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Categories of Quantum and Classical Channels

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Abstract We introduce a construction that turns a category of pure state spaces and operators into a category of observable algebras and superoperators. For example, it turns the category of finite-dimensional Hilbert spaces into the category of finite-dimensional C*-algebras and completely positive maps. In particular, the new category contains both quantum and classical channels, providing elegant abstract notions of preparation and measurement. We also consider nonstandard models, that can be used to investigate which notions from algebraic quantum information theory are operationally justifiable.

Keywords Abstract C*-algebras, categorical quantum mechanics, completely positive maps, quantum channel

Mathematics Subject Classification (2000) 81P45, 16B50, 18D35, 46L89, 46N50, 81P16

1 Introduction

Algebraic quantum information theory provides a very neat framework in which to study protocols and algorithms involving both classical and quantum systems. Instead of stacking structure on top of the base formalism of Hilbert space – to accommodate, for example, mixed states, their measurement and evolution, and classical outcomes – these basic notions are equal and first-class citizens in the algebraic approach.

The basic setup is that individual systems are modeled by C*-algebras, which can be grouped by tensor products, and can evolve along completely positive maps, also called channels. Classical systems correspond to commutative algebras. This uniformises many notions. For example, a density matrix corresponds simply to a channel from the trivial classical system $\mathbb{C}$ to a quantum system, and a positive operator valued measurement is just a channel from a quantum system to a classical one. One ends up with a category of classical and quantum systems and channels

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between them. Advanced protocols can then be modeled by combining channels in sequence as well as in parallel. For more information we refer to [21, 22].

This paper abstracts that idea away from Hilbert spaces, in an attempt to obtain a more operational formalism. The Hilbert space formalism is blessed with such an excess of structure, that many conceptually different notions coincide in this model [27]. Instead, we will take only the very basic notion of compositionality as primitive.

To be precise, we will be working within the programme of categorical quantum mechanics [1, 8]. This programme starts with so-called dagger compact categories, that assume merely a way of grouping systems together that allows for entanglement, and a way of composing operations on those systems. A surprising amount of theory already follows from these primitives, including scalars, the Born rule, quantum teleportation, and much more.

The categories initially studied mostly accommodated pure states. However, there is a beautiful construction that works on arbitrary dagger compact categories, and turns the category containing pure quantum states and operations into the category of mixed states and completely positive maps [30, 10]. The resulting categories can even be axiomatised [7, 15, 10]. Thus mixed states, and channels between quantum systems, can be studied without leaving the theory of dagger compact categories.

Another line of research within categorical quantum mechanics concerns incorporating classical systems. These can be modeled in terms of the tensor structure alone, by promoting the no-cloning theorem into an axiom: the ability to copy and delete becomes an extra feature of classical systems over quantum ones. This leads to so-called commutative Frobenius algebras within a category [13, 14, 12, 2]. Again, it is pleasantly surprising how much follows: for example, this formalism encompasses complementary observables, and measurement-based quantum computing [9].

This paper combines these two developments in representing quantum channels and classical systems, respectively. We show that (possibly noncommutative) Frobenius algebras in the category of Hilbert spaces correspond to finite-dimensional C*-algebras precisely when they are normalisable (see also [36]). This justifies regarding such algebras in arbitrary categories as *abstract C*-algebras*[1]. Then, we present a construction that turns a category (of pure states spaces) into one of channels, in such a way that the category of Hilbert spaces becomes the category of finite-dimensional C*-algebras and channels. We study the cases of "completely quantum" and "completely classical" abstract C*-algebras, showing that this so-called CP*–construction neatly combines quantum channels and classical systems.

Finally, we exemplify our constructions in nonstandard models. This provides counterexamples that separate conceptually different notions, even some that are commonly held to coincide. Our results thus form the starting point for an investigation of the foundations of quantum mechanics from an operational point of view. For example, one can show that commutativity of an algebra of observables need not imply distributivity of its accompanying quantum logic [11]. The nonstandard

[1] By an *abstract* C*-algebra we mean an object in a monoidal category satisfying certain requirements. By a *concrete* one we mean an object satisfying those requirements in the category of (finite-dimensional) Hilbert spaces. This is not to be confused with terminology from functional analysis. There, a concrete C*-algebra is a *-subalgebra of the algebra $\mathcal{B}(H)$ of bounded operators on a Hilbert space H that is uniformly closed, whereas an abstract C*-algebra is any Banach algebra with an involution satisfying $\|a^*a\| = \|a\|^2$; these notions are equivalent by the Gelfand-Naimark-Segal construction; see *e.g.* [16, Theorem I.9.12].

model of sets and relations is a satisfying example of our abstract theory, which there becomes a theory about the well-studied notion of a groupoid (see also [18]). This opens possibilities to employ "quantum reasoning" to obtain group theoretic results, and vice versa.

