A Hopfield learning rule with high capacity storage of time-correlated patterns

Amos Storkey, Romain Valabregue

Neural Systems Group

Imperial College of Science, Technology and Medicine

London SW7 2AZ

United Kingdom

Indexing terms: Hopfield network, capacity, correlated patterns.

Abstract

A new local and incremental learning rule is examined for its ability to store patterns from a time series in an attractor neural network. This learning rule has a higher capacity than the Hebb rule, and suffers significantly less capacity loss as the correlation between patterns increases.

1 Introduction

There are two rules regularly used for training Hopfield networks. The most common of these, the Hebb rule, suffers severe degradation when training patterns are correlated. As a result, practitioners have generally reverted to using the pseudo-inverse rule in such circumstances. However the pseudo-inverse method suffers from significant problems. Firstly, it is not incremental: if a new pattern is to be trained, all the old patterns have to be retrained. Secondly it is not local: the network cannot naturally be trained in a parallel manner, and so is not easily amenable to high speed parallel techniques. Lastly the training method is slow because it involves inverting an $m \times m$ matrix, where m is the number of patterns to be stored.

These problems are most significant when the patterns to be trained are a time series and real time training is needed. The slow speed of the pseudo-inverse, and more importantly its inability to add patterns incrementally, make it of little use for this task. Furthermore time series usually include significant correlations between readings from adjacent time steps. This makes the Hebb rule unsuitable.

Here, we examine the new learning rule introduced in [1] for its effectiveness with time correlated patterns. This learning rule does not suffer from the disadvantages of the pseudo-inverse: it is both local and incremental. It also has a higher capacity than the Hebb rule. We demonstrate that this new learning rule maintains its high capacity with significant correlations in patterns. This contrasts with the Hebb rule, where the capacity reduces much faster as the correlation increases. We show how this can be accounted for by the relationship between the new rule and the pseudo-inverse.

In this paper we look at a model of pattern correlations, which resemble the correlations in patterns which form a time series. Many time series are digitisations of continuous events, digitised both in the time and space domains. For example video footage consists of frames of digitised video information. Any two frames of such time series are inevitably correlated, the level of correlation depending on the speed of movement, the sampling rate, and how close in time the two frames are to one another.

2 The Hopfield network

The Hopfield network is an attractor neural network which acts as a content addressable memory. The job of the learning rule of a Hopfield network is to find some weight matrix w_{ij} which stores the required patterns as fixed points of the network.

Locality and incrementality have already been discussed in the introduction. These are both important characteristics of a learning rule. In addition to these a learning rule has a capacity. This is some measure of how many patterns can be stored in a network of a given size. More specifically, in this paper we consider the absolute capacity [2] of the network. This is given by the asymptotic ratio of the number of patterns that can be stored without error to the number of neurons, as the network size tends to infinity.

2.1 Different learning rules

Three learning rules are used in this paper. The Hebbian learning rule is local and incremental, but has a low absolute capacity of $n/(2 \ln n)$ where n is the number of neurons [2]. This capacity decreases significantly if patterns are correlated.

The pseudo inverse has a capacity of n [3], but does not have the functionality of the Hebb rule. It is not incremental or local, because it involves the calculation of an inverse.

2.1.1 The new learning rule

In order to overcome the problems of both of these learning methods we introduce the following new learning rule.

Definition 1 (The new learning rule) *The weight matrix w_{ij} of an attractor neural network is said to follow the new learning rule if it obeys*

$$w_{ij}^0 = 0 \ \forall i,j \ \textit{and} \ w_{ij}^{\nu} = w_{ij}^{\nu-1} + \frac{1}{n}\xi_i^{\nu}\xi_j^{\nu} - \frac{1}{n}\xi_i^{\nu}h_{ji}^{\nu} - \frac{1}{n}h_{ij}^{\nu}\xi_j^{\nu} \tag{1}$$

where $h_{ij}^{\mu} = \sum_{k=1,k\neq i,j}^{n} w_{ik}^{\mu-1}\xi_k^{\mu}$ is a form of local field at neuron i (the input to the neuron i) , and ξ^{μ} is the new pattern to be learnt ($\xi_i^{\mu} = \pm 1$)

This rule is local: w_{ij} depends only on information available at the two adjacent neurons (the values of ξ_i, ξ_j, h_i, h_j). It is clear from the recursive nature of (1) that it is also incremental. It has an absolute capacity of $n/\sqrt{2\ln n}$ [1].

3 Capacity

The measures of absolute capacity given above only hold for independent patterns. Here we consider what happens when patterns are correlated. We are specifically interested in the types of correlations which occur in time series. To model this we form the patterns using the following simple Markov chain.

$$P(\xi_i^{\mu} = \xi_i^{\mu-1}) = \rho; \ P(\xi_i^1 = 1) = \frac{1}{2}; \quad \frac{1}{2} \leq \rho < 1 \tag{2}$$

where $\mu = 1, 2, \ldots, m$ labels the position of the pattern in the time series and $\xi_i^{\mu} = \pm 1 \ \forall i, \mu$.

