As Natural as 0, 1, 2

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Haskell

Hindley-Milner types

Java

Girard-Reynolds types

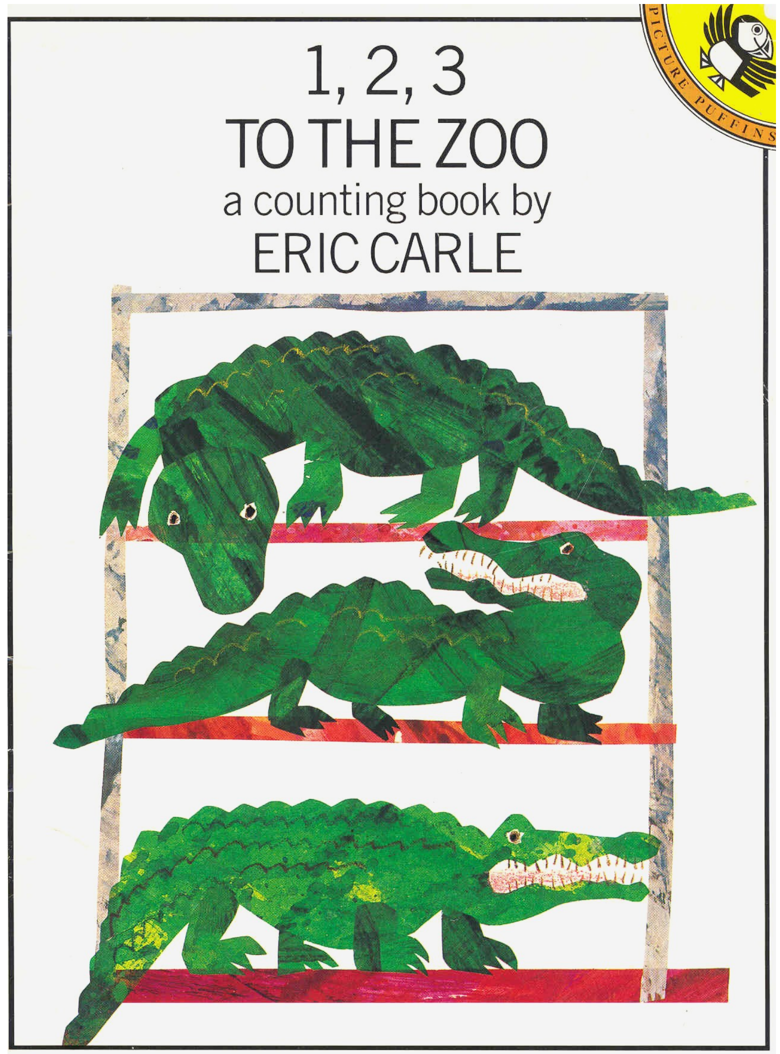
XML

Operational semantics

Part 0

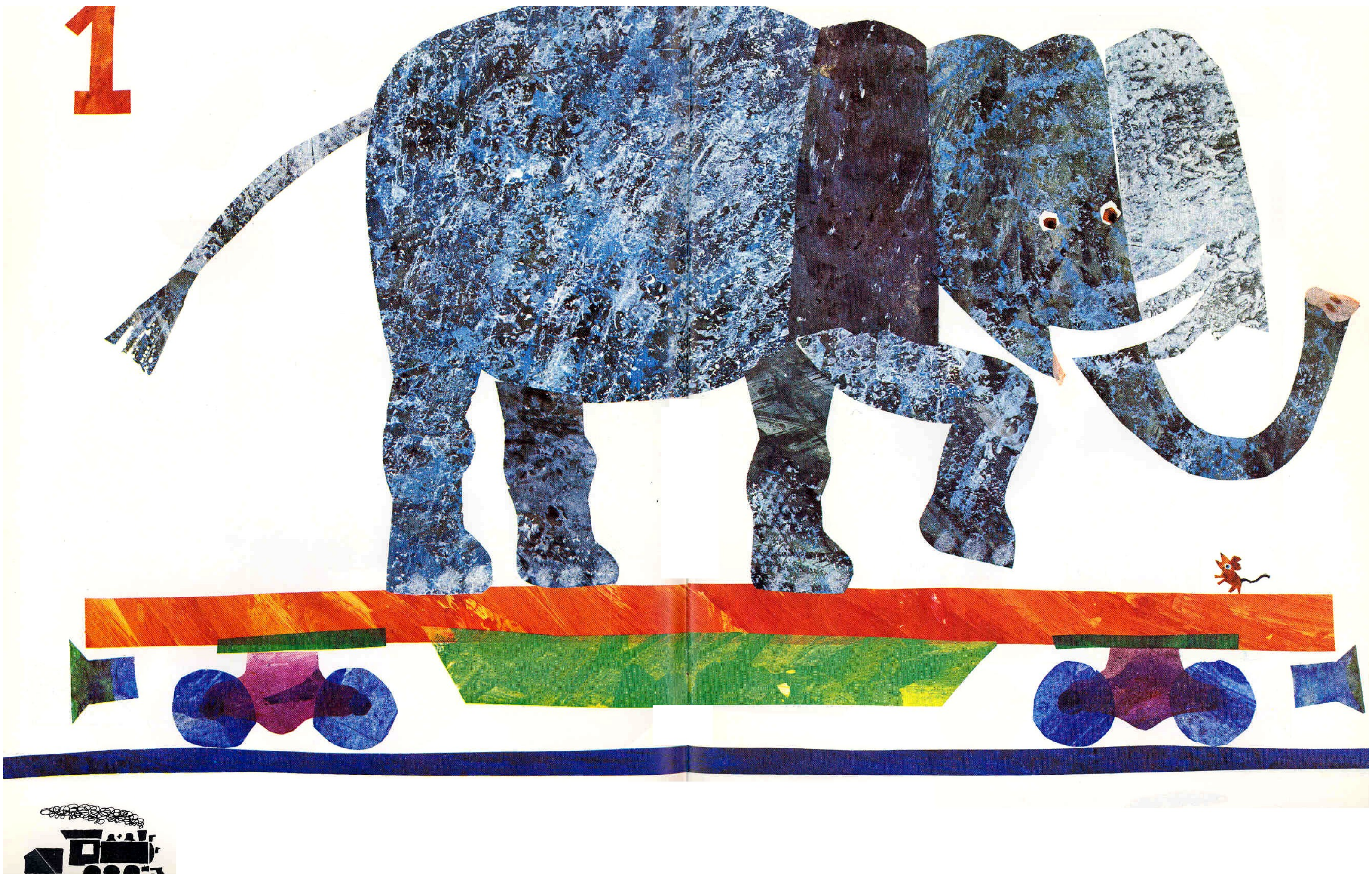
Counting starts at zero

Carle



Carle