Before giving a brief introduction to dagger compact categories, we end this introduction by reviewing related work. There have been earlier attempts to combine classical systems with quantum channels [31]. One attempt introduces biproducts to model classical information. This has the drawback that classical and quantum information no longer stand on equal footing, and that adding more primitives than merely compositionality requires operational justification. Another attempt relies on splitting idempotents. This is a clean categorical construction that does not need external ingredients, but it is not so clear that this does not capture too much. Our CP*–construction mediates between these two earlier attempts, as made precise in [19]: it needs no external structure, and it captures the right amount of objects.

A separate development adds classical data to a quantum category via a categorical construction involving the commutative Frobenius algebras in the category [12]. The notion of "classical morphism" from that work inspired the formulation of the CP*–construction, by generalising from commutative algebras to non-commutative.

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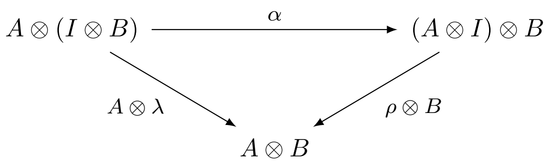
1.1 Dagger compact categories and graphical language

It is often useful to reason in a very general sense about processes and how they compose. Category theory provides the tool to do this. A category consists of a collection of objects $A, B, C, \ldots$, a collection morphisms $f, g, \cdots$, an associative operation $\circ$ for (vertical) composition, and for every object A an identity morphism 1_A. Objects can be thought of as types. They dictate which morphisms can be composed together. We shall primarily be interested in categories that have not only a vertical composition operation, but a horizontal composition as well.

Definition 1.1 A *monoidal category* consists of a category $\mathcal{V}$, an object $I \in \mathcal{V}$ called the monoidal unit, a bifunctor $\otimes : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ called the monoidal product, and natural isomorphisms $\alpha_{A,B,C} : A \otimes (B \otimes C) \to (A \otimes B) \otimes C$, $\lambda_A : I \otimes A \to A$, and $\rho_A : A \otimes I \to A$, such that $\lambda_I = \rho_I$ and the following diagrams commute:

$$\begin{array}{ccccc} A \otimes (B \otimes (C \otimes D)) & \xrightarrow{\alpha} & (A \otimes B) \otimes (C \otimes D) & \xrightarrow{\alpha} & ((A \otimes B) \otimes C) \otimes D \\ \downarrow{\scriptstyle A \otimes \alpha} & & & & \uparrow{\scriptstyle \alpha \otimes D} \\ A \otimes ((B \otimes C) \otimes D) & & \xrightarrow{\qquad\quad \alpha \qquad\quad} & & (A \otimes (B \otimes C)) \otimes D \end{array}$$



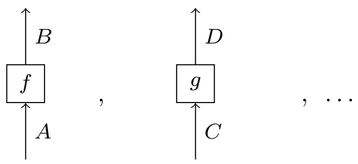
Our main example is the category **FHilb**, whose objects are finite-dimensional complex Hilbert spaces, and whose morphisms are linear functions. It becomes a monoidal category under the usual tensor product of Hilbert spaces, with unit object $\mathbb{C}$.

We often drop α, λ, and ρ when they are clear from the context. Monoidal categories where all three of these maps are actually equalities, rather than natural isomorphisms, are called *strict*. In any monoidal category, they can be used to construct a natural isomorphism from some object to any other bracketing of that object, with or without monoidal units. For example:

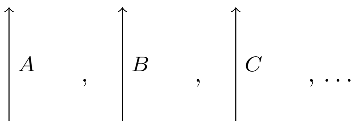
$$(A \otimes I) \otimes (B \otimes (I \otimes C)) \cong (A \otimes (B \otimes (C \otimes I))).$$

Mac Lane's *coherence theorem* proves that the equations in Definition 1.1 suffice to show that any such natural isomorphism is equal to any other one [24]. This lets us treat monoidal categories as if they were strict. That is, we may omit brackets, α, λ, and ρ without ambiguity, simply assuming they are included where necessary.