The correlation between patterns ξ_i^{μ} and $\xi_i^{\mu+t}$ is seen to be

$$\frac{E(\xi_i^{\mu}\xi_i^{\mu+t})}{(E((\xi_i^{\mu})^2)E((\xi_i^{\mu})^2))^{1/2}} = [2\rho - 1]^{|t|}$$

by application of the Chapman-Kolmogorov equations. This shows that the correlation between patterns decays as their time seperation increases. Henceforth when we use the term correlation, we refer to the value of the correlation between two consecutive patterns: $2\rho - 1$.

Here we compare the absolute capacity of the new learning rule with the Hebb rule as the correlation increases. In all cases 30 sets of patterns ξ^μ of length 500 were chosen according to (2). For each set, the patterns were presented to the network one by one until one of the patterns was no longer an attractor (i.e. absolute capacity had been reached). We plot the minimum mean and maximum capacity over the 30 sets.

The above simulations were performed, and the capacity results for the new learning rule were compared with those of the Hebb rule (Figure 1). It is clear from this graph that not only does the new rule have a significantly higher capacity for uncorrelated patterns, but it also suffers less from an increase in the correlation. The new rule seems to be unaffected by small correlations, whereas the Hebb rule capacity decays fast as the correlation increases from zero. In addition the new rule still has greater storage ability at correlations of about 0.5 than the Hebb rule does for uncorrelated patterns. The Hebb rule on the other hand is useless for correlations above that of 0.4. Results were very similar for other network sizes.

4 The relationship between the learning rules

The following relationship between the different training rules can begin to indicate why the new learning rule can store correlated patterns.

The inverse of a matrix can be calculated by the following iterative procedure

$$X^{n+1} = X^n(2I - CX^n)$$

with $X^0 = I$. The X^n converges to C^{-1} if $(I - C)$ has eigenvalues less than one. This is true for any C of the form used in the pseudo- inverse rule.

So using this with the pseudo-inverse where $C = 1/n \sum_{k=1}^{n} \xi_k^\mu \xi_k^\nu$, we can get a first approximation by iterating once only, to get a first order in C expansion:

$$w_{ij}^m = \frac{2}{n} \sum_{\mu=1}^{m} \xi_i^\mu \xi_j^\mu - \frac{1}{n} \sum_{\mu=1}^{m} \sum_{\nu=1}^{m} \xi_i^\mu C^{\mu\nu} \xi_j^\nu$$

The new rule is defined recursively If this is then expanded out we find it is also of the above form to first order in C.

The Hebbian rule is clearly the zeroth order expansion of the pseudo-inverse rule. The first order expansion itself is non-incremental, but the new learning rule provides an incremental method with a first order expansion equivalent to that of the pseudo inverse.

We can consider the new learning rule to be an intermediate between the Hebb rule and the pseudo-inverse. It inherits some of the ability of the pseudo-inverse to deal with correlated patterns.

5 Conclusions

It is little real life benefit to be able to store independent patterns in an associative network. All meaningful patterns tend to have some form of correlations. Here we have looked at a simple case, and have seen how the Hebb rule deteriorates rapidly with these types of time correlations. The new learning rule still has all the important functionality of the Hebb rule, but has both higher capacity and greater resilience to correlations. It appears that the new learning rule is the only one of the three rules considered which is suitable for real time storage of patterns that form a time series.

References

[1] A. J. Storkey. Increasing the capacity of the hopfield network without sacrificing functionality. In *International Conference on Artificial Neural Networks*, 1997.

[2] R. J. McEliece, E. C. Posner, E. R. Rodemich, and S. S. Venkatesh. The capacity of the Hopfield associative memory. *IEEE Transactions on Information Theory*, 33(4):461–482, 1987.

[3] I. Kanter and H. Sompolinsky. Associative recall of memory without errors. *Physical Review A- General Physics*, 35(1):380–392, 1987.

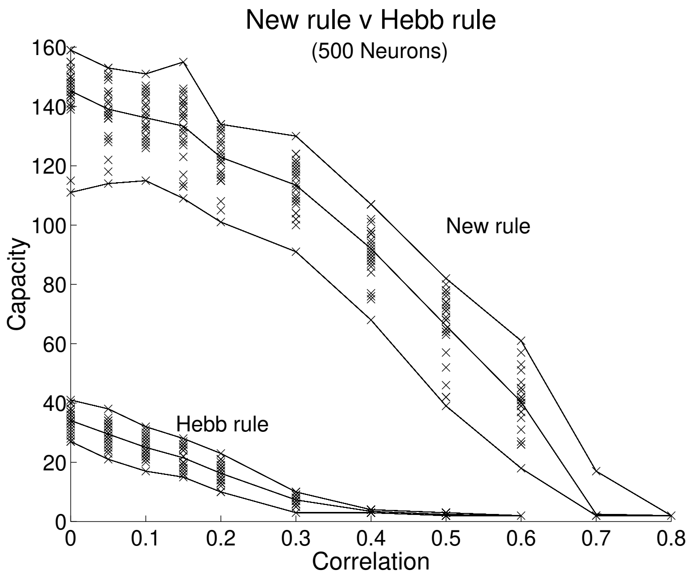

Figure 1: Capacity of new learning rule v Hebb (min-mean-max graph)