow_v & W \cdot L \\
(\eta\&) & \langle (V \cdot \text{fst}[\bar{x}]).\bar{x}, (V \cdot \text{snd}[\bar{y}]).\bar{y}\rangle \Rightarrow_v & V \\
(\eta\vee) & [x.(\langle x\rangle\text{inl} \cdot K), y.(\langle y\rangle\text{inr} \cdot K)] \Rightarrow_v & K
\end{array}
$$

Call-by-value reductions, continued

$$\text{Evaluation context} \quad E \quad ::= \quad \{\,\} \mid \langle E, M\rangle \mid \langle V, E\rangle \mid \langle E\rangle\text{inl} \mid \langle E\rangle\text{inr}$$

$$
\begin{array}{llll}
(\beta\neg) & [K]\text{not} \cdot \text{not}\langle M\rangle & \Rightarrow_v & M \cdot K \\
(\eta\neg) & [x.(V \cdot \text{not}\langle x\rangle)]\text{not} & \Rightarrow_v & V \\
(\beta\lambda) & V \cdot x.(C) & \Rightarrow_v & C\{V/x\} \\
(\eta\lambda) & x.(x \cdot K) & \Rightarrow_v & K \\
(\beta\mu) & (C).\bar{x} \cdot K & \Rightarrow_v & C\{K/\bar{x}\} \\
(\eta\mu) & (M \cdot \bar{x}).\bar{x} & \Rightarrow_v & M \\
(\text{Name}) & E\{M\} & \Rightarrow_v & (M \cdot x.(E\{x\} \cdot \bar{z})).\bar{z}, \quad \text{if } M \neq V
\end{array}
$$

Part 3

Call-by-name

Call-by-name reductions

$$\text{Covalues} \quad P, Q \quad ::= \quad \bar{x} \mid [P, Q] \mid \mathrm{fst}[P] \mid \mathrm{snd}[Q] \mid \mathrm{not}\langle M \rangle$$

$$
\begin{array}{lll}
(\beta\&_1) & \langle M, N\rangle \cdot \mathrm{fst}[P] & \Rightarrow_n \quad M \cdot P \\
(\beta\&_2) & \langle M, N\rangle \cdot \mathrm{snd}[Q] & \Rightarrow_n \quad N \cdot Q \\
(\beta\vee_1) & \langle M\rangle\mathrm{inl} \cdot [P, Q] & \Rightarrow_n \quad M \cdot P \\
(\beta\vee_2) & \langle N\rangle\mathrm{inr} \cdot [P, Q] & \Rightarrow_n \quad N \cdot Q \\
(\eta\&) & \langle (M \cdot \mathrm{fst}[\bar{x}]).\bar{x}, (M \cdot \mathrm{snd}[\bar{y}]).\bar{y}\rangle & \Rightarrow_n \quad M \\
(\eta\vee) & [x.(\langle x\rangle\mathrm{inl} \cdot P), y.(\langle y\rangle\mathrm{inr} \cdot P)] & \Rightarrow_n \quad P
\end{array}
$$

Call-by-name reductions, continued

Evaluation cocontext $F \quad ::= \quad \{\,\} \mid [F, K] \mid [P, F] \mid \mathrm{fst}[F] \mid \mathrm{snd}[F]$

$$
\begin{array}{llll}
(\beta\neg) & [K]\mathrm{not} \cdot \mathrm{not}\langle M \rangle & \Rightarrow_n & M \cdot K \\
(\eta\neg) & \mathrm{not}\langle([\bar{x}]\mathrm{not} \cdot P).\bar{x}\rangle & \Rightarrow_n & P \\
(\beta\lambda) & M \cdot x.(C) & \Rightarrow_n & C\{M/x\} \\
(\eta\lambda) & x.(x \cdot K) & \Rightarrow_n & K \\
(\beta\mu) & (C).\bar{x} \cdot P & \Rightarrow_n & C\{P/\bar{x}\} \\
(\eta\mu) & (M \cdot \bar{x}).\bar{x} & \Rightarrow_n & M \\
(\text{Name}) & F\{K\} & \Rightarrow_n & z.((z \cdot F\{\bar{x}\}).\bar{x} \cdot K), \quad \text{if } K \neq P
\end{array}
$$