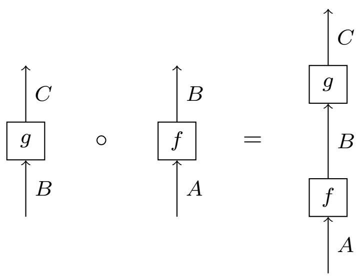
Instead of the usual algebraic notation for morphisms in monoidal categories, it is often vastly more convenient to use a graphical notation (see also [32]). Morphisms can be thought of as processes. A morphism takes something of type A and produces something of type B. We draw morphisms as:



Identity morphisms are special "do nothing" processes, which take something of type A and return the thing itself. We represent objects, and the identity morphisms on them, as empty wires:

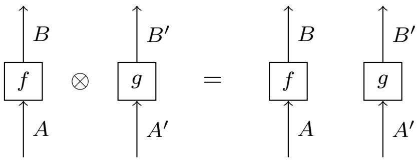


Morphisms are composed by connecting an output wire into an input wire:



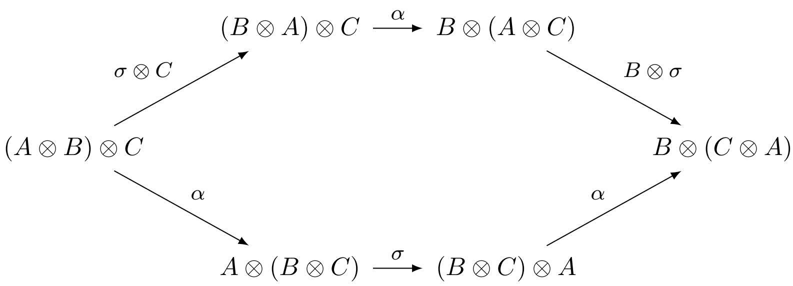
This notation neatly incorporates the assumption that composition is associative, and that composition with an identity has no effect.

The monoidal product of two morphisms is expressed as juxtaposition:

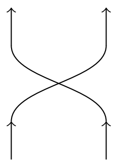


The monoidal product is also associative and unital, but possibly only up to isomorphism. The (identity on) the monoidal unit object I is denoted by the empty picture.

Definition 1.2 A *symmetric monoidal category* is a monoidal category with an additional natural isomorphism $\sigma_{A,B} : A \otimes B \to B \otimes A$, such that $\sigma^{-1}_{A,B} = \sigma_{B,A}$, $\rho_A = \lambda_A \circ \sigma_{A,I}$, and the following "hexagon" diagram commutes:

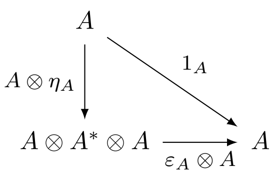


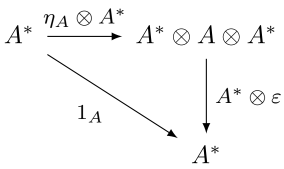
We draw symmetry maps as wire crossings:



This graphical notation unambiguously represents morphisms in symmetric monoidal categories [20]. Moreover, this representation is sound and complete with respect to the algebraic definition of a symmetric monoidal category. Our example monoidal category **FHilb** becomes symmetric by letting $\sigma_{H,K}(h \otimes k) := k \otimes h$, for all H and K.

Definition 1.3 A *compact category* is a symmetric monoidal category in which every object A comes with a *dual* object A^* and morphisms $\eta_A \colon I \to A^* \otimes A$ and $\varepsilon_A \colon A \otimes A^* \to I$ satisfying:

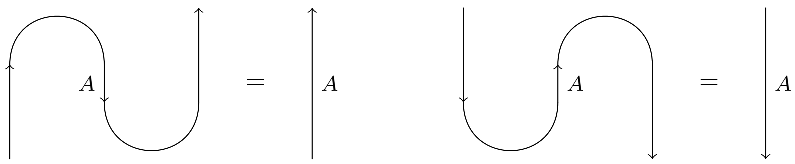




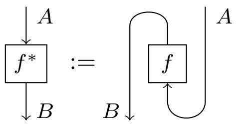
In the graphical notation, the object A^* is represented as a wire labelled A, but directed downward instead of upward:

We represent η_A as a cup, and ε_A as a cap:

The diagrams from the previous definition are called the "snake equations" because of their graphical representations:



