Down with the bureaucracy of syntax! Pattern matching for classical linear logic

Philip Wadler
University of Edinburgh
wadler@inf.ed.ac.uk

ABSTRACT

This paper introduces a new way of attaching proof terms to proof trees for classical linear logic, which bears a close resemblance to the way that pattern matching is used in programming languages. It equates the same proofs that are equated by proof nets, in the formulation of proof nets introduced by Dominic Hughes and Rob van Glabbeek; and goes beyond that formulation in handling exponentials and units. It provides a symmetric treatment of all the connectives, and may provide programmers with improved insight into connectives such as "par" and "why not" that are difficult to treat in programming languages based on an intuitionistic formulation of linear logic.

This paper uses colour to enhance its presentation. If the URL below is not blue, follow it for the colour version.

http://homepages.inf.ed.ac.uk/wadler

1. INTRODUCTION

Some distinctions are worth making, others are not.

Many proofs in sequent calculus contain inessential differences, for instance in the order of applying non-interfering rules. Here are two proofs in classical linear logic which differ in the order in which the two last rules are applied.

$$\dfrac{\dfrac{\dfrac{}{A \longrightarrow A}\,\mathrm{Id}}{A\&B \longrightarrow A}\,\&\mathrm{L}}{A\&B \longrightarrow A \oplus C}\,\oplus\mathrm{R} \qquad \dfrac{\dfrac{\dfrac{}{A \longrightarrow A}\,\mathrm{Id}}{A \longrightarrow A \oplus C}\,\oplus\mathrm{R}}{A\&B \longrightarrow A \oplus C}\,\&\mathrm{L}$$

Jean-Yves Girard refers to inessential differences in proofs as "the bureaucracy of syntax" [9], and he introduced proof nets as a way to avoid such bureacracy [8]. The treatment of the additives $\&$ and $\oplus$ in Girard's original formulation of proof nets was not very satisfactory, and an improved formulation was put forward by Dominic Hughes and Rob van Glabbeek [13].

This paper introduces a new way of attaching proof terms to proof trees for classical linear logic, which bears a close resemblance to the way that pattern matching is used in programming languages. It equates the same proofs that are equated by proof nets, in the formulation of proof nets introduced by Hughes and van Glabbeek [13]; and goes beyond that formulation in handling exponentials and units. It provides a symmetric treatment of all the connectives, and may provide programmers with improved insight into connectives such as "par" ($\invamp$) and "why not" ($?$) that are difficult to treat in programming languages based on an intuitionistic formulation of linear logic.

As is well known, proofs in the intuitionistic natural deduction of Gentzen [7] can be labeled with terms of the simply-typed lambda calculus of Church [4, 5], in such a way that reduction of terms corresponds to simplification of proofs [11, 12]. The same idea extends to other logics and computational calculi. Proofs in the intuitionistic natural deduction formulation of the linear logic of Girard [8] can be labelled with terms in calculi devised by Abramsky [1], Benton, Bierman, Hyland, and de Paiva [2], and Wadler [16]; and proofs in the classical sequent calculus of Gentzen [7] can be labelled with terms in calculi devised by Curien and Herbelin [6] and Wadler [17]. This paper proposes proof terms corresponding to a sequent formulation of classical linear logic, combining ideas from the works cited above.

This paper presumes some familiarity with linear logic and the use of the "formulas as types" to relate proofs and programs. For a tutorial introduction, see [16].

The organization of this paper is as follows. Section 2 reviews programming languages for intuitionistic linear logic. Section 3 reviews the sequent formulation of classical linear logic. Section 4 introduces proof terms for these sequents. Section 5 introduces reduction and relates it to cut elimination. Section 6 sketches the relation to proof nets. Section 7 sketches an alternative formulation of proof terms.

2. PATTERN MATCHING

Here we give an intuitive introduction to the use of pattern matching in proof terms for linear logic, using an intuitionistic rather than a classical formulation.

Recall that linear logic provides two products. From the additive product, $A\&B$, one may extract either the first component or the second component, but not both. We indicate this with pattern matching, using a dash to indicate which component is ignored.

$$\dfrac{\Gamma \longrightarrow^{M} A \qquad \Gamma \longrightarrow^{N} B}{\Gamma \longrightarrow^{M\&N} A\&B}\,\&\mathrm{I}$$

Draft of 1 April 2004

$$\dfrac{\Gamma \longrightarrow {}^{M} A\&B \qquad {}^{x}A, \Sigma \longrightarrow {}^{N} C}{\Gamma, \Sigma \longrightarrow {}^{\text{case } M \text{ of } x\&-.\,N} C}\;\&\text{E}$$

$$\dfrac{\Gamma \longrightarrow {}^{M} A\&B \qquad {}^{y}B, \Sigma \longrightarrow {}^{N} C}{\Gamma, \Sigma \longrightarrow {}^{\text{case } M \text{ of } -\&y.\,N} C}\;\&\text{E}$$

(Here the types are formulas of linear logic, which are labeled with variables x, y, z on the left, and proof terms M, N, O on the right.) We have the following reduction rules.

$$\begin{array}{lcl} \text{case } M\&N \text{ of } x\&-.\,O & \Longrightarrow & O[M/x] \\ \text{case } M\&N \text{ of } -\&y.\,O & \Longrightarrow & O[N/y] \end{array}$$

From the multiplicative product, $A \otimes B$, one must extract both components.

$$\dfrac{\Gamma \longrightarrow {}^{M} A \qquad \Sigma \longrightarrow {}^{N} B}{\Gamma, \Sigma \longrightarrow {}^{M\otimes N} A \otimes B}\;\otimes\text{I}$$

$$\dfrac{\Gamma \longrightarrow {}^{M} A \otimes B \qquad {}^{x}A, {}^{y}B, \Sigma \longrightarrow {}^{N} C}{\Gamma, \Sigma \longrightarrow {}^{\text{case } M \text{ of } x\otimes y.\,N} C}\;\otimes\text{E}$$

We have the following reduction rule.

$$\text{case } M \otimes N \text{ of } x \otimes y.\,O \quad \Longrightarrow \quad O[M/x, N/y]$$

Finally, for a sum one requires a case analysis.

$$\dfrac{\Gamma \longrightarrow {}^{M} A}{\Gamma \longrightarrow {}^{M\oplus -} A \oplus B}\;\oplus\text{R}$$

$$\dfrac{\Gamma \longrightarrow {}^{M} B}{\Gamma \longrightarrow {}^{-\oplus M} A \oplus B}\;\oplus\text{R}$$

$$\dfrac{{}^{\Gamma}\!\longrightarrow {}^{M} A \oplus B \qquad {}^{x}A, \Sigma \longrightarrow {}^{N} C \qquad {}^{y}B, \Sigma \longrightarrow {}^{O} C}{\Gamma, \Sigma \longrightarrow {}^{\text{case } M \text{ of } x\oplus -.\,N/-\oplus y.\,O} C}\;\oplus\text{L}$$

We have the following reduction rules.

$$\begin{array}{lcl} \text{case } M \oplus - \text{ of } x \oplus -.\,N/- \oplus y.\,O & \Longrightarrow & N[M/x] \\ \text{case } - \oplus M \text{ of } x \oplus -.\,N/- \oplus y.\,O & \Longrightarrow & O[M/y] \end{array}$$

The two proof trees from the introduction are in sequent style, rather than natural deduction style, but roughly speaking we can assign them proof terms as follows.

$$\dfrac{\dfrac{\dfrac{}{{}^{x}A \longrightarrow {}^{x}A}\;\text{Id}}{{}^{z}A\&B \longrightarrow {}^{\text{case } z \text{ of } x\&-.\,x} A}\;\&\text{L}}{{}^{z}A\&B \longrightarrow {}^{(\text{case } z \text{ of } x\&-.\,x)\oplus -} A \oplus C}\;\oplus\text{R}$$

$$\dfrac{\dfrac{\dfrac{}{{}^{x}A \longrightarrow {}^{x}A}\;\text{Id}}{{}^{x}A \longrightarrow {}^{x\oplus -} A \oplus C}\;\oplus\text{R}}{{}^{z}A\&B \longrightarrow {}^{\text{case } z \text{ of } x\&-.\,(x\oplus -)} A \oplus C}\;\&\text{L}$$

These proofs do not differ in any essential way. This is reresented by the following *commuting conversion*.

$$\begin{array}{l} (\text{case } M \text{ of } x\&-.\,N) \oplus - \\ \quad \Longrightarrow \quad \text{case } M \text{ of } x\&-.\,(N \oplus -) \end{array}$$

Note that this commuting conversion is valid in a strict language, but not a lazy one. This is one reason why this paper uses a linear language, since linearity implies strictness.