Part 4

Duality

Duality

$$
\begin{aligned}
(X)^\circ &\equiv X \\
(A \mathbin{\&} B)^\circ &\equiv A^\circ \vee B^\circ \\
(A \vee B)^\circ &\equiv A^\circ \mathbin{\&} B^\circ \\
(\neg A)^\circ &\equiv \neg A^\circ
\end{aligned}
$$

$$
\begin{aligned}
(x)^\circ &\equiv \bar{x} \\
(\langle M, N\rangle)^\circ &\equiv [M^\circ, N^\circ] \\
(\langle M\rangle \mathrm{inl})^\circ &\equiv \mathrm{fst}[M^\circ] \\
(\langle N\rangle \mathrm{inr})^\circ &\equiv \mathrm{snd}[M^\circ] \\
([K]\mathrm{not})^\circ &\equiv \mathrm{not}\langle K^\circ\rangle \\
((C).\bar{x})^\circ &\equiv x.(C^\circ)
\end{aligned}
$$

$$
\begin{aligned}
(\bar{x})^\circ &\equiv x \\
([K, L])^\circ &\equiv \langle K^\circ, L^\circ\rangle \\
(\mathrm{fst}[K])^\circ &\equiv \langle K^\circ\rangle \mathrm{inl} \\
(\mathrm{snd}[L])^\circ &\equiv \langle K^\circ\rangle \mathrm{inr} \\
(\mathrm{not}\langle M\rangle)^\circ &\equiv [M^\circ]\mathrm{not} \\
(x.(C))^\circ &\equiv (C^\circ).\bar{x}
\end{aligned}
$$

$$
(M \cdot K)^\circ \equiv K^\circ \cdot M^\circ
$$

Call-by-value is dual to call-by-name

$$\left.\begin{array}{r} \Gamma \rhd M : A; \Delta \\ \Gamma; K : A \lhd \Delta \\ \Gamma \rhd C \lhd \Delta \end{array}\right\} \quad \text{iff} \quad \left\{\begin{array}{l} \Delta^\circ; M^\circ : A^\circ \lhd \Gamma^\circ \\ \Delta^\circ \rhd K^\circ : A^\circ; \Gamma^\circ \\ \Delta^\circ \rhd C^\circ \lhd \Gamma^\circ \end{array}\right.$$

$$M^{\circ\circ} \equiv M$$

$$\left.\begin{array}{r} M \Rightarrow_v N \\ K \Rightarrow_v L \\ C \Rightarrow_v D \end{array}\right\} \quad \text{iff} \quad \left\{\begin{array}{l} M^\circ \Rightarrow_n N^\circ \\ K^\circ \Rightarrow_n L^\circ \\ C^\circ \Rightarrow_n D^\circ \end{array}\right.$$

Part 5

Call-by-value
Continuation-passing style

Continuation-passing style

$$
\begin{aligned}
(X)^\dagger &\equiv X \\
(A \mathbin{\&} B)^\dagger &\equiv A^\dagger \times B^\dagger \\
(A \vee B)^\dagger &\equiv A^\dagger + B^\dagger \\
(\neg A)^\dagger &\equiv A^\dagger \to R
\end{aligned}
$$