Carle



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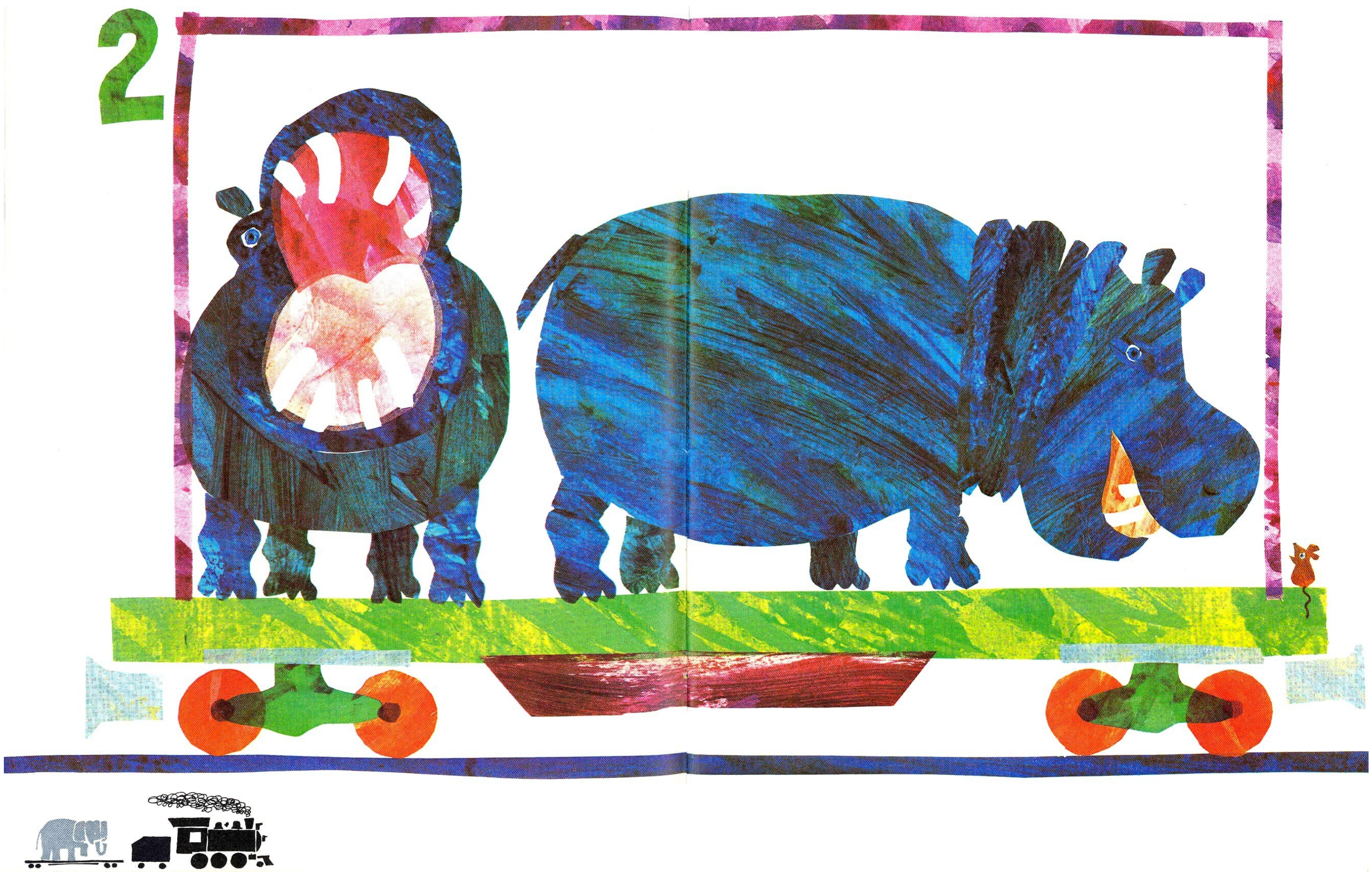

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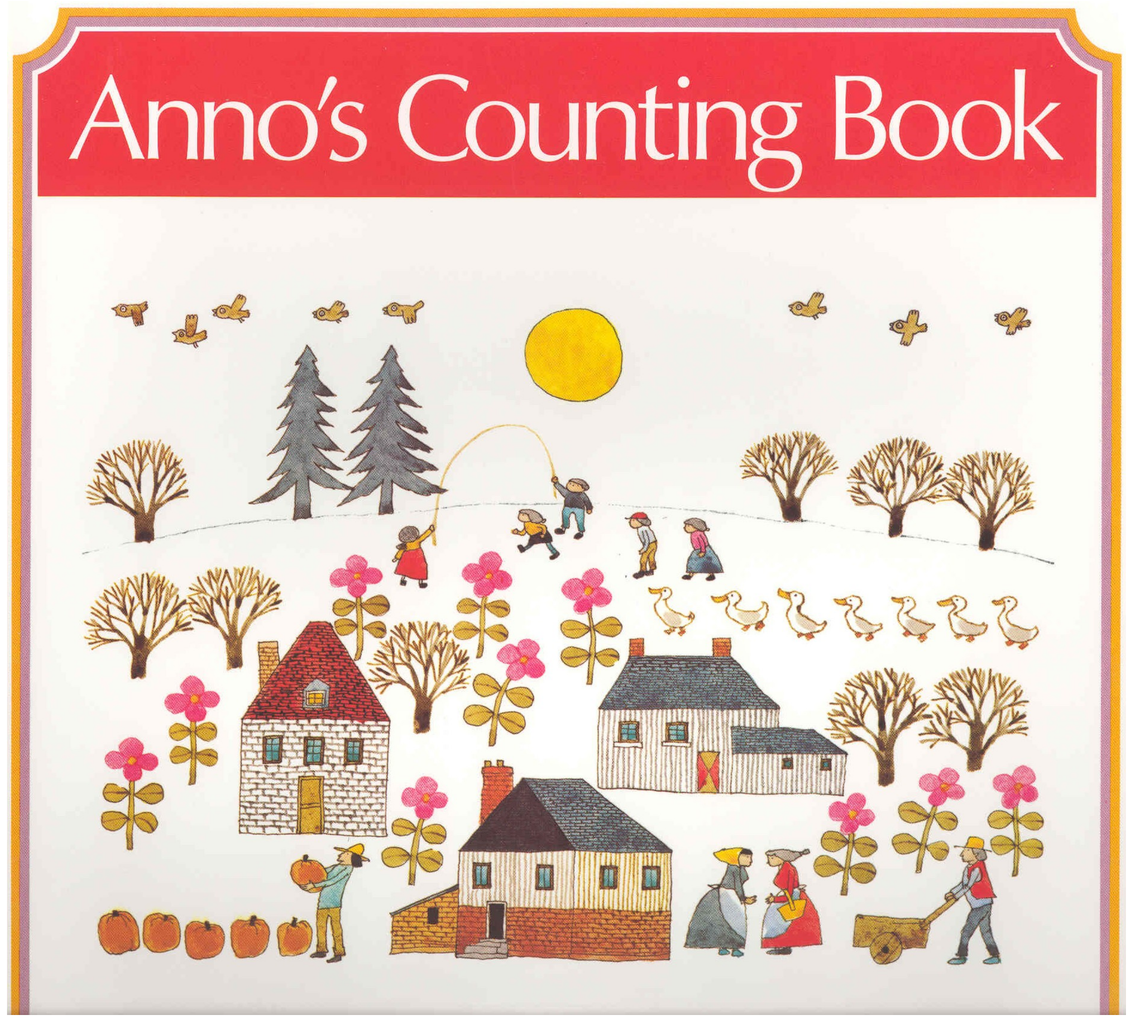

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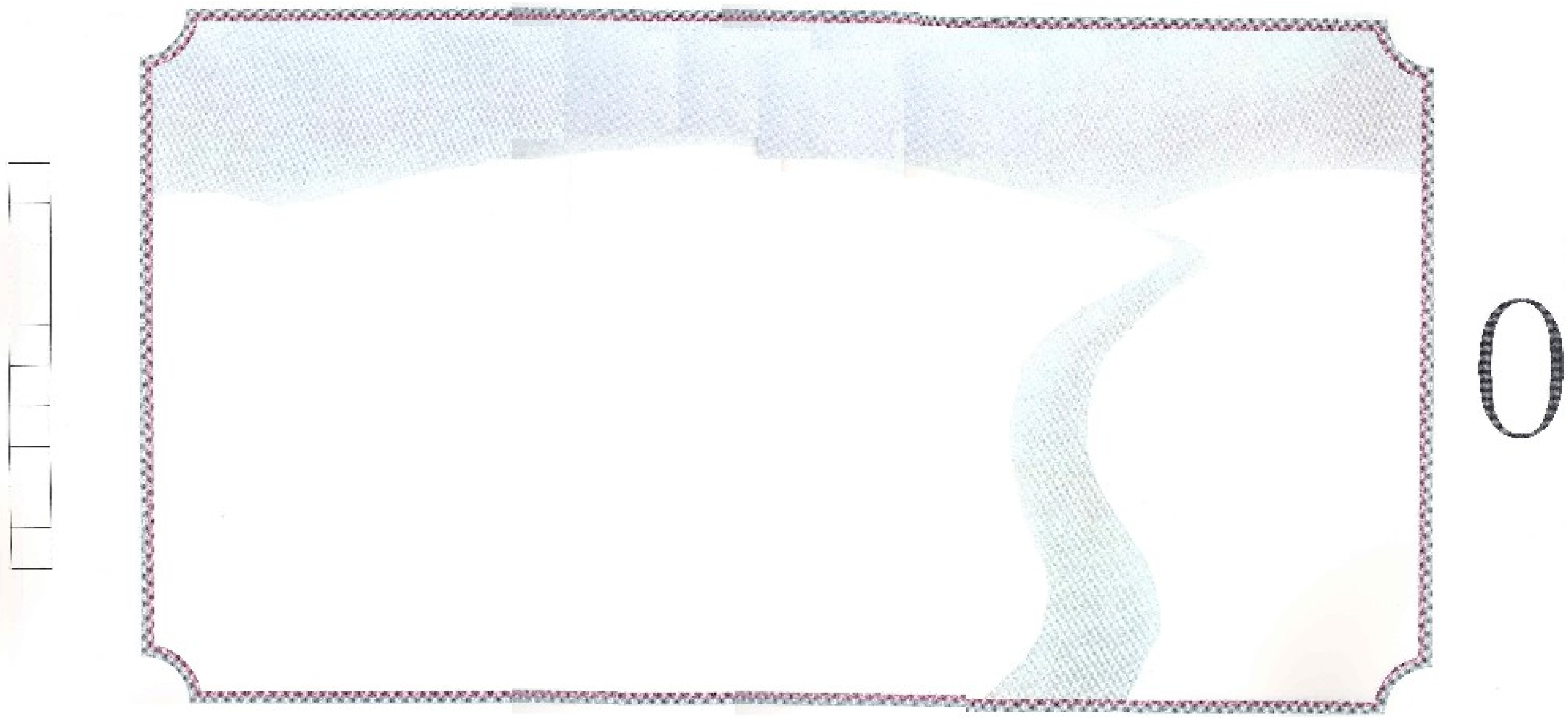