Two proofs may also end in unrelated left rules, the order of which does not matter.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{{}^{x}A \longrightarrow {}^{x}A}\;\text{Id} \qquad \dfrac{}{{}^{y}B \longrightarrow {}^{y}B}\;\text{Id}}{{}^{x}A, {}^{y}B \longrightarrow {}^{x\otimes y} A \otimes B}\;\otimes\text{R}}{{}^{x}A, {}^{v}B\&D \longrightarrow {}^{\text{case } v \text{ of } y\&-.\,x\otimes y} A \otimes B}\;\&\text{L}}{{}^{u}A\&C, {}^{v}B\&D \longrightarrow {}^{\text{case } u \text{ of } x\&-.\,(\text{case } v \text{ of } y\&-.\,x\otimes y)} A \otimes B}\;\&\text{L}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{{}^{x}A \longrightarrow {}^{x}A}\;\text{Id} \qquad \dfrac{}{{}^{y}B \longrightarrow {}^{y}B}\;\text{Id}}{{}^{x}A, {}^{y}B \longrightarrow {}^{x\otimes y} A \otimes B}\;\otimes\text{R}}{{}^{u}A\&C, {}^{y}B \longrightarrow {}^{\text{case } u \text{ of } x\&-.\,x\otimes y} A \otimes B}\;\&\text{L}}{{}^{u}A\&C, {}^{v}B\&D \longrightarrow {}^{\text{case } v \text{ of } y\&-.\,(\text{case } u \text{ of } x\&-.\,x\otimes y)} A \otimes B}\;\&\text{L}$$

This is indicated by a commuting conversion such as the following.

$$\begin{array}{l} \text{case } M \text{ of } x\&-.\,(\text{case } N \text{ of } y\&-.\,O) \\ \quad \Longrightarrow \quad \text{case } N \text{ of } y\&-.\,(\text{case } M \text{ of } x\&-.\,O) \end{array}$$

Again, note that this is valid in a strict language, but not a lazy one.

In what follows, we build the notion of pattern matching and case analysis into our proof terms, in a way that avoids the need for commuting conversions. Above we have used intuitionistic sequents, in which variables appear to the left of the arrow and a single type labeled with a proof term appears to the right. In what follows, we use a more symmetric classical linear logic, in which any number of types can appear to either the left or right of the arrow.

3. CLASSICAL LINEAR LOGIC

This section presents classical linear logic with the full complement of connectives as introduced by Jean-Yves Girard [8]. As noted by Benton, Bierman, Hyland, and de Paiva [2] and Wadler [15, 16], some care is required in the formulation of the rules for exponentials to ensure that substitution works smoothly. This paper uses a formulation in which structural boxes delimit the use of weakening and contraction, based upon the two-zone "Logic of Unity" formulation by Girard [10], and using ideas taken from Pfenning [14] and Wadler [16].

The syntax of formulas and sequences appears on the first three lines of Figure 1. A formula is a literal X, an additive $A\&B$ or $A \oplus B$, a multiplicative $A \otimes B$ or $A \invamp B$, an involution $A^{\perp}$, or an exponential $!A$ or $?A$. A formula sequence contains zero or more formulas, where each formula may be unboxed A or boxed $[A]$; only boxed formulas will be subject to weakening and contraction. We write Γ, Δ for concatenation of sequences and $()$ for the empty sequence. Concatenation of sequences is associative and and has the empty sequence as left and right unit.

A sequent formulation of classical linear logic is shown in Figure 2. Sequents take the form

$$\Gamma \longrightarrow \Delta$$

where Γ and Δ are formula sequences. The axiom rule, the exchange rule, and the rules for additives, multiplicatives, and involution are entirely standard. The rules for exponentials are expressed using boxes to impose the structural constraints, and there are weakening, contraction, and dereliction rules for boxes. Finally, there are three forms of Cut, as used by Girard [10] and Pfenning [14]. The notation $[\Gamma]$ stands for a formula sequence in which all the formulas are boxed.

If desired, the number of rules could be halved by eliminating rules whose names contain $\oplus$, 0, $\parr$, $\bot$, and ?, and replacing these connectives by their definitions as de Morgan duals.

$$\begin{array}{ll} A \oplus B = (A^\perp \& B^\perp)^\perp & 0 = \top^\perp \\ A \parr B = (A^\perp \otimes B^\perp)^\perp & \bot = 1^\perp \\ \multicolumn{2}{c}{?A = (!A^\perp)^\perp} \end{array}$$

4. PROOF TERMS

A sequent formulation of classical linear logic with proof terms is shown in Figure 3. Sequents now take the form

$$^G\Gamma \xrightarrow{S} {}^I\Delta$$

Here Γ and Δ are sequences of (possibly boxed) formulas, as before; G is a matrix of patterns, I is a matrix of values, and S is stack of statements. Patterns are deconstructors, values are constructors, and statements are sequences of zero or more cuts, where each cut matches a constructor with a deconstructor.

For the simples sequents, we label each formula on the left with a pattern p_i, the middle with a statement s, and each formula on the right with a value v_j.

$$^{p_1}A_1, \ldots, {}^{p_m}A_m \xrightarrow{s} {}^{v_1}B_1, \ldots, {}^{v_n}B_n$$

A sequent in this form will be called a *row sequent*. (The formulas may also be boxed, but for simplicity we don't show this.) More generally, we label each formula on the left with a stack of patterns P_i, the middle with a stack of statements S, and each formula on the right with a stack of values V_j.

$$^{P_1}A_1, \ldots, {}^{P_m}A_m \xrightarrow{S} {}^{V_1}B_1, \ldots, {}^{V_n}B_n$$

Each of the stacks P_i, S, and V_j has the same height. Hence, we can regard the left as labeled with a matrix of patterns G, the middle as labeled with a stack S, and the right as labeled with a matrix of values I.

$$^G\Gamma \xrightarrow{S} {}^I\Delta$$

Here G, S, and I all have the same height, and the width of G is the same as the length of Γ, and the width of I is the same as the length of Δ. (The general notation restores the possibility that formulas are boxed, since Γ and Δ may contain both unboxed and boxed formulas.) As we shall see, this matrix formulation corresponds to a case analysis, where each row corresponds to one case.

Let p, q range over patterns, v, w range over values, and s, t range over statements. A pattern is a variable x, an additive pattern $p\&-$ or $-\&q$ (note there are two ways to deconstruct an additive), a multiplicative pattern $p \otimes q$ (note there is just one way to deconstruct a multiplicative), an involutive pattern $v^\perp$ (note this contains a value, not a pattern!), an exponential pattern $!p$ or $?[p.t]$ (note how the latter "boxes up" an enclosed statement) and patterns for contraction $p \mathbin{@} q$, weakening $-$, and dereliction $[p]$. Values are exactly dual. A value is a covariable $\bar{x}$, an additive value $v\&-$ or $-\&w$ (note there are two ways to construct an additive), a multiplicative value $v \otimes w$ (note there is just one way to deconstruct a multiplicative), an involutive pattern $p^\perp$ (note this contains a pattern, not a value!), an exponential pattern $![s.v]$ or $?p$ (note how the former "boxes up" an enclosed statement) and values for contraction $v \mathbin{@} w$, weakening $-$, and dereliction $[v]$. A statement is a sequence of cuts, where cut matches a value against a pattern, $v \bullet p$. There are also two variants of cut, that closely resemble the boxed forms of exponentials, $[s.v] \bullet p$ and $v \bullet [p.t]$. The forms $![s.v], [s.v], ?[t.p], [t.p]$ are binding forms; the free variables and covariables are the free variables and covariables of s or t without the free variables of v or p.

In each row of a sequent, no variable or covariable will appear free more than once, and a variable x appears if and only if its matching covariable $\bar{x}$ appears, hence there are always an equal number of variables and covariables in each row.

Sequence concatenation is written with a comma; it is associative and has () as left and right unit. Stack concatenation is written with a slash; it is associative and has $\emptyset$ as a left and right unit.

We extend operations on patterns, values, and statements to operations on stacks in a natural way. When the two stacks come from the same sequent (and thus must have the same height) then the operation is applied pairwise, while when the two stacks come from different sequents (and thus may have different heights) then the operation is computed as a cartesian product. Thus, in rule ($\otimes$L) we have

$$P \otimes Q = \begin{array}{ll} p_1 \otimes q_1 & / \\ \cdots & / \\ p_n \otimes q_n & \end{array}$$

where P and Q are both stacks of height n, while in rule ($\otimes$R) we have

$$V \otimes W = \begin{array}{ll} v_1 \otimes w_1 & / \\ \cdots & / \\ v_1 \otimes w_n & / \\ \cdots & / \\ v_m \otimes w_1 & / \\ \cdots & / \\ v_m \otimes w_n & \end{array}$$

where V is a stack of height m and W is a stack of height n. The most complex use of this convention is in rule (Cut), in which we have

$$S, V \bullet P, T = \begin{array}{ll} s_1,\ v_1 \bullet p_1,\ t_1 & / \\ \cdots & / \\ s_1,\ v_1 \bullet p_n,\ t_n & / \\ \cdots & / \\ s_m,\ v_m \bullet p_1,\ t_1 & / \\ \cdots & / \\ s_m,\ v_m \bullet p_n,\ t_n & \end{array}$$

where S and V are stacks of height m and P and T are stacks of height n. Because of these cartesian products, the size of the stacks may grow exponentially in the size of the derivation. (We will discuss possible remedies for this in the last section.)

Note that in the additive rules (&R) and ($\oplus$L) the height of the stack in the conclusion is the sum of the heights of the stacks in the hypotheses, while in the multiplicative rules ($\otimes$R) and ($\parr$L) the height of the stack in the conclusion is the product of the heights of the stacks in the hypotheses.