In a compact category, any map $f\colon A \to B$ can also be considered as a map $f^*\colon A^* \to B^*$ by using caps and cups to "bend the wires" around:



Our example category **FHilb** is compact closed. For a finite-dimensional Hilbert space H, let H^* be the dual Hilbert space. Any orthonormal basis e_i for H then induces a basis $\overline{e_i}$ for H^*. Define $\varepsilon_H(e_i \otimes \overline{e_j}) = \delta_{ij}$ and $\eta_H(1) = \sum_i \overline{e_i} \otimes e_j$. These maps satisfy the snake equations and do not depend on the choice of basis e_i. Computing $f^* : B^* \to A^*$ in terms of ε and η yields the (operator) transpose of f, *i.e.* $f^*(\xi) = \xi \circ f$. This is not to be confused with the matrix transpose, which is basis-dependent (as it depends on fixing a *particular* isomorphism $A^* \cong A$).

Finally, we abstract the notion of conjugate-transpose.

Definition 1.4 A *dagger* on a category $\mathbf{V}$ is a contravariant functor $\dagger\colon \mathbf{V}^{\mathrm{op}} \to \mathbf{V}$ satisfying $A^\dagger = A$ for all objects A, and $f^{\dagger\dagger} = f$ for all morphisms f. A *unitary* in a dagger category is a map $u\colon A \to B$ with $u \circ u^\dagger = 1_B$ and $u^\dagger \circ u = 1_A$.

In particular, †-categories are always isomorphic with their opposite category. As the notation suggests, the † functor is an abstract version of the conjugate-transpose of a complex linear map. Thus, for linear maps, the abstract notion of unitary is precisely the usual one.

Definition 1.5 A *dagger compact category* is a compact category that comes with a dagger such that $(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger$, the structure maps α_A, λ_A, and $\sigma_{A,B}$ are all unitary, and $\varepsilon_A^\dagger = \eta_{A^*}$.

The role of conjugation in a dagger compact category is played by the *lower-star* operation: $f_* : A^* \to B^*$, which is defined as:

$$f_* := (f^\dagger)^* = (f^*)^\dagger$$

Our example category **FHilb** is dagger compact via the formula for adjoints: $\langle f(h) \,|\, k \rangle = \langle h \,|\, f^\dagger(k) \rangle$.

Finally, we will need the following notion of structure-preserving functor between dagger compact categories.

Definition 1.6 A functor $F\colon \mathbf{C} \to \mathbf{D}$ between dagger symmetric monoidal categories is a *dagger symmetric monoidal functor* when $F \circ \dagger = \dagger \circ F$ and it comes with an isomorphism $\psi\colon I \to F(I)$ and a natural isomorphism $\varphi_{A,B}\colon FA \otimes FB \to F(A \otimes B)$ making the following diagrams commute:

$$\begin{array}{ccccc}
(FA \otimes FB) \otimes FC & \xrightarrow{\varphi \otimes FC} & F(A \otimes B) \otimes FC & \xrightarrow{\varphi} & F((A \otimes B) \otimes C) \\
{\scriptstyle \alpha}\downarrow & & & & \downarrow{\scriptstyle F(\alpha)} \\
FA \otimes (FB \otimes FC) & \xrightarrow{FA \otimes \varphi} & FA \otimes F(B \otimes C) & \xrightarrow{\varphi} & F(A \otimes (B \otimes C))
\end{array}$$

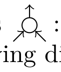
$$\begin{array}{ccc}
FA \otimes I & \xrightarrow{\rho} & FA \\
{\scriptstyle FA \otimes \psi}\downarrow & & \downarrow{\scriptstyle F(\rho^{-1})} \\
FA \otimes FI & \xrightarrow{\varphi} & F(A \otimes I)
\end{array}
\qquad
\begin{array}{ccc}
I \otimes FA & \xrightarrow{\lambda} & FA \\
{\scriptstyle \psi \otimes FA}\downarrow & & \downarrow{\scriptstyle F(\lambda^{-1})} \\
FI \otimes FA & \xrightarrow{\varphi} & F(I \otimes A)
\end{array}$$

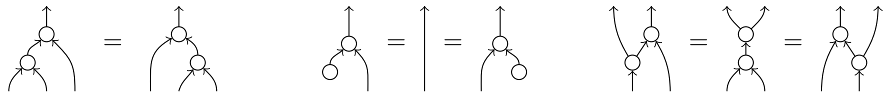
$$\begin{array}{ccc}
FA \otimes FB & \xrightarrow{\sigma_{FA,FB}} & FB \otimes FA \\
\downarrow{\scriptstyle \varphi_{A,B}} & & \downarrow{\scriptstyle \varphi_{B,A}} \\
F(A \otimes B) & \xrightarrow{F\sigma_{A,B}} & F(B \otimes A)
\end{array}$$

Preserving the dagger and the monoidal structure suffices to preserve the compact structure [17].