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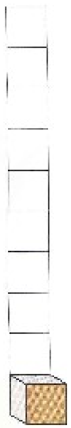
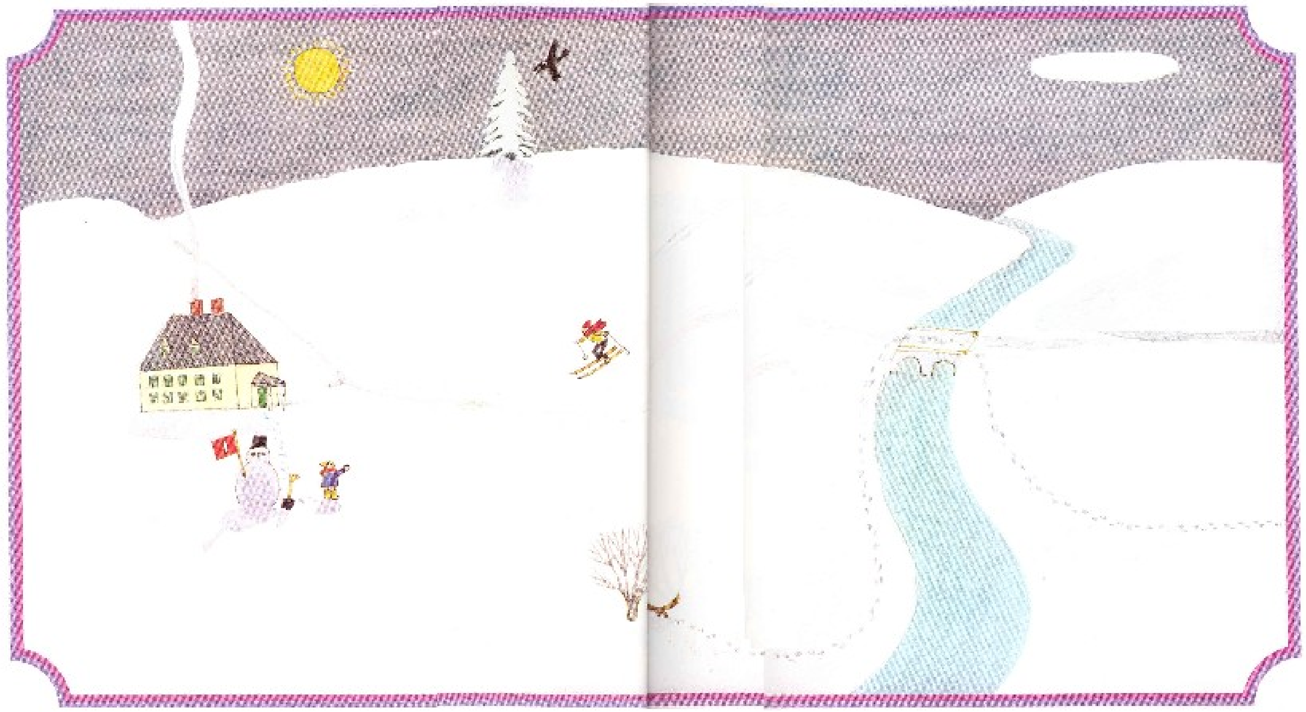
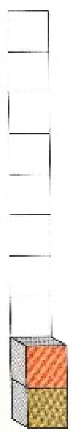