Example derivations without cuts are shown in Figure 4.

Part (a) shows two derivations of the sequent $A\&B \longrightarrow A \oplus C$ from the introduction and Section 2, that differ in an inessential way and yield identically labelled sequents.

Part (b) shows two derivations of the sequent $A\&C, B\&D \longrightarrow A \otimes B$ from Section 2, that differ in an inessential way and yield identically labelled sequents.

Part (c) shows a derivation of the distributive law, $A \otimes (B\&C) \longrightarrow (A \otimes B)\&(A \otimes C)$.

Part (d) shows a derivation of the sequent $A, B \longrightarrow (A\&A)\otimes(B\&B)$. There are two sub-derivations with a stack of height two, and the final derivation involves the cartesian product of these, so has a stack of height four.

Example derivations with cuts are shown in Figure 5.

Part (a) shows derivations of the sequents $A \otimes B \longrightarrow B \otimes A$ and $B \otimes A \longrightarrow A \otimes B$, each of which exchanges the components of a tensor product. Cutting these against each other yields a derivation of the sequent $A \otimes B \longrightarrow A \otimes B$, which exchanges the components twice, and hence simplifies to the identity.

Part (b) shows a similar derivation, where the multiplicative $\otimes$ is replaced by the additive $\&$. This time each sub-derivation involves a stack of height two, and the final result requires a stack of four cuts. Two of these are consistent, $-\&\bar{y} \bullet -\&v$ and $\bar{x}\&- \bullet u\&-$, and two are inconsistent, $-\&\bar{y} \bullet u\&-$ and $\bar{x}\&- \bullet -\&v$. Reduction, as described in the next section, eliminates the inconsistent cuts from the stack.

Part (c) shows a derivation of $!(A^{\perp} \parr A) \longrightarrow\ !(A^{\perp} \parr A)$ that corresponds to the Church numeral two, in that it uses the left-hand side twice in the proof of the right hand side. Part (d) shows the result of cutting this derivation against a copy of itself, which corresponds to two plus two, the Church numeral four.

5. REDUCTIONS

Reduction rules for the calculus are shown in Figure 6. Each statement in a stack of statements is reduced separately. Because a reduction may replace a single cut by more or fewer cuts, each statement in a stack of statements may contain a different number of cuts.

The rule

$$(v \otimes w) \bullet (p \otimes q) \Longrightarrow v \bullet p,\ w \bullet q$$

is straightforward. It replaces one cut by a pair of smaller cuts. An example of the application of this rule is shown in Figure 5, part (a).

The rule

$$(v\&-) \bullet (-\&q) \Longrightarrow \emptyset$$

uses $\emptyset$ to indicate that this cut is inconsistent, and the corresponding row should be eliminated from the stack of sequents. Note that a corresponding consistent cut, such as

$$(v\&-) \bullet (p\&-) \Longrightarrow v \bullet p$$

will occur elsewhere in the same stack. An example of the application of this rule is shown in Figure 5, part (b).

The rule

$$[s.v] \bullet p \mathbin{@} q \Longrightarrow X \bullet \bar{X}'@\bar{X}'',\ [s'.v'] \bullet p,\ [s''.v''] \bullet q$$

performs a contraction. We write X to stand for all the free variables and covariables in $[s.v]$, that is, all of the free variables and covariables in s without those in v. If X is $x_1, \ldots, x_m, \bar{y}, \ldots, \bar{y}_n$, then $X \bullet \bar{X}'@\bar{X}''$ stands for $x_1 \bullet \bar{x}'_1@\bar{x}'_1, \ldots, x_m \bullet \bar{x}'_m@\bar{x}'_m,\ \bar{y}_1 \bullet y'_1@y''_1, \ldots, \bar{y}_n \bullet y'_n@y''_n$. Primes indicate a renaming all free variables and covariables in a pattern, value, or statement. An example of the application of this rule is shown in Figure 5, part (d).

Similarly, the rule

$$[s.v] \bullet - \Longrightarrow X \bullet -$$

performs weakening. As before, we write X to stand for all the free variables and covariables in $[s.v]$. If X is $x_1, \ldots, x_m, \bar{y}, \ldots, \bar{y}_n$, then $X\bullet-$ stands for $x_1\bullet-, \ldots, x_m\bullet -,\ \bar{y}_1 \bullet -, \ldots, \bar{y}_n \bullet -$.

Finally, the rule

$$[s.v] \bullet [p] \Longrightarrow s,\ v \bullet p$$

performs dereliction. It removes the box that had surrounded the statement s.

The rule

$$^{g}\Gamma \xrightarrow{s,\, \bar{x}\bullet p,\, t} {}^{i}\Delta \Longrightarrow {}^{g[p/x]}\Gamma \xrightarrow{s[p/x],t[p/x]} {}^{i[p/x]}\Delta$$

substitutes a pattern for a variable. Here $g = p_1, \ldots, p_m$ is a row of patterns and $i = v_1, \ldots, v_n$ is a row of values. We write $g[p/x], s[p/x], t[p/x], i[p/x]$ to indicate substition of the pattern p for the free occurrence of the variable x in g, s, t, i; note that because of linearity, the variable will appear in exactly one of these. Figure 5, parts (a), (b), and (d) all demonstrate the application of this rule.

Reducing proof terms in this way corresponds precisely to cut elimination.

PROPOSITION 1. *Let π be a proof annotated with proof terms ending in the sequent*

$$^{G}\Gamma \xrightarrow{S} {}^{I}\Delta.$$

Then there is a reduction of the corresponding proof term to normal form

$$^{G}\Gamma \xrightarrow{S} {}^{I}\Delta \Longrightarrow {}^{H}\Gamma \xrightarrow{()} {}^{J}\Delta$$

if and only if eliminating all cuts in pi yields a proof annotated with proofterms ending in the sequent

$$^{H}\Gamma \xrightarrow{()} {}^{J}\Delta.$$

6. PROOF NETS

This way of labelling sequents corresponds closely to the form of proof nets described by Hughes and van Glabbeek [13].

Hughes and van Glabbeek represent a proof net by a set of *linkings*, consisting of *axiom links* on *additive resolutions*. Their axiom links correspond to our pairing of a variable and a covariable, x and $\bar{x}$. Their additive resolutions choose one argument of each additive, and correspond to our patterns $p\&-$ and $-\&q$ and our values $v\&-$ and $-\&w$. Each of their linkings corresponds to a row in one of our derivations.

Hughes and van Glabbeek restrict their consideration, as is usual in proof nets, to one-sided sequents where involution is applied only to literals. More significantly, they do not handle exponentials or units. In contrast, our formulation handles all of classical linear logic.

For the subset of linear logic treated by Hughes and van Glabbeek, our formulation corresponds precisely to theirs.

PROPOSITION 2. *Let π and π' be two proofs in the one-sided subset of classical linear logic considered by Hughes and van Glabbeek, ending in the two annotated sequents*

$$\xrightarrow{S} {}^{I}\Delta \qquad \text{and} \qquad \xrightarrow{T} {}^{J}\Delta$$

Then π and π' are assigned the same proof net if and only if the annotations are identical up to permutation of rows and columns.

7. ALTERNATIVE FORMULATION

A drawback of the formulation proposed here is that the height of the stack can grow exponentially in the size of the derivation. This is not a problem for multiplicatives or exponentials, but is a problem for any proof containing additives. This section introduces an alternative formulation which avoids that problem; but it introduces another, in that although there is an initial correspondence between proof trees and proof terms, it is not clear how to preserve this correspondence as reduction proceeds.

The alternative formulation uses patterns and values as before, and a slightly revised definition of statements which includes not only sequence concatenation and the empty sequence as algebraic operations on statements, but also stack concatentation and the empty stack.

$$s, t \quad ::= \quad v \bullet p \mid [s.v] \bullet p \mid v \bullet [p.t] \mid s,\ t \mid () \mid s/t \mid \emptyset$$

When we write s/t, both s and t must contain the same free variables.

Figure 7 shows the new assignment of proof terms to sequents. Sequents now have the form

$$^{x_1}A_1, \ldots, {}^{x_m}A_m \xrightarrow{s} {}^{\bar{y}_1}B_1, \ldots, {}^{\bar{y}_n}B_n$$

where each fomula to the left is labeled with a variable (rather than a pattern) and each formula to the right is labeled with a covariable (rather than a value). As before, formulas may also be boxed.

The reduction rules are as in Figure 6, except the rules for substitution are simplified so that one only substitutes a pattern or value into a statement, not into the surrounding sequent.

$$s,\ \bar{x} \bullet p \implies s[p/x] \qquad v \bullet x,\ t \implies t[v/\bar{x}]$$

Here s must contain x and t must contain $\bar{x}$. In addition, we now have the following distributive laws.

$$\begin{array}{lcl} s,\ (t/u) & \implies & (s,\ t)/(s,\ u) \\ s,\ \emptyset & \implies & \emptyset \end{array}$$

We also add equivalences to capture the fact that the concatentation of sequences and stacks are associative with units () and $\emptyset$, and that the order of sequences and stacks is now irrelevant.

$$\begin{array}{lcl@{\qquad}lcl} s,\ (t,\ u) & = & (s,\ t),\ u & s/(t/u) & = & (s/t)/u \\ s,\ () & = & s & s/\emptyset & = & s \\ s,\ t & = & t,\ s & s/t & = & t/s \end{array}$$

The resulting set of rules is similar in some ways to that of the Chemical Abstract Machine [3], but with the addition of a stacking operator to handle additives.