2 Abstract C*-algebras

This section defines so-called normalisable dagger Frobenius algebras. The running example investigates these structures in the category of finite-dimensional Hilbert spaces. As will turn out, they are precisely finite-dimensional C*-algebras. Therefore, we will think of normalisable dagger Frobenius algebras in arbitrary dagger compact categories as *abstract C*-algebras.*

Definition 2.1 A *dagger Frobenius algebra* is an object A in a dagger monoidal category together with morphisms  $: A \otimes A \to A$ and  $: I \to A$, called multiplication and unit, satisfying the following diagrammatic equations:



These identities are called associativity, unitality, and the Frobenius law. The maps (comultiplication) and and (counit) are defined as $(\;)^\dagger$ and $(\;)^\dagger$, respectively. They automatically satisfy coassociativity and counitality, which are the upside-down versions of associativity and unitality.

Example 2.2 An important example is the set $A = \mathbb{M}_n$ of n-by-n matrices with complex entries. This set is clearly an algebra: defining as $(a, b) \mapsto ab$ and $: \mathbb{C} \to A$ by $1 \mapsto 1_A$ satisfies associativity and unitality. The algebra A becomes a Hilbert space under the Hilbert–Schmidt inner product $\langle a \,|\, b\rangle = \mathrm{Tr}(a^\dagger b)$. It has a canonical orthonormal basis $\{e_{ij} \mid i, j = 1, \ldots, n\}$, where e_{ij} is the matrix all of whose entries vanish except for a one at location (i, j). We can now compute

$$(e_{ij}) = \langle (e_{ij}) \,|\, 1\rangle = \langle e_{ij} \,|\, (1)\rangle = \langle e_{ij} \,|\, 1_A\rangle = \mathrm{Tr}(e_{ji}) = \delta_{ij},$$

so that $: a \mapsto \mathrm{Tr}(a)$ by linearity. Similarly,

$$\langle (e_{ij}) \,|\, e_{kl} \otimes e_{pq}\rangle = \langle e_{ij} \,|\, (e_{kl} \otimes e_{pq})\rangle = \langle e_{ij} \,|\, \delta_{lp} e_{kq}\rangle = \delta_{ik}\delta_{jq}\delta_{lp},$$

whence $(e_{ij}) = \sum_l e_{il} \otimes e_{lj}$. With these explicit expressions it is easy to see

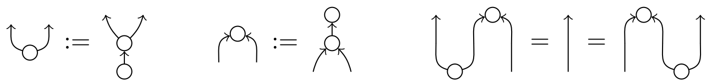
$$\begin{aligned} \circ (e_{ij} \otimes e_{kl}) &= (\delta_{jk} e_{il}) = \delta_{jk} \sum_p e_{ip} \otimes e_{pl} \\ &= \sum_p (\otimes \,)(e_{ij} \otimes e_{kp} \otimes e_{pl}) \\ &= (\otimes \,)(\sum_p e_{ij} \otimes e_{kp} \otimes e_{pl}) \\ &= (\otimes \,) \circ (\, \otimes)(e_{ij} \otimes e_{kl}). \end{aligned}$$

Similarly,

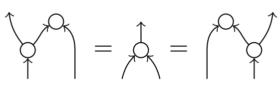
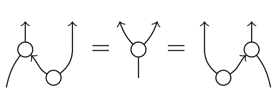
$$\circ (e_{ij} \otimes e_{kl}) = (\, \otimes) \circ (\otimes \,)(e_{ij} \otimes e_{kl}).$$

Linearity now shows that $(A, \;, \;)$ is a dagger Frobenius algebra in **FHilb**.

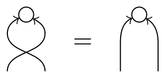
Any dagger Frobenius algebra defines a cap and a cup satisfying the snake identities.