1

Anno



2

Zero appears in India, 7th century CE

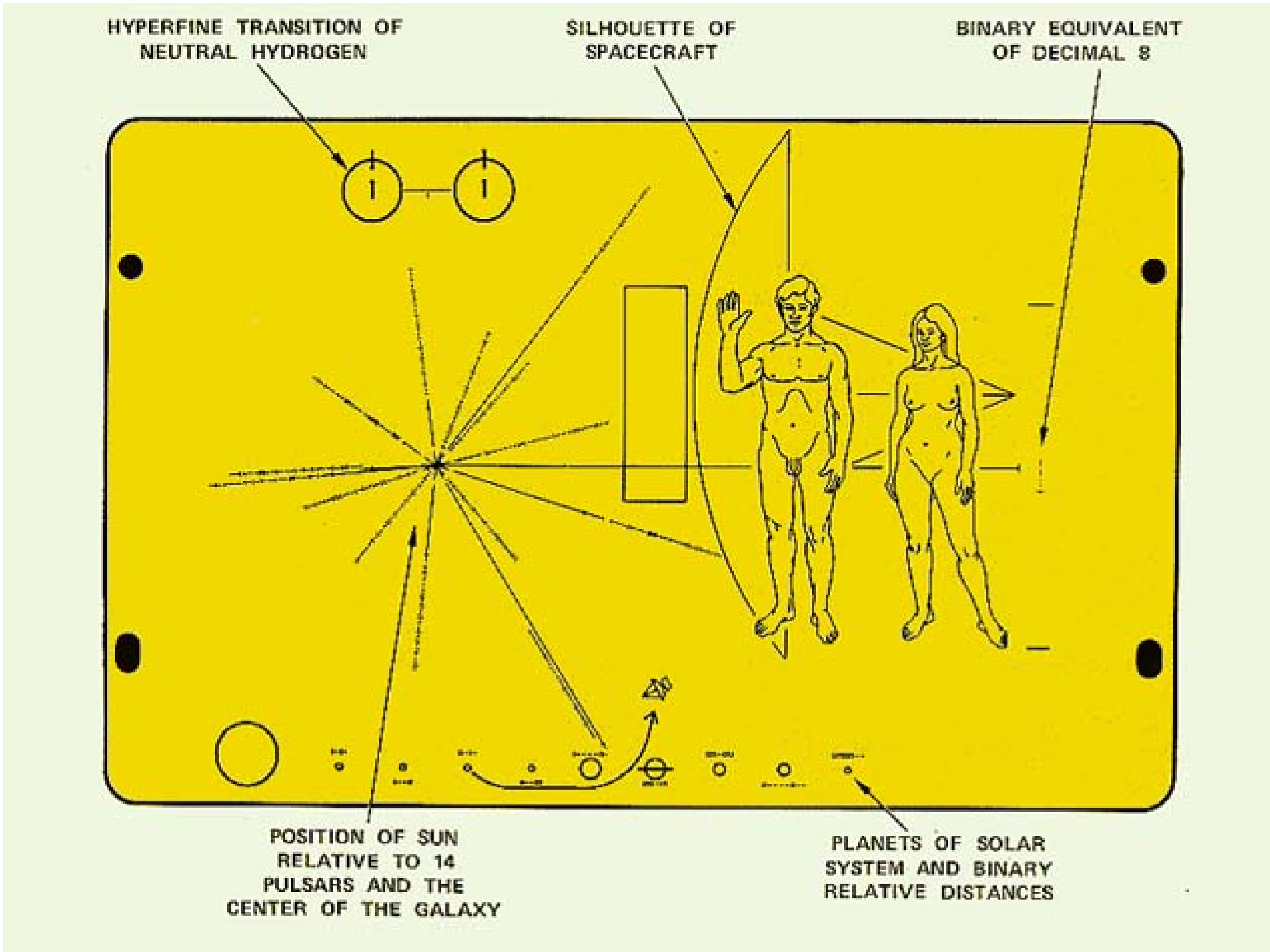
Counting in Japanese

0	leā	零	零
1	īchī	一	一
2	nī	二	二
3	san	三	三
4	shī	四	四
5	gō	五	五

Part 1

Aliens

How to talk to aliens



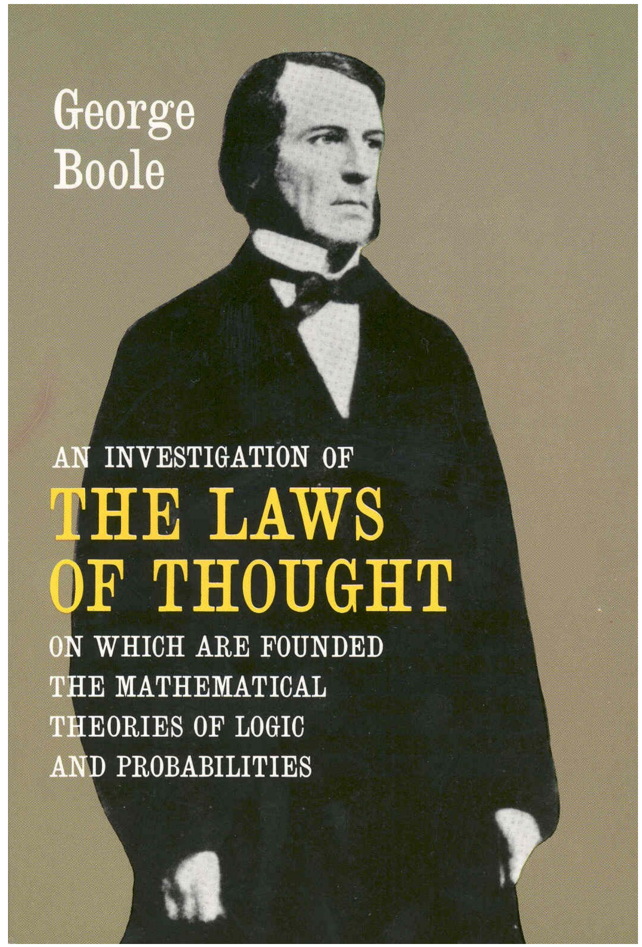
Independence Day

A universal programming language?

Part 2

Boolean algebra

George Boole (1815–1864)



Boole 1847: Mathematical analysis of logic

The primary canonical forms already determined for the expression of Propositions, are

All Xs are Ys,	$x(1-y)=0,$	....A.
No Xs are Ys,	$xy=0,$	....E.
Some Xs are Ys,	$v=xy,$	....I.
Some Xs are not Ys,	$v=x(1-y)$	....O.

On examining these, we perceive that E and I are symmetrical with respect to x and y, so that x being changed into y, and y into x, the equations remain unchanged. Hence E and I may be interpreted into

No Ys are Xs,
Some Ys are Xs,

respectively. Thus we have the known rule of the Logicians, that particular
26|27 affirmative and universal negative Propositions admit of simple conversion. |

Boole 1854: Laws of Thought

PROPOSITION IV.

That axiom of metaphysicians which is termed the principle of contradiction, and which affirms that it is impossible for any being to possess a quality, and at the same time not to possess it, is a consequence of the fundamental law of thought, whose expression is $x^2 = x$.

Let us write this equation in the form

$$x - x^2 = 0,$$

whence we have

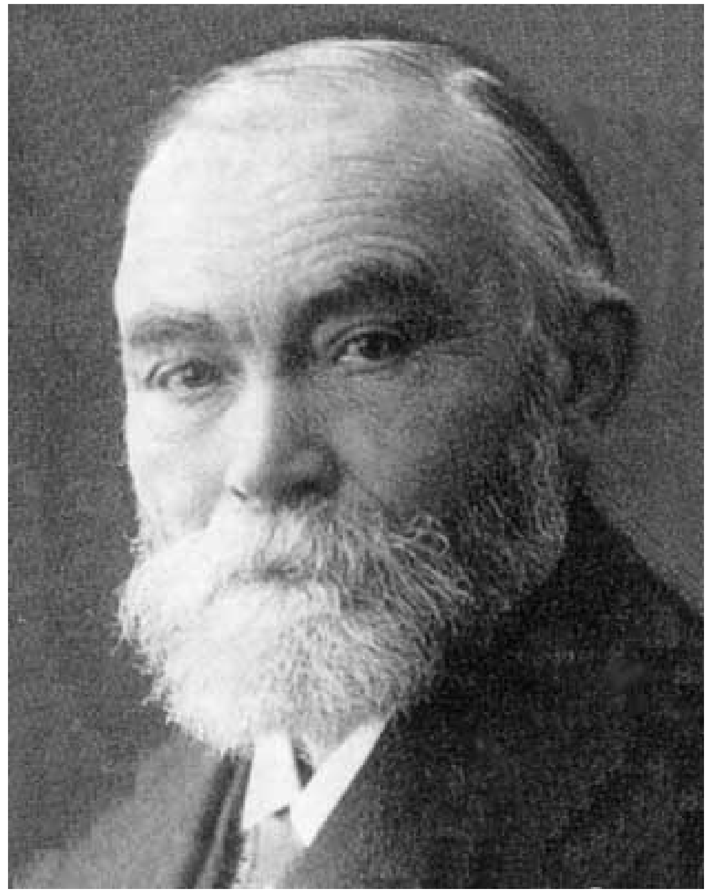
$$x(1 - x) = 0; \qquad (1)$$

both these transformations being justified by the axiomatic laws of combination and transposition (II. 13). Let us, for simplicity

Part 3

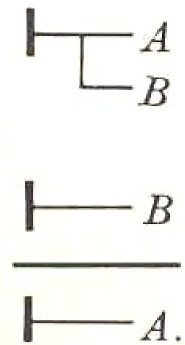
Frege's *Begriffsschrift*

Gotlob Frege (1848–1925)



Frege 1879

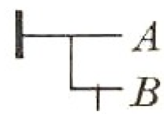
We could write this inference perhaps as follows:



This would become awkward if long expressions were to take the places of A and B, since each of them would have to be written twice. That is why I use the following

Frege 1879

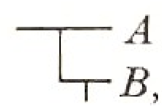
In view of the preceding it is easy to state what the significance of each of the three parts of the horizontal stroke to the left of A is.



means "The case in which A is denied and the negation of B is affirmed does not obtain", or "A and B cannot both be denied". Only the following possibilities remain:

A is affirmed and B is affirmed;
A is affirmed and B is denied;
A is denied and B is affirmed;

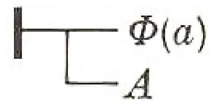
A and B together exhaust all possibilities. Now the words "or" and "either—or" are used in two ways: "A or B" means, in the first place, just the same as



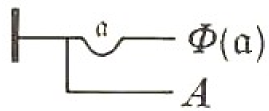
hence it means that no possibility other than A and B is thinkable. For example, if

Frege 1879

It is clear also that from

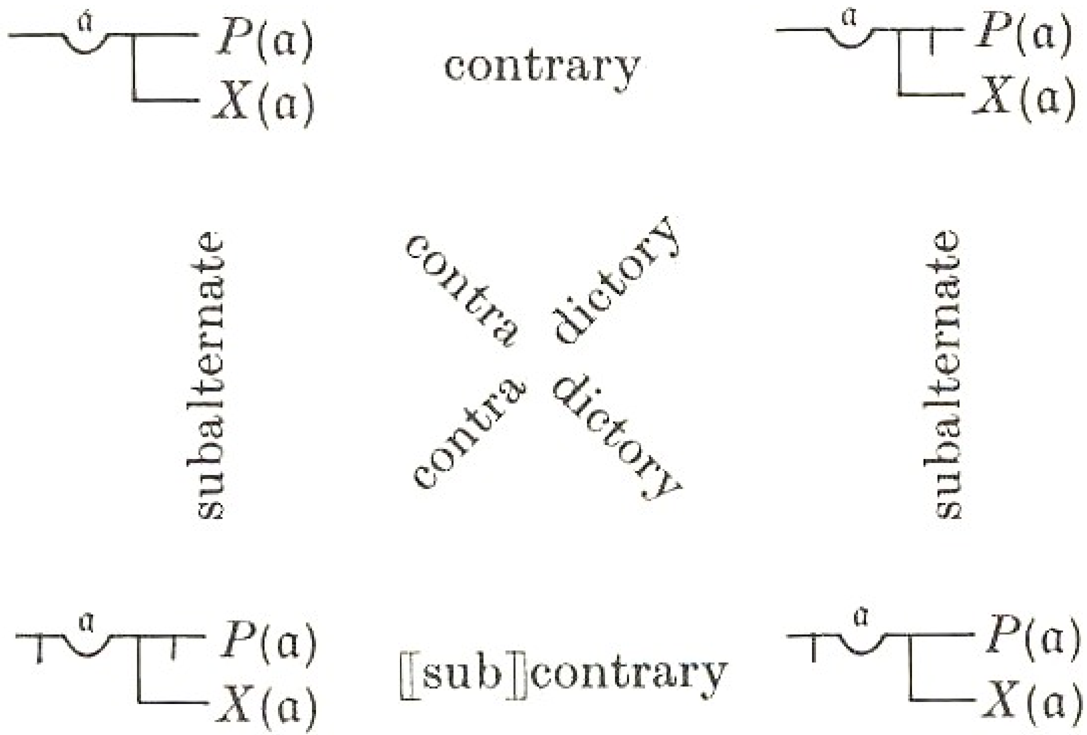


we can derive



if A is an expression in which a does not occur and if a stands only in the argument places of $\Phi(a)$.[14] If  is denied, we must be able to specify a meaning for a such that $\Phi(a)$ will be denied. If, therefore,  were to be denied and