Note that after reduction some cuts may remain, but these will all be of the form $\bar{x} \bullet p$ where x labels a formula on the left of the sequent, or $v \bullet x$ where $\bar{x}$ labels a formula on the right of the sequent.

The problem with this alternative formulation is that although it is clear how initally a proof tree may be assigned a proof term, and it is clear how to reduce the proof terms, it is not clear how to maintain the correspondence between proof trees and proof terms as reduction proceeds. It is hoped it may be possible to remove this difficulty in the near future.

Acknowledgments. Thanks to Rob van Glabbeek, Geoffrey Washburn, Stephanie Weirich, and Steve Zdancewic for discussions related to this work.

8. REFERENCES

[1] S. Abramsky, Computational interpretations of linear logic. *Theoretical Computer Science*, 111:3–57, 1993.

[2] P. N. Benton, G. M. Bierman, J. M. E. Hyland, and V. C. V. de Paiva, A term calculus for intuitionistic linear logic. In M. Bezem and J. F. Groote, editors, *Typed Lambda Calculi and Applications*, (Springer-Verlag LNCS 664), 1993.

[3] Gerard Berry and Gerard Boudol, The chemical abstract machine. *17th ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages*, San Francisco, 1989, pages 81–94.

[4] Alonzo Church, A set of postulates for the foundation of logic. *Annals of Mathematics*, II.33:346–366, 1932.

[5] Alonzo Church, A formulation of the simple theory of types. *Journal of Symbolic Logic*, 5:56–68, 1940.

[6] P.-L. Curien and H. Herbelin, The duality of computation. In *5'th International Conference on Functional Programming*, pages 233–243, ACM, September 2000.

[7] Gerhard Gentzen, Investigations into Logical Deduction. *Mathematische Zeitschrift* 39:176–210,405–431, 1935. Reprinted in M. E. Szabo, editor, *The Collected Papers of Gerhard Gentzen*, North-Holland, 1969.

[8] Jean-Yves Girard, Linear logic. *Theoretical Computer Science*, 50:1–102, 1987.

[9] Jean-Yves Girard, Towards a geometry of interaction. In J. W. Gray and Andre Scedrov, editors, *Categories in Computer Science and Logic*, volume 92 of *Contemporary Mathematics*, pages 69–108, American Mathematical Society, 1989.

[10] Jean-Yves Girard. On the unity of logic. *Annals of Pure and Applied Logic*, 59:201–217, 1993.

[11] Jean-Yves Girard, Yves Lafont, and Paul Taylor, *Proofs and types.* Cambridge University Press, 1989.

[12] W. A. Howard, The formulae-as-types notion of contstruction. In J. P. Seldin and J. R. Hindley, editors, *To H. B. Curry: Essays on Combinatory Logic, Lambda Calculus, and Formalism*, Academic Press, 1980.

[13] Dominic Hughes and Robert van Glabbeek, Proof Nets for Unit-free Multiplicative-Additive Linear Logic. *18th Annual IEEE Symposium on Logic in Computer Science* (LICS 2003), Ottawa, Canada, June 2003.

[14] Frank Pfenning, Structural cut elimination in linear logic. Technical Report CMU-CS-94-222, Department of Computer Science, Carnegie Mellon University, December 1994.

[15] Philip Wadler, There's no substitute for linear logic.

Literal	X, Y, Z		
Formula	A, B, C	$::=$	$X \mid A \& B \mid \top \mid A \oplus B \mid 0 \mid A \otimes B \mid 1 \mid A \parr B \mid \bot \mid A^{\perp} \mid !A \mid ?A$
Formula sequence	$\Gamma, \Delta, \Sigma, \Theta$	$::=$	$A \mid [A] \mid \Gamma, \Sigma \mid ()$
Variable	x, y, z		
Covariable	$\bar{x}, \bar{y}, \bar{z}$		
Pattern	p, q	$::=$	$x \mid p \& - \mid - \& q \mid p \oplus - \mid - \oplus q \mid p \otimes q \mid 1 \mid p \parr q \mid \bot \mid v^{\perp} \mid !p \mid ?[p.t] \mid p \mathbin{@} q \mid - \mid [p]$
Value	v, w	$::=$	$\bar{x} \mid v \& - \mid - \& w \mid v \oplus - \mid - \& w \mid v \otimes w \mid 1 \mid v \parr w \mid \bot \mid p^{\perp} \mid ![s.v] \mid ?v \mid v \mathbin{@} w \mid - \mid [v]$
Statement	s, t	$::=$	$v \bullet p \mid [s.v] \bullet p \mid v \bullet [p.t] \mid s,\ t \mid ()$
Pattern stack	P, Q	$::=$	$p \mid P/Q \mid \emptyset$
Value stack	V, W	$::=$	$v \mid V/W \mid \emptyset$
Statement stack	S, T	$::=$	$s \mid S/T \mid \emptyset$
Pattern matrix	G, H	$::=$	$P \mid G, H \mid ()$
Value matrix	I, J	$::=$	$V \mid I, J \mid ()$

Figure 1: Syntax

Workshop on Mathematical Foundations of Programming Semantics, Oxford, April 1992.

[16] Philip Wadler, A taste of linear logic. *Mathematical Foundations of Computer Science* (Springer-Verlag LNCS 711), Gdansk, August-September 1993.

[17] Philip Wadler, Call-by-value is dual to call-by-name. *8'th ACM International Conference on Functional Programming*, Uppsala, September 2003.

$$\frac{}{A \longrightarrow A}\,\mathrm{Id} \qquad \frac{\Gamma, \Sigma \longrightarrow \Delta, \Theta}{\Sigma, \Gamma \longrightarrow \Theta, \Delta}\,\mathrm{Exch}$$

$$\frac{\Gamma \longrightarrow \Delta, A \qquad \Gamma \longrightarrow \Delta, B}{\Gamma \longrightarrow \Delta, A\&B}\,\&\mathrm{R} \qquad \frac{A, \Gamma \longrightarrow \Delta}{A\&B, \Gamma \longrightarrow \Delta} \qquad \frac{B, \Gamma \longrightarrow \Delta}{A\&B, \Gamma \longrightarrow \Delta}\,\&\mathrm{L}$$

$$\frac{}{\Gamma \longrightarrow \Delta, \top}\,\top\mathrm{R} \qquad \text{No } \top\mathrm{L} \text{ rule}$$

$$\frac{\Gamma \longrightarrow \Delta, A}{\Gamma \longrightarrow \Delta, A \oplus B} \qquad \frac{\Gamma \longrightarrow \Delta, B}{\Gamma \longrightarrow \Delta, A \oplus B}\,\oplus\mathrm{R} \qquad \frac{A, \Gamma \longrightarrow \Delta \qquad B, \Gamma \longrightarrow \Delta}{A \oplus B, \Gamma \longrightarrow \Delta}\,\oplus\mathrm{L}$$

$$\text{No } 0\mathrm{R} \text{ rule} \qquad \frac{}{0, \Gamma \longrightarrow \Delta}\,0\mathrm{L}$$

$$\frac{\Gamma \longrightarrow \Delta, A \qquad \Sigma \longrightarrow \Theta, B}{\Gamma, \Sigma \longrightarrow \Delta, \Theta, A \otimes B}\,\otimes\mathrm{R} \qquad \frac{A, B, \Gamma \longrightarrow \Delta}{A \otimes B, \Gamma \longrightarrow \Delta}\,\otimes\mathrm{L}$$

$$\frac{}{\longrightarrow 1}\,1\mathrm{R} \qquad \frac{\Gamma \longrightarrow \Delta}{1, \Gamma \longrightarrow \Delta}\,1\mathrm{L}$$

$$\frac{\Gamma \longrightarrow \Delta, A, B}{\Gamma \longrightarrow \Delta, A \parr B}\,\parr\mathrm{R} \qquad \frac{A, \Gamma \longrightarrow \Delta \qquad B, \Sigma \longrightarrow \Theta}{A \parr B, \Gamma, \Sigma \longrightarrow \Delta, \Theta}\,\parr\mathrm{L}$$

$$\frac{\Gamma \longrightarrow \Delta}{\Gamma \longrightarrow \Delta, \bot}\,\bot\mathrm{R} \qquad \frac{}{\bot \longrightarrow}\,\bot\mathrm{L}$$

$$\frac{A, \Gamma \longrightarrow \Delta}{\Gamma \longrightarrow \Delta, A^{\bot}}\,{}^{\bot}\mathrm{R} \qquad \frac{\Gamma \longrightarrow \Delta, A}{A^{\bot}, \Gamma \longrightarrow \Delta}\,{}^{\bot}\mathrm{L}$$

$$\frac{[\Gamma] \longrightarrow [\Delta], A}{[\Gamma] \longrightarrow [\Delta], !A}\,!\mathrm{R} \qquad \frac{[A], \Gamma \longrightarrow \Delta}{!A, \Gamma \longrightarrow \Delta}\,!\mathrm{L}$$