$$
\begin{array}{lcl}
(x)^* & \equiv & \hat{\lambda}\bar{z}.\, \bar{z}\, x \\
(\langle M, N\rangle)^* & \equiv & \hat{\lambda}\bar{z}.\, M^*\, (\hat{\lambda}x.\, N^*\, (\hat{\lambda}y.\, \bar{z}\,(x, y))) \\
(\langle M\rangle\mathrm{inl})^* & \equiv & \hat{\lambda}\bar{z}.\, M^*\, (\hat{\lambda}x.\, \bar{z}\,(\mathrm{inl}(x))) \\
(\langle N\rangle\mathrm{inr})^* & \equiv & \hat{\lambda}\bar{z}.\, N^*\, (\hat{\lambda}y.\, \bar{z}\,(\mathrm{inr}(y))) \\
([K]\mathrm{not})^* & \equiv & \hat{\lambda}\bar{z}.\, \bar{z}\, K^* \\
((C).\bar{x})^* & \equiv & \lambda\bar{x}.\, C^* \\
\\
(\bar{x})^* & \equiv & \hat{\lambda}z.\, \bar{x}\, z \\
([K, L])^* & \equiv & \hat{\lambda}z.\, \mathrm{case}\ z\ \mathrm{of}\ [\mathrm{inl}(x) \rightarrow K^*\, x,\ \mathrm{inr}(y) \rightarrow L^*\, y] \\
(\mathrm{fst}[K])^* & \equiv & \hat{\lambda}z.\, \mathrm{case}\ z\ \mathrm{of}\ (x,-) \rightarrow K^*\, x \\
(\mathrm{snd}[L])^* & \equiv & \hat{\lambda}z.\, \mathrm{case}\ z\ \mathrm{of}\ (-,y) \rightarrow L^*\, y \\
(\mathrm{not}\langle M\rangle)^* & \equiv & \hat{\lambda}z.\, K^*\, z \\
(x.(C))^* & \equiv & \lambda x.\, C^* \\
\\
(M \cdot K)^* & \equiv & M^*\, K^*
\end{array}
$$

CPS preserves types and reductions

$$\left.\begin{array}{r} \Gamma \triangleright M : A;\ \Delta \\ \Gamma;\ K : A \triangleleft \Delta \\ \Gamma \triangleright C \triangleleft \Delta \end{array}\right\} \quad \text{iff} \quad \left\{\begin{array}{l} (\Gamma)^\dagger, (\neg\Delta)^\dagger \triangleright M^* : (\neg\neg A)^\dagger \\ (\Gamma)^\dagger, (\neg\Delta)^\dagger \triangleright K^* : (\neg A)^\dagger \\ (\Gamma)^\dagger, (\neg\Delta)^\dagger \triangleright C^* : R \end{array}\right.$$

$$\left.\begin{array}{r} M \Rightarrow_v N \\ K \Rightarrow_v L \\ C \Rightarrow_v D \end{array}\right\} \quad \text{iff} \quad \left\{\begin{array}{l} M^* \Rightarrow N^* \\ K^* \Rightarrow L^* \\ C^* \Rightarrow D^* \end{array}\right.$$

Part 6

Call-by-name
Continuation-passing style

Continuation-passing style

$$
\begin{aligned}
(X)^{\dagger} &\equiv X \\
(A \mathbin{\&} B)^{\dagger} &\equiv A^{\dagger} + B^{\dagger} \\
(A \vee B)^{\dagger} &\equiv A^{\dagger} \times B^{\dagger} \\
(\neg A)^{\dagger} &\equiv A^{\dagger} \to R
\end{aligned}
$$

$$
\begin{array}{lcl}
(x)^* & \equiv & \hat{\lambda}\bar{z}.\, x\ \bar{z} \\
(\langle M, N\rangle)^* & \equiv & \hat{\lambda}\bar{z}.\, \mathrm{case}\ \bar{z}\ \mathrm{of}\ [\mathrm{inl}(\bar{x}) \longrightarrow M^*\ \bar{x},\ \mathrm{inr}(\bar{y}) \longrightarrow N^*\ \bar{y}] \\
(\langle M\rangle\mathrm{inl})^* & \equiv & \hat{\lambda}\bar{z}.\, \mathrm{case}\ \bar{z}\ \mathrm{of}\ (\bar{x}, -) \longrightarrow M^*\ \bar{x} \\
(\langle N\rangle\mathrm{inr})^* & \equiv & \hat{\lambda}\bar{z}.\, \mathrm{case}\ \bar{z}\ \mathrm{of}\ (-, \bar{y}) \longrightarrow N^*\ \bar{y} \\
([K]\mathrm{not})^* & \equiv & \hat{\lambda}\bar{z}.\, M^*\ \bar{z} \\
((C).\bar{x})^* & \equiv & \lambda\bar{x}.\, C^*
\end{array}
$$