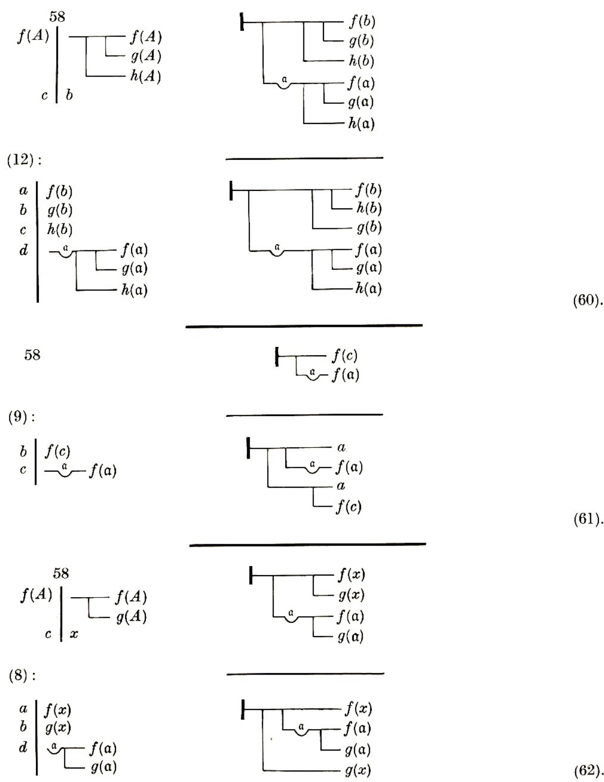
Frege 1879



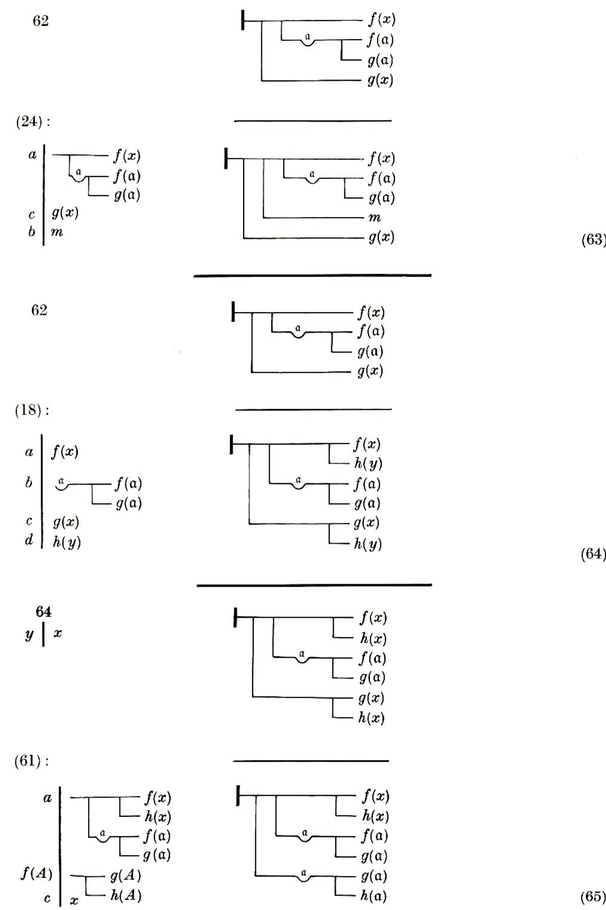
P(a)
X(a)
contrary
P(a)
X(a)
subalternate
contra
dictory
contra
dictory
subalternate
P(a)
X(a)
[[sub]]contrary
P(a)
X(a)

Frege 1879

We see how this judgment replaces one mode of inference, namely, Felapton or Fesapo, between which we do not distinguish here since no subject has been singled out.



This judgment replaces the mode of inference Barbara when the minor premiss, $g(x)$, has a particular content.



Frege in modern notation

$$\frac{\vdash B \to A \qquad \vdash B}{\vdash A}$$

$$\vdash A \to (B \to A)$$

$$\vdash (C \to (B \to A)) \to ((C \to B) \to (C \to A))$$

$$\vdash (C \to (B \to A)) \to (B \to (C \to A))$$

$$\vdash (\neg(\neg A)) \to A$$

$$\vdash A \to (\neg(\neg A))$$

$$A \,\&\, B \;=\; \neg(A \to (\neg B))$$

$$A \vee B \;=\; (\neg A) \to B$$

Part 4

Gentzen's Natural Deduction

Gerhard Gentzen (1909–1945)

Gentzen 1934: Natural Deduction

$\&\text{–}I$: $$\frac{\mathfrak{A} \quad \mathfrak{B}}{\mathfrak{A} \,\&\, \mathfrak{B}}$$

$\&\text{–}E$: $$\frac{\mathfrak{A} \,\&\, \mathfrak{B}}{\mathfrak{A}} \qquad \frac{\mathfrak{A} \,\&\, \mathfrak{B}}{\mathfrak{B}}$$

$\vee\text{–}I$: $$\frac{\mathfrak{A}}{\mathfrak{A} \vee \mathfrak{B}} \qquad \frac{\mathfrak{B}}{\mathfrak{A} \vee \mathfrak{B}}$$

$\vee\text{–}E$: $$\frac{\mathfrak{A} \vee \mathfrak{B} \quad \begin{matrix}[\mathfrak{A}] \\ \mathfrak{C}\end{matrix} \quad \begin{matrix}[\mathfrak{B}] \\ \mathfrak{C}\end{matrix}}{\mathfrak{C}}$$

$\forall\text{–}I$: $$\frac{\mathfrak{F}\mathfrak{a}}{\forall \mathfrak{x}\, \mathfrak{F}\mathfrak{x}}$$

$\forall\text{–}E$: $$\frac{\forall \mathfrak{x}\, \mathfrak{F}\mathfrak{x}}{\mathfrak{F}\mathfrak{a}}$$

$\exists\text{–}I$: $$\frac{\mathfrak{F}\mathfrak{a}}{\exists \mathfrak{x}\, \mathfrak{F}\mathfrak{x}}$$

$\exists\text{–}E$: $$\frac{\exists \mathfrak{x}\, \mathfrak{F}\mathfrak{x} \quad \begin{matrix}[\mathfrak{F}\mathfrak{a}] \\ \mathfrak{C}\end{matrix}}{\mathfrak{C}}$$

$\supset\text{–}I$: $$\frac{\begin{matrix}[\mathfrak{A}] \\ \mathfrak{B}\end{matrix}}{\mathfrak{A} \supset \mathfrak{B}}$$

$\supset\text{–}E$: $$\frac{\mathfrak{A} \quad \mathfrak{A} \supset \mathfrak{B}}{\mathfrak{B}}$$

$\neg\text{–}I$: $$\frac{\begin{matrix}[\mathfrak{A}] \\ \curlywedge\end{matrix}}{\neg \mathfrak{A}}$$

$\neg\text{–}E$: $$\frac{\mathfrak{A} \quad \neg \mathfrak{A}}{\curlywedge} \qquad \frac{\curlywedge}{\mathfrak{D}} \;.$$

Gentzen 1934: Natural Deduction

$$
\dfrac{
\dfrac{
\overset{2}{X \vee (Y \,\&\, Z)}
\quad
\dfrac{
\dfrac{\overset{1}{X}}{X \vee Y}\,\vee\text{–}I
\quad
\dfrac{\overset{1}{X}}{X \vee Z}\,\vee\text{–}I
}{(X \vee Y) \,\&\, (X \vee Z)}\,\&\text{–}I
\quad
\dfrac{
\dfrac{\dfrac{\overset{1}{Y \,\&\, Z}}{Y}\,\&\text{–}E}{X \vee Y}\,\vee\text{–}I
\quad
\dfrac{\dfrac{\overset{1}{Y \,\&\, Z}}{Z}\,\&\text{–}E}{X \vee Z}\,\vee\text{–}I
}{(X \vee Y) \,\&\, (X \vee Z)}\,\&\text{–}I
}{(X \vee Y) \,\&\, (X \vee Z)}\,\vee\text{–}E_1
}{(X \vee (Y \,\&\, Z)) \supset ((X \vee Y) \,\&\, (X \vee Z))}\,\supset\text{–}I_2 .
$$