$$\frac{\Gamma \longrightarrow \Delta, [A]}{\Gamma \longrightarrow \Delta, ?A}\,?\mathrm{R} \qquad \frac{A, [\Gamma] \longrightarrow [\Delta]}{?A, [\Gamma] \longrightarrow [\Delta]}\,?\mathrm{L}$$

$$\frac{[A], [A], \Gamma \longrightarrow \Delta}{[A], \Gamma \longrightarrow \Delta}\,!\mathrm{C} \qquad \frac{\Gamma \longrightarrow \Delta}{[A], \Gamma \longrightarrow \Delta}\,!\mathrm{W} \qquad \frac{A, \Gamma \longrightarrow \Delta}{[A], \Gamma \longrightarrow \Delta}\,!\mathrm{D}$$

$$\frac{\Gamma \longrightarrow \Delta, [A], [A]}{\Gamma \longrightarrow \Delta, [A]}\,?\mathrm{C} \qquad \frac{\Gamma \longrightarrow \Delta}{\Gamma \longrightarrow \Delta, [A]}\,?\mathrm{W} \qquad \frac{\Gamma \longrightarrow \Delta, A}{\Gamma \longrightarrow \Delta, [A]}\,?\mathrm{D}$$

$$\frac{\Gamma \longrightarrow \Delta, A \qquad A, \Sigma \longrightarrow \Theta}{\Gamma, \Sigma \longrightarrow \Delta, \Theta}\,\mathrm{Cut}$$

$$\frac{[\Gamma] \longrightarrow [\Delta], A \qquad [A], \Sigma \longrightarrow \Theta}{[\Gamma], \Sigma \longrightarrow [\Delta], \Theta}\,!\mathrm{Cut} \qquad \frac{\Gamma \longrightarrow \Delta, [A] \qquad A, [\Sigma] \longrightarrow [\Theta]}{\Gamma, [\Sigma] \longrightarrow \Delta, [\Theta]}\,?\mathrm{Cut}$$

Figure 2: Classical linear logic

$$\dfrac{}{{}^{x}A \xrightarrow{()} {}^{\bar{x}}A}\ \text{Id} \qquad \dfrac{{}^{G}\Gamma, {}^{H}\Sigma \xrightarrow{S} {}^{I}\Delta, {}^{J}\Theta}{{}^{H}\Sigma, {}^{G}\Gamma \xrightarrow{S} {}^{J}\Theta, {}^{I}\Delta} \qquad \dfrac{{}^{G/H}\Gamma \xrightarrow{S/T} {}^{I/J}\Delta}{{}^{H/G}\Gamma \xrightarrow{T/S} {}^{J/I}\Delta}\ \text{Exch}$$

$$\dfrac{{}^{G}\Gamma \xrightarrow{S} {}^{I}\Delta, {}^{V}A \qquad {}^{H}\Gamma \xrightarrow{T} {}^{J}\Delta, {}^{W}B}{{}^{G/H}\Gamma \xrightarrow{S/T} {}^{H/I}\Delta, {}^{V\&-/-\&W}A\&B}\ \&\text{R} \qquad \dfrac{{}^{P}A, {}^{G}\Gamma \xrightarrow{S} {}^{I}\Delta}{{}^{P\&-}A\&B, {}^{G}\Gamma \xrightarrow{S} {}^{I}\Delta} \qquad \dfrac{{}^{Q}B, {}^{G}\Gamma \xrightarrow{S} {}^{I}\Delta}{{}^{Q\&-}A\&B, {}^{G}\Gamma \xrightarrow{S} {}^{I}\Delta}\ \&\text{L}$$

$$\dfrac{}{{}^{\emptyset}\Gamma \xrightarrow{\emptyset} {}^{\emptyset}\Delta, {}^{\emptyset}\top}\ \top\text{R} \qquad \text{No } \top\text{L rule}$$

$$\dfrac{{}^{G}\Gamma \xrightarrow{S} {}^{I}\Delta, {}^{V}A}{{}^{G}\Gamma \xrightarrow{S} {}^{I}\Delta, {}^{V\oplus-}A\oplus B} \qquad \dfrac{{}^{G}\Gamma \xrightarrow{S} {}^{I}\Delta, {}^{W}B}{{}^{G}\Gamma \xrightarrow{S} {}^{I}\Delta, {}^{W\oplus-}A\oplus B}\ \oplus\text{L} \qquad \dfrac{{}^{P}A, {}^{G}\Gamma \xrightarrow{S} {}^{I}\Delta \qquad {}^{Q}B, {}^{H}\Gamma \xrightarrow{T} {}^{J}\Delta}{{}^{P\oplus-/-\&Q}A\&B, {}^{G/H}\Gamma \xrightarrow{S/T} {}^{H/I}\Delta}\ \oplus\text{R}$$

$$\text{No 0R rule} \qquad \dfrac{}{{}^{\emptyset}0, {}^{\emptyset}\Gamma \xrightarrow{\emptyset} {}^{\emptyset}\Delta}\ 0\text{L}$$

$$\dfrac{{}^{G}\Gamma \xrightarrow{S} {}^{I}\Delta, {}^{V}A \qquad {}^{H}\Sigma \xrightarrow{T} {}^{J}\Theta, {}^{W}B}{{}^{G,H}\Gamma, \Sigma \xrightarrow{S,T} {}^{I,J}\Delta, \Theta, {}^{V\otimes W}A\otimes B}\ \otimes\text{R} \qquad \dfrac{{}^{P}A, {}^{Q}B, {}^{G}\Gamma \xrightarrow{S} {}^{I}\Delta}{{}^{P\otimes Q}A\otimes B, {}^{G}\Gamma \xrightarrow{S} {}^{I}\Delta}\ \otimes\text{L}$$

$$\dfrac{}{\xrightarrow{()} {}^{1}1}\ 1\text{R} \qquad \dfrac{{}^{G}\Gamma \xrightarrow{S} {}^{I}\Delta}{{}^{1}1, {}^{G}\Gamma \xrightarrow{S} {}^{I}\Delta}\ 1\text{L}$$

$$\dfrac{{}^{G}\Gamma \xrightarrow{S} {}^{I}\Delta, {}^{V}A, {}^{W}B}{{}^{G}\Gamma \xrightarrow{S} {}^{I}\Delta, {}^{V\parr W}A\parr B}\ \parr\text{R} \qquad \dfrac{{}^{P}A, {}^{G}\Gamma \xrightarrow{S} {}^{I}\Delta \qquad {}^{Q}B, {}^{H}\Sigma \xrightarrow{T} {}^{J}\Theta}{{}^{P\parr Q}A\parr B, {}^{G,H}\Gamma, \Sigma \xrightarrow{S,T} {}^{I,J}\Delta, \Theta}\ \parr\text{L}$$

$$\dfrac{{}^{G}\Gamma \xrightarrow{S} {}^{I}\Delta}{{}^{G}\Gamma \xrightarrow{S} {}^{I}\Delta, {}^{\bot}\bot}\ \bot\text{R} \qquad \dfrac{}{{}^{\bot}\bot \xrightarrow{()}}\ \bot\text{L}$$

$$\dfrac{{}^{P}A, {}^{G}\Gamma \xrightarrow{S} {}^{I}\Delta}{{}^{G}\Gamma \xrightarrow{S} {}^{I}\Delta, {}^{P^{\bot}}A^{\bot}}\ {}^{\bot}\text{R} \qquad \dfrac{{}^{G}\Gamma \xrightarrow{S} {}^{I}\Delta, {}^{V}A}{{}^{V^{\bot}}A^{\bot}, {}^{G}\Gamma \xrightarrow{S} {}^{I}\Delta}\ {}^{\bot}\text{L}$$

$$\dfrac{{}^{G}[\Gamma] \xrightarrow{S} {}^{I}[\Delta], {}^{V}A}{{}^{G}[\Gamma] \xrightarrow{()} {}^{I}[\Delta], {}^{![S.V]}!A}\ !\text{R} \qquad \dfrac{{}^{P}[A], {}^{G}\Gamma \xrightarrow{S} {}^{I}\Delta}{{}^{!P}!A, {}^{G}\Gamma \xrightarrow{S} {}^{I}\Delta}\ !\text{L}$$

$$\dfrac{{}^{G}\Gamma \xrightarrow{S} {}^{I}\Delta, {}^{V}[A]}{{}^{G}\Gamma \xrightarrow{S} {}^{I}\Delta, {}^{?V}?A}\ ?\text{R} \qquad \dfrac{{}^{P}A, {}^{G}[\Gamma] \xrightarrow{T} {}^{I}[\Delta]}{{}^{?[P.T]}?A, {}^{G}[\Gamma] \xrightarrow{()} {}^{I}[\Delta]}\ ?\text{L}$$

$$\dfrac{{}^{P}[A], {}^{Q}[A], {}^{G}\Gamma \xrightarrow{S} {}^{I}\Delta}{{}^{P@Q}[A], {}^{G}\Gamma \xrightarrow{S} {}^{I}\Delta}\ !\text{C} \qquad \dfrac{{}^{G}\Gamma \xrightarrow{S} {}^{I}\Delta}{{}^{-}[A], {}^{G}\Gamma \xrightarrow{S} {}^{I}\Delta}\ !\text{W} \qquad \dfrac{{}^{P}A, {}^{G}\Gamma \xrightarrow{S} {}^{I}\Delta}{{}^{[P]}[A], {}^{G}\Gamma \xrightarrow{S} {}^{I}\Delta}\ !\text{D}$$