$$
\begin{array}{lcl}
(\bar{x})^* & \equiv & \hat{\lambda}z.\, z\ \bar{x} \\
([K, L])^* & \equiv & \hat{\lambda}z.\, K^*\ (\hat{\lambda}\bar{x}.\, L^*\ (\hat{\lambda}\bar{y}.\, z\ (\bar{x}, \bar{y}))) \\
(\mathrm{fst}[K])^* & \equiv & \hat{\lambda}z.\, K^*\ (\hat{\lambda}\bar{x}.\, z\ (\mathrm{inl}(\bar{x}))) \\
(\mathrm{snd}[L])^* & \equiv & \hat{\lambda}z.\, L^*\ (\hat{\lambda}\bar{y}.\, z\ (\mathrm{inr}(\bar{y}))) \\
(\mathrm{not}\langle M\rangle)^* & \equiv & \hat{\lambda}z.\, z\ M^* \\
(x.(C))^* & \equiv & \lambda x.\, C^*
\end{array}
$$

$$
(M \cdot K)^* \equiv K^*\ M^*
$$

CPS preserves types and reductions

$$\left.\begin{array}{r} \Gamma \rhd M : A;\, \Delta \\ \Gamma;\, K : A \lhd \Delta \\ \Gamma \rhd C \lhd \Delta \end{array}\right\} \quad \text{iff} \quad \left\{\begin{array}{l} (\Delta)^\dagger, (\neg\Gamma)^\dagger \rhd M^* : (\neg A)^\dagger \\ (\Delta)^\dagger, (\neg\Gamma)^\dagger \rhd K^* : (\neg\neg A)^\dagger \\ (\Delta)^\dagger, (\neg\Gamma)^\dagger \rhd C^* : R \end{array}\right.$$

$$\left.\begin{array}{r} M \Rightarrow_n N \\ K \Rightarrow_n L \\ C \Rightarrow_n D \end{array}\right\} \quad \text{iff} \quad \left\{\begin{array}{l} M^* \Rightarrow N^* \\ K^* \Rightarrow L^* \\ C^* \Rightarrow D^* \end{array}\right.$$

Part 7

Functions

Functions

$$\frac{\Gamma, x : A \triangleright N : B;\ \Delta}{\Gamma \triangleright \lambda x.\, N : A \to B;\ \Delta}\ \to\text{-R} \qquad \frac{\Gamma \triangleright M : A;\ \Delta \qquad \Gamma;\ L : B \triangleleft \Delta}{\Gamma;\ M \mathbin{@} L : A \to B \triangleleft \Delta}\ \to\text{-L}$$

$$\begin{array}{llll} (\beta \to) & (\lambda x.\, N) \cdot (M \mathbin{@} L) & \Rightarrow_v & M \cdot x.(N \cdot L) \\ (\eta \to) & \lambda x.\, (V \cdot x \mathbin{@} \bar{y}).\bar{y} & \Rightarrow_v & V \end{array}$$

$$\begin{array}{lcl} A \to B & \equiv & \neg A \vee B \\ \lambda x.\, N & \equiv & (\langle [x.(\langle N \rangle \mathrm{inr} \cdot \bar{z})]\mathrm{not} \rangle \mathrm{inl} \cdot \bar{z}).\bar{z} \\ M \mathbin{@} L & \equiv & [\mathrm{not}\langle M \rangle, L] \end{array}$$

Part 8

Conclusions

Related work

- Filinski (1989)
  Symmetric λ-calculus
- Griffin (1990)
  Computational interpretation of classical logic
- Parigot (1992)
  $\lambda\mu$-calculus
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