Gentzen 1934: Sequent Calculus

$$\&\text{–}IS\colon\ \frac{\Gamma \to \Theta, \mathfrak{A} \qquad \Gamma \to \Theta, \mathfrak{B}}{\Gamma \to \Theta, \mathfrak{A} \,\&\, \mathfrak{B}},$$

$$\&\text{–}IA\colon\ \frac{\mathfrak{A}, \Gamma \to \Theta}{\mathfrak{A} \,\&\, \mathfrak{B}, \Gamma \to \Theta} \qquad \frac{\mathfrak{B}, \Gamma \to \Theta}{\mathfrak{A} \,\&\, \mathfrak{B}, \Gamma \to \Theta},$$

$$\vee\text{–}IA\colon\ \frac{\mathfrak{A}, \Gamma \to \Theta \qquad \mathfrak{B}, \Gamma \to \Theta}{\mathfrak{A} \vee \mathfrak{B}, \Gamma \to \Theta},$$

$$\vee\text{–}IS\colon\ \frac{\Gamma \to \Theta, \mathfrak{A}}{\Gamma \to \Theta, \mathfrak{A} \vee \mathfrak{B}} \qquad \frac{\Gamma \to \Theta, \mathfrak{B}}{\Gamma \to \Theta, \mathfrak{A} \vee \mathfrak{B}},$$

$$\forall\text{–}IS\colon\ \frac{\Gamma \to \Theta, \mathfrak{F}\mathfrak{a}}{\Gamma \to \Theta, \forall \mathfrak{x}\, \mathfrak{F}\mathfrak{x}},$$

$$\exists\text{–}IA\colon\ \frac{\mathfrak{F}\mathfrak{a}, \Gamma \to \Theta}{\exists \mathfrak{x}\, \mathfrak{F}\mathfrak{x}, \Gamma \to \Theta}.$$

Gentzen 1934: Natural Deduction

$$\frac{}{A \vdash A}\ \text{Id}$$

$$\frac{\Gamma,\ A \vdash B}{\Gamma \vdash A \rightarrow B}\ {\rightarrow}\text{-I} \qquad \frac{\Gamma \vdash A \rightarrow B \qquad \Delta \vdash A}{\Gamma,\ \Delta \vdash B}\ {\rightarrow}\text{-E}$$

$$\frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma,\ \Delta \vdash A \,\&\, B}\ \&\text{-I} \qquad \frac{\Gamma \vdash A \,\&\, B}{\Gamma \vdash A}\ \&\text{-E}_0 \qquad \frac{\Gamma \vdash A \,\&\, B}{\Gamma \vdash B}\ \&\text{-E}_1$$

Simplifying proofs

$$\dfrac{\dfrac{\begin{array}{c}\dfrac{}{A \vdash A}\,\text{Id}\\ \vdots\; u\\ \Gamma, A \vdash B\end{array}}{\Gamma \vdash A \to B}\,{\to}\text{-I} \qquad \begin{array}{c}\vdots\; t\\ \Delta \vdash A\end{array}}{\Gamma, \Delta \vdash B}\,{\to}\text{-E} \qquad \Rightarrow \qquad \begin{array}{c}\vdots\; t\\ \Delta \vdash A\\ \vdots\; u\\ \Gamma, \Delta \vdash B\end{array}$$

$$\dfrac{\dfrac{\begin{array}{c}\vdots\; t\\ \Gamma \vdash A\end{array} \qquad \begin{array}{c}\vdots\; u\\ \Delta \vdash B\end{array}}{\Gamma, \Delta \vdash A \,\&\, B}\,\&\text{-I}}{\Gamma, \Delta \vdash A}\,\&\text{-E}_0 \qquad \Rightarrow \qquad \begin{array}{c}\vdots\; t\\ \Gamma \vdash A\end{array}$$

A proof

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{B \mathbin{\&} A \vdash B \mathbin{\&} A}\,\mathrm{Id}}{B \mathbin{\&} A \vdash A}\,\&\text{-E}_1 \qquad \dfrac{\dfrac{}{B \mathbin{\&} A \vdash B \mathbin{\&} A}\,\mathrm{Id}}{B \mathbin{\&} A \vdash B}\,\&\text{-E}_0}{B \mathbin{\&} A \vdash A \mathbin{\&} B}\,\&\text{-I}}{\vdash (B \mathbin{\&} A) \to (A \mathbin{\&} B)}\,\to\text{-I} \qquad \dfrac{\dfrac{}{B \vdash B}\,\mathrm{Id} \qquad \dfrac{}{A \vdash A}\,\mathrm{Id}}{A,\ B \vdash B \mathbin{\&} A}\,\&\text{-I}}{A,\ B \vdash A \mathbin{\&} B}\,\to\text{-E}$$

Simplifying a proof

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{B \,\&\, A \vdash B \,\&\, A}\,\text{Id}}{B \,\&\, A \vdash A}\,\&\text{-E}_1 \qquad \dfrac{\dfrac{}{B \,\&\, A \vdash B \,\&\, A}\,\text{Id}}{B \,\&\, A \vdash B}\,\&\text{-E}_0}{B \,\&\, A \vdash A \,\&\, B}\,\&\text{-I}}{\vdash (B \,\&\, A) \to (A \,\&\, B)}\,\to\text{-I} \qquad \dfrac{\dfrac{}{B \vdash B}\,\text{Id} \qquad \dfrac{}{A \vdash A}\,\text{Id}}{A,\ B \vdash B \,\&\, A}\,\&\text{-I}}{A,\ B \vdash A \,\&\, B}\,\to\text{-E}$$

$$\Downarrow$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{B \vdash B}\,\text{Id} \qquad \dfrac{}{A \vdash A}\,\text{Id}}{A,\ B \vdash B \,\&\, A}\,\&\text{-I}}{A,\ B \vdash A}\,\&\text{-E}_1 \qquad \dfrac{\dfrac{\dfrac{}{B \vdash B}\,\text{Id} \qquad \dfrac{}{A \vdash A}\,\text{Id}}{A,\ B \vdash B \,\&\, A}\,\&\text{-I}}{A,\ B \vdash B}\,\&\text{-E}_0}{A,\ B \vdash A \,\&\, B}\,\&\text{-I}$$

Simplifying a proof

$$\dfrac{\dfrac{\dfrac{\dfrac{}{B \vdash B}\,\text{Id} \qquad \dfrac{}{A \vdash A}\,\text{Id}}{A,\ B \vdash B \,\&\, A}\,\&\text{-I}}{A,\ B \vdash A}\,\&\text{-E}_1 \qquad \dfrac{\dfrac{\dfrac{}{B \vdash B}\,\text{Id} \qquad \dfrac{}{A \vdash A}\,\text{Id}}{A,\ B \vdash B \,\&\, A}\,\&\text{-I}}{A,\ B \vdash B}\,\&\text{-E}_0}{A,\ B \vdash A \,\&\, B}\,\&\text{-I}$$

$$\Downarrow$$

$$\dfrac{\dfrac{}{A \vdash A}\,\text{Id} \qquad \dfrac{}{B \vdash B}\,\text{Id}}{A,\ B \vdash A \,\&\, B}\,\&\text{-I}$$

Part 5

Church's Lambda Calculus

Alonzo Church (1903–1995)

Church 1932: Lambda Calculus

Annals of Mathematics, ser. 2 vol. 33 (1932)

A SET OF POSTULATES FOR THE FOUNDATION OF LOGIC.[1]

BY ALONZO CHURCH.[2]

1. **Introduction.** In this paper we present a set of postulates for the foundation of formal logic, in which we avoid use of the free, or real, variable, and in which we introduce a certain restriction on the law of excluded middle as a means of avoiding the paradoxes connected with the mathematics of the transfinite.

Church 1932: Lambda Calculus

In consequence of this abstract character of the system which we are about to formulate, it is not admissible, in proving theorems of the system, to make use of the meaning of any of the symbols, although in the application which is intended the symbols do acquire meanings. The initial set of postulates must of themselves define the system as a formal structure, and in developing this formal structure reference to the proposed application must be held irrelevant. There may, indeed, be other applications of the system than its use as a logic.

Church 1932: Lambda Calculus

An occurrence of a variable **x** in a given formula is called an occurrence of **x** as a *bound variable* in the given formula if it is an occurrence of **x** in a part of the formula of the form λ**x**[**M**]; that is, if there is a formula **M** such that λ**x**[**M**] occurs in the given formula and the occurrence of **x** in question is an occurrence in λ**x**[**M**]. All other occurrences of a variable in a formula are called occurrences as a *free variable*.