$$\dfrac{{}^{G}\Gamma \xrightarrow{S} {}^{I}\Delta, {}^{V}[A], {}^{W}[A]}{{}^{G}\Gamma \xrightarrow{S} {}^{I}\Delta, {}^{V@W}[A]}\ ?\text{C} \qquad \dfrac{{}^{G}\Gamma \xrightarrow{S} {}^{I}\Delta}{{}^{G}\Gamma \xrightarrow{S} {}^{I}\Delta, {}^{-}[A]}\ ?\text{W} \qquad \dfrac{{}^{G}\Gamma \xrightarrow{S} {}^{I}\Delta, {}^{V}A}{{}^{G}\Gamma \xrightarrow{S} {}^{I}\Delta, {}^{[V]}[A]}\ ?\text{D}$$

$$\dfrac{{}^{G}\Gamma \xrightarrow{S} {}^{I}\Delta, {}^{V}A \qquad {}^{P}A, {}^{H}\Sigma \xrightarrow{T} {}^{J}\Theta}{{}^{G,H}\Gamma, \Sigma \xrightarrow{S, V\bullet P, T} {}^{I,J}\Delta, \Theta}\ \text{Cut}$$

$$\dfrac{{}^{G}[\Gamma] \xrightarrow{S} {}^{I}[\Delta], {}^{V}A \qquad {}^{P}[A], {}^{H}\Sigma \xrightarrow{T} {}^{J}\Theta}{{}^{G,H}[\Gamma], \Sigma \xrightarrow{[S.V]\bullet P, T} {}^{I,J}[\Delta], \Theta}\ !\text{Cut} \qquad \dfrac{{}^{G}\Gamma \xrightarrow{S} {}^{I}\Delta, {}^{V}[A] \qquad {}^{P}A, {}^{H}[\Sigma] \xrightarrow{T} {}^{J}[\Theta]}{{}^{G,H}\Gamma, [\Sigma] \xrightarrow{S, V\bullet[P.T]} {}^{I,J}\Delta, [\Theta]}\ ?\text{Cut}$$

Figure 3: Classical linear logic with proof terms

(a)

$$
\dfrac{\dfrac{\dfrac{}{{}^{x}A \longrightarrow {}^{\bar{x}}A}\;\text{Id}}{{}^{x\&-}A\&B \longrightarrow {}^{\bar{x}}A}\;\&\text{L}}{{}^{x\&-}A\&B \longrightarrow {}^{\bar{x}\&-}A \oplus C}\;\oplus\text{R}
\qquad
\dfrac{\dfrac{\dfrac{}{{}^{x}A \longrightarrow {}^{\bar{x}}A}\;\text{Id}}{{}^{x}A \longrightarrow {}^{\bar{x}\&-}A \oplus C}\;\oplus\text{R}}{{}^{x\&-}A\&B \longrightarrow {}^{\bar{x}\&-}A \oplus C}\;\&\text{L}
$$

(b)

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{{}^{x}A \longrightarrow {}^{x}A}\;\text{Id} \qquad \dfrac{}{{}^{y}B \longrightarrow {}^{y}B}\;\text{Id}}{{}^{x}A, {}^{y}B \longrightarrow {}^{x\otimes y}A \otimes B}\;\otimes\text{R}}{{}^{x}A, {}^{y\&-}B\&D \longrightarrow {}^{x\otimes y}A \otimes B}\;\&\text{L}}{{}^{x\&-}A\&C, {}^{y\&-}B\&D \longrightarrow {}^{x\otimes y}A \otimes B}\;\&\text{L}
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{}{{}^{x}A \longrightarrow {}^{x}A}\;\text{Id} \qquad \dfrac{}{{}^{y}B \longrightarrow {}^{y}B}\;\text{Id}}{{}^{x}A, {}^{y}B \longrightarrow {}^{x\otimes y}A \otimes B}\;\otimes\text{R}}{{}^{x\&-}A\&C, {}^{y}B \longrightarrow {}^{x\otimes y}A \otimes B}\;\&\text{L}}{{}^{x\&-}A\&C, {}^{y\&-}B\&D \longrightarrow {}^{x\otimes y}A \otimes B}\;\&\text{L}
$$

(c)

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{{}^{x}A \longrightarrow {}^{\bar{x}}A}\;\text{Id} \qquad \dfrac{\dfrac{}{{}^{y}B \longrightarrow {}^{\bar{y}}B}\;\text{Id}}{{}^{y\&-}B\&C \longrightarrow {}^{\bar{y}}C}\;\&\text{L}}{{}^{x}A, {}^{y\&-}B\&C \longrightarrow {}^{\bar{x}\otimes\bar{y}}A \otimes C}\;\otimes\text{R} \qquad \dfrac{\dfrac{}{{}^{x}A \longrightarrow {}^{\bar{x}}A}\;\text{Id} \qquad \dfrac{\dfrac{}{{}^{z}C \longrightarrow {}^{\bar{z}}C}\;\text{Id}}{{}^{-\&z}B\&C \longrightarrow {}^{\bar{z}}C}\;\&\text{L}}{{}^{x}A, {}^{-\&z}B\&C \longrightarrow {}^{\bar{x}\otimes\bar{z}}A \otimes C}\;\otimes\text{R}}{{}^{x/x}A, {}^{y\&-/-\&z}B\&C \longrightarrow {}^{(\bar{x}\otimes\bar{y})\&-/-\&(\bar{x}\otimes\bar{z})}(A \otimes B)\&(A \otimes C)}\;\&\text{R}}{{}^{x\otimes(y\&-)/x\otimes(-\&z)}A \otimes (B\&C) \longrightarrow {}^{(\bar{x}\otimes\bar{y})\&-/-\&(\bar{x}\otimes\bar{z})}(A \otimes B)\&(A \otimes C)}\;\otimes\text{L}
$$

(d)

$$
\dfrac{\dfrac{\dfrac{}{{}^{x}A \longrightarrow {}^{x}A}\;\text{Id} \qquad \dfrac{}{{}^{x}A \longrightarrow {}^{x}A}\;\text{Id}}{{}^{x/x}A \longrightarrow {}^{x\&-/-\&x}A\&A}\;\&\text{R} \qquad \dfrac{\dfrac{}{{}^{y}B \longrightarrow {}^{y}B}\;\text{Id} \qquad \dfrac{}{{}^{y}B \longrightarrow {}^{y}B}\;\text{Id}}{{}^{y/y}B \longrightarrow {}^{y\&-/-\&y}B\&B}\;\&\text{R}}{{}^{x/x/x/x}A, {}^{y/y/y/y}B \longrightarrow {}^{(x\&-)\otimes(y\&-)/(x\&-)\otimes(-\&y)/(-\&x)\otimes(y\&-)/(-\&x)\otimes(-\&y)}(A\&A) \otimes (B\&B)}\;\otimes\text{R}
$$

Figure 4: Example derivations

(a)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{{}^{y}B \longrightarrow {}^{\bar y}B}\,\text{Id} \qquad \dfrac{}{{}^{x}A \longrightarrow {}^{\bar x}A}\,\text{Id}}{{}^{y}B, {}^{x}A \longrightarrow {}^{\bar y\otimes\bar x}B\otimes A}\,\otimes\text{R}}{{}^{x}A, {}^{y}B \longrightarrow {}^{\bar y\otimes\bar x}B\otimes A}\,\text{Exch}}{{}^{x\otimes y}A\otimes B \longrightarrow {}^{\bar y\otimes\bar x}B\otimes A}\,\otimes\text{L} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{{}^{u}A \longrightarrow {}^{\bar u}A}\,\text{Id} \qquad \dfrac{}{{}^{v}B \longrightarrow {}^{\bar v}B}\,\text{Id}}{{}^{u}A, {}^{v}B \longrightarrow {}^{\bar u\otimes\bar v}A\otimes B}\,\otimes\text{R}}{{}^{v}B, {}^{u}A \longrightarrow {}^{\bar u\otimes\bar v}A\otimes B}\,\text{Exch}}{{}^{v\otimes u}B\otimes A \longrightarrow {}^{\bar u\otimes\bar v}A\otimes B}\,\otimes\text{L}}{{}^{x\otimes y}A\otimes B \xrightarrow{\bar y\otimes\bar x\bullet v\otimes u} {}^{\bar u\otimes\bar v}A\otimes B}\,\text{Cut}$$

$$\Longrightarrow \quad \dfrac{\dfrac{\dfrac{}{{}^{x}A \longrightarrow {}^{\bar x}A}\,\text{Id} \qquad \dfrac{}{{}^{y}B \longrightarrow {}^{\bar y}B}\,\text{Id}}{{}^{x}A, {}^{y}B \longrightarrow {}^{\bar x\otimes\bar y}A\otimes B}\,\otimes\text{R}}{{}^{x\otimes y}A\otimes B \longrightarrow {}^{\bar x\otimes\bar y}A\otimes B}\,\otimes\text{L}$$