A formula is said to be *well-formed* if it is a variable, or if it is one

Church 1932: Lambda Calculus

$$(\lambda x.\, x + x)\, 2 \Rightarrow 2 + 2$$

Church-Rosser Theorem

$$
\begin{array}{ccccc}
 & & (\lambda x.\, x + x)\,((\lambda y.\, y + 1)\,1) & & \\
 & \swarrow & & \searrow & \\
(\lambda x.\, x + x)\,(1 + 1) & & & & ((\lambda y.\, y + 1)\,1) + ((\lambda y.\, y + 1)\,1) \\
 & \searrow & & \swarrow & \\
 & & (1 + 1) + (1 + 1) & &
\end{array}
$$

Reduction rules

$$
\begin{aligned}
(\lambda x.\, u)\, t \quad &\Rightarrow \quad u[t/x] \\
\langle t, u\rangle_0 \quad &\Rightarrow \quad t \\
\langle t, u\rangle_1 \quad &\Rightarrow \quad u
\end{aligned}
$$

Reducing a term

$$(\lambda z.\,\langle z_1, z_0\rangle)\,\langle y, x\rangle$$

$$\Downarrow$$

$$\langle\langle y, x\rangle_1, \langle y, x\rangle_0\rangle$$

$$\Downarrow$$

$$\langle x, y\rangle$$

Church 1940: Typed Lambda Calculus

$$\dfrac{}{x : A \vdash x : A}\ \text{Id}$$

$$\dfrac{\Gamma,\ x : A \vdash u : B}{\Gamma \vdash \lambda x.\, u : A \to B}\ \to\text{-I} \qquad \dfrac{\Gamma \vdash s : A \to B \qquad \Delta \vdash t : A}{\Gamma,\ \Delta \vdash s\, t : B}\ \to\text{-E}$$

$$\dfrac{\Gamma \vdash t : A \qquad \Delta \vdash u : B}{\Gamma,\ \Delta \vdash \langle t, u \rangle : A \,\&\, B}\ \&\text{-I} \qquad \dfrac{\Gamma \vdash s : A \,\&\, B}{\Gamma \vdash s_0 : A}\ \&\text{-E}_0 \qquad \dfrac{\Gamma \vdash s : A \,\&\, B}{\Gamma \vdash s_1 : B}\ \&\text{-E}_1$$

Simplifying programs

$$\dfrac{\dfrac{\begin{array}{c}\dfrac{}{x : A \vdash x : A}\ \text{Id}\\ \vdots\ u\\ \Gamma, x : A \vdash u : B\end{array}}{\Gamma \vdash \lambda x.\, u : A \to B}\ \to\text{-I} \qquad \begin{array}{c}\vdots\ t\\ \Delta \vdash t : A\end{array}}{\Gamma, \Delta \vdash (\lambda x.\, u)\, t : B}\ \to\text{-E} \quad \Rightarrow \quad \begin{array}{c}\vdots\ t\\ \Delta \vdash t : A\\ \vdots\ u\\ \Gamma, \Delta \vdash u[t/x] : B\end{array}$$

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ t\\ \Gamma \vdash t : A\end{array} \qquad \begin{array}{c}\vdots\ u\\ \Delta \vdash u : B\end{array}}{\Gamma, \Delta \vdash \langle t, u\rangle : A \,\&\, B}\ \&\text{-I}}{\Gamma, \Delta \vdash \langle t, u\rangle_0 : A}\ \&\text{-E}_0 \quad \Rightarrow \quad \begin{array}{c}\vdots\ t\\ \Gamma \vdash t : A\end{array}$$

A program

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{}{z{:}B \mathbin{\&} A \vdash z{:}B \mathbin{\&} A}\,\text{Id}
    }{z{:}B \mathbin{\&} A \vdash z_1{:}A}\,\&\text{-E}_1
    \quad
    \dfrac{
      \dfrac{}{z{:}B \mathbin{\&} A \vdash z{:}B \mathbin{\&} A}\,\text{Id}
    }{z{:}B \mathbin{\&} A \vdash z_0{:}B}\,\&\text{-E}_0
  }{z{:}B \mathbin{\&} A \vdash \langle z_1, z_0 \rangle{:}A \mathbin{\&} B}\,\&\text{-I}
  }{\vdash \lambda z.\, \langle z_1, z_0 \rangle{:}(B \mathbin{\&} A) \to (A \mathbin{\&} B)}\,\to\text{-I}
  \quad
  \dfrac{
    \dfrac{}{y{:}B \vdash y{:}B}\,\text{Id}
    \quad
    \dfrac{}{x{:}A \vdash x{:}A}\,\text{Id}
  }{x{:}A, y{:}B \vdash \langle y, x \rangle{:}B \mathbin{\&} A}\,\&\text{-I}
}{x{:}A, y{:}B \vdash (\lambda z.\, \langle z_1, z_0 \rangle)\, \langle y, x \rangle{:}A \mathbin{\&} B}\,\to\text{-E}
$$

Simplifying a program

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{\dfrac{}{z{:}B \,\&\, A \vdash z{:}B \,\&\, A}\,\text{Id}}{z{:}B \,\&\, A \vdash z_1{:}A}\,\&\text{-E}_1
      \qquad
      \dfrac{\dfrac{}{z{:}B \,\&\, A \vdash z{:}B \,\&\, A}\,\text{Id}}{z{:}B \,\&\, A \vdash z_0{:}B}\,\&\text{-E}_0
    }{z{:}B \,\&\, A \vdash \langle z_1, z_0\rangle{:}A \,\&\, B}\,\&\text{-I}
  }{\vdash \lambda z.\, \langle z_1, z_0\rangle{:}(B \,\&\, A) \to (A \,\&\, B)}\,\to\text{-I}
  \qquad
  \dfrac{
    \dfrac{}{y{:}B \vdash y{:}B}\,\text{Id}
    \qquad
    \dfrac{}{x{:}A \vdash x{:}A}\,\text{Id}
  }{x{:}A, y{:}B \vdash \langle y, x\rangle{:}B \,\&\, A}\,\&\text{-I}
}{x{:}A, y{:}B \vdash (\lambda z.\, \langle z_1, z_0\rangle)\, \langle y, x\rangle{:}A \,\&\, B}\,\to\text{-E}
$$

$$\Downarrow$$

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{}{y{:}B \vdash y{:}B}\,\text{Id}
      \qquad
      \dfrac{}{x{:}A \vdash x{:}A}\,\text{Id}
    }{x{:}A, y{:}B \vdash \langle y, x\rangle{:}B \,\&\, A}\,\&\text{-I}
  }{x{:}A, y{:}B \vdash \langle y, x\rangle_1{:}A}\,\&\text{-E}_1
  \qquad
  \dfrac{
    \dfrac{
      \dfrac{}{y{:}B \vdash y{:}B}\,\text{Id}
      \qquad
      \dfrac{}{x{:}A \vdash x{:}A}\,\text{Id}
    }{x{:}A, y{:}B \vdash \langle y, x\rangle{:}B \,\&\, A}\,\&\text{-I}
  }{x{:}A, y{:}B \vdash \langle y, x\rangle_0{:}B}\,\&\text{-E}_0
}{x{:}A, y{:}B \vdash \langle \langle y, x\rangle_1, \langle y, x\rangle_0\rangle{:}A \,\&\, B}\,\&\text{-I}
$$

Simplifying a program

$$\dfrac{\dfrac{\dfrac{\dfrac{}{y{:}B \vdash y{:}B}\,\text{Id} \qquad \dfrac{}{x{:}A \vdash x{:}A}\,\text{Id}}{x{:}A, y{:}B \vdash \langle y, x\rangle{:}B \mathbin{\&} A}\,\&\text{-I}}{x{:}A, y{:}B \vdash \langle y, x\rangle_1{:}A}\,\&\text{-E}_1 \qquad \dfrac{\dfrac{\dfrac{}{y{:}B \vdash y{:}B}\,\text{Id} \qquad \dfrac{}{x{:}A \vdash x{:}A}\,\text{Id}}{x{:}A, y{:}B \vdash \langle y, x\rangle{:}B \mathbin{\&} A}\,\&\text{-I}}{x{:}A, y{:}B \vdash \langle y, x\rangle_0{:}B}\,\&\text{-E}_0}{x{:}A, y{:}B \vdash \langle \langle y, x\rangle_1, \langle y, x\rangle_0\rangle{:}A \mathbin{\&} B}\,\&\text{-I}$$

$$\Downarrow$$

$$\dfrac{\dfrac{}{x{:}A \vdash x{:}A}\,\text{Id} \qquad \dfrac{}{y{:}B \vdash y{:}B}\,\text{Id}}{x{:}A, y{:}B \vdash \langle x, y\rangle{:}A \mathbin{\&} B}\,\&\text{-I}$$

Part 6

The Curry-Howard Isomorphism

Haskell Curry (1900–1982) / William Howard

Howard 1980

THE FORMULAE-AS-TYPES NOTION OF CONSTRUCTION

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Dedicated to H. B. Curry on the occasion of his 80th birthday.

The following consists of notes which were privately circulated in 1969. Since they have been referred to a few times in the literature, it seems worth while to publish them. They have been rearranged for easier reading, and some inessential corrections have been made.

Howard 1980

1. Formulation of the sequent calculus

Let $P(\supset)$ denote positive implicational propositional logic. The prime formulae of $P(\supset)$ are propositional variables. If α and β are formulae, so is $\alpha \supset \beta$. A *sequent* has the form $\Gamma \to \beta$, where Γ is a (possibly empty) finite sequence of formulae and β is a formula. The axioms and rules of inference of $P(\supset)$ are as follows.

(1.1) Axioms: all sequents of the form
$$\alpha \to \alpha$$

(1.2)
$$\frac{\Gamma, \alpha \to \beta}{\Gamma \to \alpha \supset \beta}$$

(1.3)
$$\frac{\Gamma \to \alpha \qquad \Delta \to \alpha \supset \beta}{\Gamma, \Delta \to \beta}$$

(1.4) Thinning, permutation and contraction rules

Howard 1980

2. *Type symbols, terms and constructions*

By a type symbol is meant a formula of $P(\supset)$. We will consider a λ-formalism in which each term has a type symbol α as a superscript (which we may not always write); the term is said to be of type α. The rules of term formation are as follows.

(2.1) Variables X^α, Y^β, ... are terms

(2.2) λ-abstraction: from F^β get

$$(\lambda X^\alpha . F^\beta)^{\alpha \supset \beta} .$$

(2.3) Application: from $G^{\alpha \supset \beta}$ and H^α

get $(G^{\alpha \supset \beta} H^\alpha)^\beta$.

Part 7

Programs and Proofs

Programs

- Lisp (McCarthy, 1960)
- Iswim (Landin, 1966)
- Scheme (Steele and Sussman, 1975)
- ML (Milner, Gordon, Wadsworth, 1979)
- Miranda (Turner, 1985)
- Haskell (Hudak, Peyton Jones, and Wadler, 1987)
- O'Caml (Leroy, 1996)

Proofs

- Automath (de Bruijn, 1970)
- Type Theory (Martin Löf, 1975)
- ML/LCF (Milner, Gordon, and Wadsworth, 1979)
- HOL (Gordon and Melham, 1988)
- CoQ (Huet and Coquand, 1988)
- Isabelle (Paulson, 1993)

Proofs/Programs

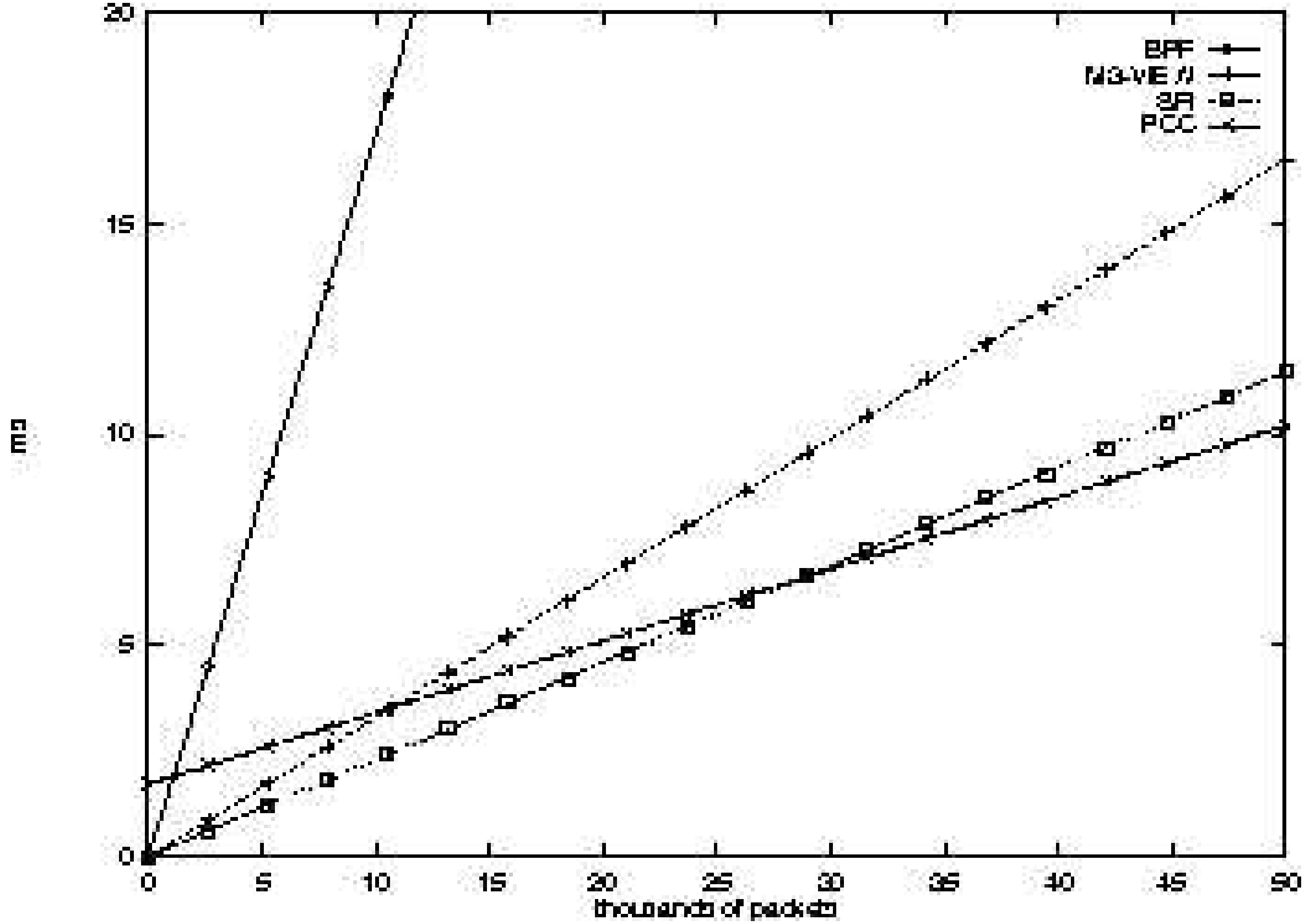
- Hindley/Milner (1969/1975)
- Girard/Reynolds (1972/1975)
- Linear Logic/Syntactic Control of Interference (1985)
- Classical Logic/Continuation-Passing Style (1990)
- And dual to Or/Call-by-value dual to Call-by-name (2000)

Part 8

Programs and Proofs on the Web

Java (Gosling, Joy, and Steele, 1996)

Proof-Carrying Code (Necula and Lee, 1996)



Typed Assembly Language (Morrisett et al 1998)

Typed Assembly Language (Morrisett et al 1998)

What do you want to type check today?

Part 9

Conclusion

Russell's paradox

Let w be the predicate: to be a predicate that cannot be predicated of itself. Can w be predicated of itself? From each answer the opposite follows.

— Bertrand Russell to Gottlob Frege, 16 June 1902

$$w = \{x \mid x \notin x\}$$

$$w \in w \quad \text{iff} \quad w \notin w$$

Russell on Frege

"As I think about acts of integrity and grace, I realise there is nothing in my knowledge to compare to Frege's dedication to truth. His entire life's work was on the verge of completion, much of his work has been ignored to the benefit of men infinitely less capable, his second volume was about to be published, and upon finding that his fundamental assumption was in error, he responded with intellectual pleasure, clearly submerging any feelings of personal disappointment. It was almost superhuman, and a telling indication of that which men are capable if their dedication is to creative work and knowledge instead of cruder efforts to dominate and be known."

— Bertrand Russell, 23 November 1962

Frege

Undone by Russell's Paradox

Church and Curry

Attended 1982 Conference on
Lisp and Functional Programming

Gentzen

"He once confided in me that he was really
quite contented since now he had at last time to think
about a consistency proof for analysis."

Frege

Undone by Russell's Paradox

Church and Curry

Attended 1982 Conference on
Lisp and Functional Programming

Gentzen

"He once confided in me that he was really
quite contented since now he had at last time to think
about a consistency proof for analysis."
Died in prison, 4 August 1945

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