(b)

$$\dfrac{\dfrac{\dfrac{\dfrac{}{{}^{y}B \longrightarrow {}^{\bar y}B}\,\text{Id}}{{}^{-\&y}A\&B \longrightarrow {}^{\bar y}B}\,\&\text{L} \qquad \dfrac{\dfrac{}{{}^{x}A \longrightarrow {}^{\bar x}A}\,\text{Id}}{{}^{x\&-}A\&B \longrightarrow {}^{\bar x}A}\,\&\text{L}}{{}^{-\&y/x\&-}A\&B \longrightarrow {}^{\bar y\&-/-\&\bar x}B\&A}\,\&\text{R} \qquad \dfrac{\dfrac{\dfrac{}{{}^{u}A \longrightarrow {}^{\bar u}A}\,\text{Id}}{{}^{-\&u}B\&A \longrightarrow {}^{\bar u}A}\,\&\text{L} \qquad \dfrac{\dfrac{}{{}^{v}B \longrightarrow {}^{\bar v}B}\,\text{Id}}{{}^{v\&-}B\&A \longrightarrow {}^{\bar v}B}\,\&\text{L}}{{}^{-\&u/v\&-}B\&A \longrightarrow {}^{\bar u\&-/-\&\bar v}A\&B}\,\&\text{R}}{{}^{-\&y/x\&-/-\&y/x\&-}A\&B \xrightarrow{\bar y\&-\bullet-\&u/\bar y\&-\bullet v\&-/-\&\bar x\bullet-\&u/-\&\bar x\bullet v\&-} {}^{\bar u\&-/\bar u\&-/-\&\bar v/-\&\bar v}A\&B}\,\text{Cut}$$

$$\Longrightarrow \quad \dfrac{\dfrac{\dfrac{}{{}^{x}A \longrightarrow {}^{\bar x}A}\,\text{Id}}{{}^{x\&-}A\&B \longrightarrow {}^{\bar x}A}\,\&\text{L} \qquad \dfrac{\dfrac{}{{}^{y}B \longrightarrow {}^{\bar y}B}\,\text{Id}}{{}^{-\&y}A\&B \longrightarrow {}^{\bar y}B}\,\&\text{L}}{{}^{x\&-/-\&y}A\&B \longrightarrow {}^{x\&-/-\&y}A\&B}\,\&\text{R}$$

(c)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{{}^{a}A \longrightarrow {}^{a}A}\,\text{Id}}{{}^{\bar a^\perp}A^\perp, {}^{a}A \longrightarrow}\,{}^\perp\text{L} \qquad \dfrac{}{{}^{b}A \longrightarrow {}^{\bar b}A}\,\text{Id}}{{}^{\bar a^\perp\parr b}A^\perp\parr A, {}^{a}A \longrightarrow {}^{\bar b}A}\,\parr\text{L}}{{}^{[\bar a^\perp\parr b]}[A^\perp\parr A], {}^{a}A \longrightarrow {}^{\bar b}A}\,!\text{D} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{{}^{c}A \longrightarrow {}^{c}A}\,\text{Id}}{{}^{\bar c^\perp}A^\perp, {}^{c}A \longrightarrow}\,{}^\perp\text{L} \qquad \dfrac{}{{}^{d}A \longrightarrow {}^{\bar d}A}\,\text{Id}}{{}^{\bar c^\perp\parr d}A^\perp\parr A, {}^{c}A \longrightarrow {}^{\bar d}A}\,\parr\text{L}}{{}^{[\bar c^\perp\parr d]}[A^\perp\parr A], {}^{c}A \longrightarrow {}^{\bar d}A}\,!\text{D}}{{}^{[\bar a^\perp\parr b]}[A^\perp\parr A], {}^{[\bar c^\perp\parr d]}[A^\perp\parr A], {}^{a}A \xrightarrow{\bar b\bullet c} {}^{\bar d}A}\,\text{Cut}}{{}^{[\bar a^\perp\parr b]@[\bar c^\perp\parr d]}[A^\perp\parr A], {}^{a}A \xrightarrow{\bar b\bullet c} {}^{\bar d}A}\,!\text{C}}{{}^{[\bar a^\perp\parr b]@[\bar c^\perp\parr d]}[A^\perp\parr A] \xrightarrow{\bar b\bullet c} {}^{a^\perp}A^\perp, {}^{\bar d}A}\,{}^\perp\text{R}}{{}^{[\bar a^\perp\parr b]@[\bar c^\perp\parr d]}[A^\perp\parr A] \xrightarrow{\bar b\bullet c} {}^{a^\perp\parr\bar d}A^\perp\parr A}\,\parr\text{L}}{{}^{[\bar a^\perp\parr b]@[\bar c^\perp\parr d]}[A^\perp\parr A] \longrightarrow {}^{![\bar b\bullet c.\, a^\perp\parr\bar d]}!(A^\perp\parr A)}\,!\text{R}}{{}^{!([\bar a^\perp\parr b]@[\bar c^\perp\parr d])}!(A^\perp\parr A) \longrightarrow {}^{![\bar b\bullet c.\, a^\perp\parr\bar d]}!(A^\perp\parr A)}\,!\text{L}$$

(d)

$$\dfrac{{}^{!([\bar a^\perp\parr b]@[\bar c^\perp\parr d])}!(A^\perp\parr A) \longrightarrow {}^{![\bar b\bullet c.\, a^\perp\parr\bar d]}!(A^\perp\parr A) \qquad {}^{!([\bar e^\perp\parr f]@[\bar g^\perp\parr h])}!(A^\perp\parr A) \longrightarrow {}^{![\bar f\bullet g.\, e^\perp\parr\bar h]}!(A^\perp\parr A)}{{}^{!([\bar a^\perp\parr b]@[\bar c^\perp\parr d])}!(A^\perp\parr A) \xrightarrow{![\bar b\bullet c.\, a^\perp\parr\bar d]\bullet!([\bar e^\perp\parr f]@[\bar g^\perp\parr h])} {}^{![\bar f\bullet g.\, e^\perp\parr\bar h]}!(A^\perp\parr A)}\,\text{Cut}$$

$$\Longrightarrow \quad {}^{!([\bar{a'}^\perp\parr b']@[\bar{a''}^\perp\parr b'']@[\bar{c'}^\perp\parr d']@[\bar{c''}^\perp\parr d''])}!(A^\perp\parr A) \xrightarrow{\bar{b'}\bullet c', a'^\perp\parr\bar{d'}\bullet\bar e^\perp\parr f, \bar{b''}\bullet c'', a''^\perp\parr\bar{d''}\bullet\bar g^\perp\parr h} {}^{![\bar f\bullet g.\, e^\perp\parr\bar h]}!(A^\perp\parr A)$$

$$\Longrightarrow \quad {}^{!([\bar{a'}^\perp\parr b']@[\bar{a''}^\perp\parr b'']@[\bar{b'}^\perp\parr d']@[\bar{b''}^\perp\parr d''])}!(A^\perp\parr A) \longrightarrow {}^{![\bar{d'}\bullet a''.\, a'^\perp\parr\bar{d''}]}!(A^\perp\parr A)$$

Figure 5: Example derivations with cut

$$
\begin{array}{cccccc}
(v \& -) \bullet (p \& -) & \Longrightarrow & v \bullet p & (v \oplus -) \bullet (p \oplus -) & \Longrightarrow & v \bullet p \\
(v \& -) \bullet (- \& q) & \Longrightarrow & \emptyset & (v \oplus -) \bullet (- \oplus q) & \Longrightarrow & \emptyset \\
(- \& w) \bullet (p \& -) & \Longrightarrow & \emptyset & (- \oplus w) \bullet (p \oplus -) & \Longrightarrow & \emptyset \\
(- \& w) \bullet (- \& q) & \Longrightarrow & w \bullet q & (- \oplus w) \bullet (- \oplus q) & \Longrightarrow & w \bullet q \\
 & & & & & \\
(v \otimes w) \bullet (p \otimes q) & \Longrightarrow & v \bullet p,\ w \bullet q & (v \parr w) \bullet (p \parr q) & \Longrightarrow & v \bullet p,\ w \bullet q \\
1 \bullet 1 & \Longrightarrow & () & \bot \bullet \bot & \Longrightarrow & () \\
 & & & & & \\
p^{\bot} \bullet v^{\bot} & \Longrightarrow & v \bullet p & & & \\
 & & & & & \\
![s.v] \bullet !p & \Longrightarrow & [s.v] \bullet p & ?v \bullet ?[p.t] & \Longrightarrow & v \bullet [p.t] \\
[s.v] \bullet p \mathbin{@} q & \Longrightarrow & X \bullet \bar{X}' @ \bar{X}'',\ [s'.v'] \bullet p,\ [s''.v''] \bullet q & (v \mathbin{@} w) \bullet [p.t] & \Longrightarrow & X \bullet \bar{X}' @ \bar{X}'',\ v \bullet [p'.t'],\ w \bullet [p''.t''] \\
[s.v] \bullet - & \Longrightarrow & X \bullet - & - \bullet [p.t] & \Longrightarrow & X \bullet - \\
[s.v] \bullet [p] & \Longrightarrow & s,\ v \bullet p & [v] \bullet [p.t] & \Longrightarrow & v \bullet p,\ t \\
 & & & & & \\
{}^{g}\Gamma \xrightarrow{s,\, \bar{x} \bullet p,\, t} {}^{i}\Delta & \Longrightarrow & {}^{g[p/x]}\Gamma \xrightarrow{s[p/x],\, t[p/x]} {}^{i[p/x]}\Delta & {}^{g}\Gamma \xrightarrow{s,\, v \bullet x,\, t} {}^{i}\Delta & \Longrightarrow & {}^{g[v/\bar{x}]}\Gamma \xrightarrow{s[v/\bar{x}],\, t[v/\bar{x}]} {}^{i[v/\bar{x}]}\Delta
\end{array}
$$

Figure 6: Reductions

$$\frac{}{{}^{x}A \xrightarrow{()} {}^{\bar{x}}A}\,\text{Id} \qquad \frac{\Gamma, \Sigma \xrightarrow{s} \Delta, \Theta}{\Sigma, \Gamma \xrightarrow{s} \Theta, \Delta}\,\text{Exch}$$

$$\frac{\Gamma \xrightarrow{s} \Delta, {}^{\bar{x}}A \qquad \Gamma \xrightarrow{t} \Delta, {}^{\bar{y}}B}{\Gamma \xrightarrow{s,\, \bar{x}\&-\bullet z/t,\, -\&\bar{y}\bullet z} \Delta, {}^{\bar{z}}A\&B}\,\&\text{R} \qquad \frac{{}^{x}A, \Gamma \xrightarrow{s} \Delta}{{}^{z}A\&B, \Gamma \xrightarrow{\bar{z}\bullet x\&-,\, s} \Delta} \qquad \frac{{}^{y}B, \Gamma \xrightarrow{s} \Delta}{{}^{z}A\&B, \Gamma \xrightarrow{\bar{z}\bullet -\&y,\, s} \Delta}\,\&\text{L}$$

$$\frac{}{\Gamma \xrightarrow{\emptyset} \Delta, {}^{\bar{z}}\top}\,\top\text{R} \qquad \text{No } \top\text{L rule}$$

$$\frac{\Gamma \xrightarrow{s} \Delta, {}^{\bar{x}}A}{\Gamma \xrightarrow{s,\, \bar{x}\&-\bullet z} \Delta, {}^{z}A \oplus B} \qquad \frac{\Gamma \xrightarrow{s} \Delta, {}^{\bar{y}}B}{\Gamma \xrightarrow{s,\, -\&\bar{y}\bullet z} \Delta, {}^{\bar{z}}A \oplus B}\,\oplus\text{L} \qquad \frac{{}^{x}A, \Gamma \xrightarrow{s} \Delta \qquad {}^{y}B, \Gamma \xrightarrow{t} \Delta}{{}^{z}A\&B, \Gamma \xrightarrow{\bar{z}\bullet x\oplus-,\, s/\bar{z}\bullet -\oplus y,\, t} \Delta}\,\oplus\text{R}$$

$$\text{No 0R rule} \qquad \frac{}{{}^{z}0, \Gamma \xrightarrow{\emptyset} \Delta}\,0\text{L}$$

$$\frac{\Gamma \xrightarrow{s} \Delta, {}^{\bar{x}}A \qquad \Sigma \xrightarrow{t} \Theta, {}^{\bar{y}}B}{\Gamma, \Sigma \xrightarrow{s,\, t,\, \bar{x}\otimes\bar{y}\bullet z} \Delta, \Theta, {}^{\bar{z}}A \otimes B}\,\otimes\text{R} \qquad \frac{{}^{x}A, {}^{y}B, \Gamma \xrightarrow{s} \Delta}{{}^{z}A \otimes B, \Gamma \xrightarrow{\bar{z}\bullet x\otimes y,\, s} \Delta}\,\otimes\text{L}$$

$$\frac{}{\xrightarrow{1\bullet z} {}^{\bar{z}}1}\,1\text{R} \qquad \frac{\Gamma \xrightarrow{s} \Delta}{{}^{z}1, \Gamma \xrightarrow{\bar{z}\bullet 1,\, s} \Delta}\,1\text{L}$$

$$\frac{\Gamma \xrightarrow{s} \Delta, {}^{\bar{x}}A, {}^{\bar{y}}B}{\Gamma \xrightarrow{s,\, \bar{x}\parr\bar{y}\bullet z} \Delta, {}^{\bar{z}}A\parr B}\,\parr\text{R} \qquad \frac{{}^{x}A, \Gamma \xrightarrow{s} \Delta \qquad {}^{y}B, \Sigma \xrightarrow{t} \Theta}{{}^{z}A\parr B, \Gamma, \Sigma \xrightarrow{\bar{z}\bullet x\parr y,\, s,\, t} \Delta, \Theta}\,\parr\text{L}$$

$$\frac{\Gamma \xrightarrow{s} \Delta}{\Gamma \xrightarrow{s,\, \bot\bullet z} \Delta, {}^{\bar{z}}\bot}\,\bot\text{R} \qquad \frac{}{{}^{z}\bot \xrightarrow{\bar{z}\bullet\bot}}\,\bot\text{L}$$

$$\frac{{}^{x}A, \Gamma \xrightarrow{s} \Delta}{\Gamma \xrightarrow{s,\, x^{\bot}\bullet z} \Delta, {}^{\bar{z}}A^{\bot}}\,{}^{\bot}\text{R} \qquad \frac{\Gamma \xrightarrow{s} \Delta, {}^{\bar{x}}A}{{}^{z}A^{\bot}, \Gamma \xrightarrow{\bar{z}\bullet\bar{x}^{\bot},\, s} \Delta}\,{}^{\bot}\text{L}$$

$$\frac{[\Gamma] \xrightarrow{s} [\Delta], {}^{\bar{x}}A}{[\Gamma] \xrightarrow{![s.\bar{x}]\bullet z} [\Delta], {}^{\bar{z}}!A}\,!\text{R} \qquad \frac{{}^{x}[A], \Gamma \xrightarrow{s} \Delta}{{}^{z}!A, \Gamma \xrightarrow{\bar{z}\bullet !x,\, s} \Delta}\,!\text{L}$$

$$\frac{\Gamma \xrightarrow{s} \Delta, {}^{\bar{x}}[A]}{\Gamma \xrightarrow{s,\, ?\bar{x}\bullet z} \Delta, {}^{\bar{z}}?A}\,?\text{R} \qquad \frac{{}^{x}A, [\Gamma] \xrightarrow{t} [\Delta]}{{}^{z}?A, [\Gamma] \xrightarrow{\bar{z}\bullet ?[x.t]} [\Delta]}\,?\text{L}$$

$$\frac{{}^{x}[A], {}^{y}[A], \Gamma \xrightarrow{s} \Delta}{{}^{z}[A], \Gamma \xrightarrow{\bar{z}\bullet x@y,\, s} \Delta}\,!\text{C} \qquad \frac{\Gamma \xrightarrow{s} \Delta}{{}^{z}[A], \Gamma \xrightarrow{\bar{z}\bullet -,\, s} \Delta}\,!\text{W} \qquad \frac{{}^{x}A, \Gamma \xrightarrow{s} \Delta}{{}^{z}[A], \Gamma \xrightarrow{\bar{z}\bullet [x],\, s} \Delta}\,!\text{D}$$

$$\frac{\Gamma \xrightarrow{s} \Delta, {}^{\bar{x}}[A], {}^{\bar{y}}[A]}{\Gamma \xrightarrow{s,\, \bar{x}@\bar{y}\bullet z} \Delta, {}^{\bar{z}}[A]}\,?\text{C} \qquad \frac{\Gamma \xrightarrow{s} \Delta}{\Gamma \xrightarrow{s,\, -\bullet z} \Delta, {}^{\bar{z}}[A]}\,?\text{W} \qquad \frac{\Gamma \xrightarrow{s} \Delta, {}^{\bar{x}}A}{\Gamma \xrightarrow{s,\, [\bar{x}]\bullet z} \Delta, {}^{\bar{z}}[A]}\,?\text{D}$$

$$\frac{\Gamma \xrightarrow{s} \Delta, {}^{\bar{x}}A \qquad {}^{x}A, \Sigma \xrightarrow{t} \Theta}{\Gamma, \Sigma \xrightarrow{s,\, \bar{x}\bullet x,\, t} \Delta, \Theta}\,\text{Cut}$$

$$\frac{[\Gamma] \xrightarrow{s} [\Delta], {}^{\bar{x}}A \qquad {}^{x}[A], \Sigma \xrightarrow{T} \Theta}{[\Gamma], \Sigma \xrightarrow{[s.\bar{x}]\bullet x,\, t} [\Delta], \Theta}\,!\text{Cut} \qquad \frac{\Gamma \xrightarrow{s} \Delta, {}^{\bar{x}}[A] \qquad {}^{x}A, [\Sigma] \xrightarrow{t} [\Theta]}{\Gamma, [\Sigma] \xrightarrow{s,\, \bar{x}\bullet[x.t]} \Delta, [\Theta]}\,?\text{Cut}$$

Figure 7: Classical linear logic with proof terms, alternative formulation