 (1)

This cup and cap provide an alternative form of the Frobenius law that is sometimes more convenient:

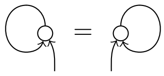
 or, equivalently 

Definition 2.3 A dagger Frobenius algebra $(A, \;, \;)$ is *symmetric* when it satisfies the following equation:

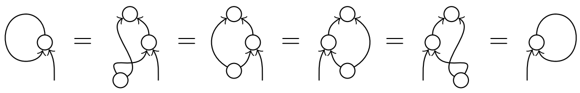


The dagger Frobenius algebra $\mathbb{M}_n$ in **FHilb** is symmetric by the cyclic property of the trace: $\mathrm{Tr}(ba) = \mathrm{Tr}(ab)$.

Proposition 2.4 *For any symmetric Frobenius algebra:*

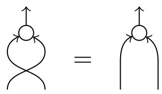


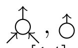
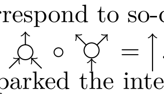
Proof Symmetry can be used to interchange traces with Frobenius caps and cups.

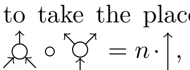


□

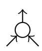
A dagger Frobenius algebra is certainly symmetric when it is *commutative*, *i.e.* when it satisfies the following equation:

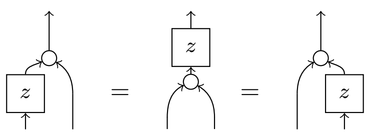


Being commutative is strictly stronger than being symmetric. For example, in **FHilb**, the algebra $\mathbb{M}_n$ is commutative precisely when $n = 1$. Nevertheless, there are plenty of commutative dagger Frobenius algebras in **FHilb**. For example, consider the subalgebra A of $\mathbb{M}_n$ consisting of matrices that are diagonal in some fixed orthogonal basis. It turns out that this is the only example: commutative dagger Frobenius algebras (A, ) in **FHilb** are in one-to-one correspondence with orthogonal bases of A; see [14]. Orthonormal bases correspond to so-called *special* algebras, *i.e.* those whose multiplication is an isometry: . This abstract characterisation of orthonormal bases is what first sparked the interest in Frobenius algebras in categorical quantum mechanics [13].

We will combine symmetric and commutative algebras as follows. If A and B are dagger Frobenius algebras in **FHilb**, then so is their direct sum $A \oplus B$. If A and B are symmetric or commutative, then so is $A \oplus B$. However, not many interesting, non-commutative algebras in **FHilb** are special, so we need to find a condition to take the place of specialness. Investigating matrix algebras $\mathbb{M}_n$, we note that , so we might think it suffices to consider algebras that are special *up to a scaling factor*. However, "scaled specialness", unlike specialness, is not preserved by direct sum. For example, if $A = \mathbb{M}_m$ and $B = \mathbb{M}_n$ the induced Frobenius algebra on of $A \oplus B$ is only special up to a scalar when $n = m$.

For this reason we will consider a more general condition, called *normalisability*. Before defining this concept, we introduce the notion of a central map.

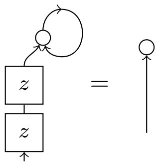
Definition 2.5 A map $z\colon A \to A$ is *central* for a multiplication  on A when:



The terminology derives from the usual notion of centre for *e.g.* a group, ring, algebra, etc. Left (or right) multiplication  $\circ (a \otimes -) \colon A \to A$ with an element $a \colon I \to A$ is a central map precisely when a is in the centre $Z(A) = \{a \in A \mid \forall b \in A \colon ab = ba\}$. Furthermore, all central maps of a Frobenius algebra arise this way.

A map $g \colon A \to A$ in a dagger category is called *positive* when $g = h^\dagger \circ h$ for some h. It is called *positive definite* if it is a positive isomorphism. Using these conditions, we can define normalisability as a well-behavedness property of the "loop"  $\circ$ .

Definition 2.6 A dagger Frobenius algebra $(A,$ , $)$ is *normalisable* when it comes with a central, positive definite $z \colon A \to A$ such that



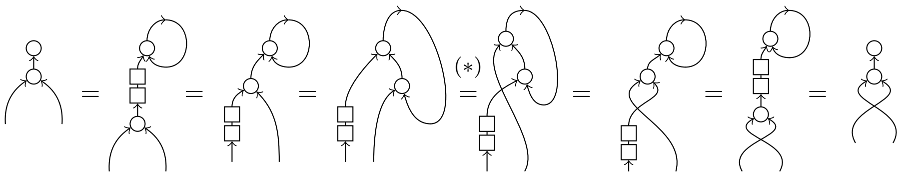
The map z is called the *normaliser*, and we will often depict it simply as .

The equation above uniquely fixes the map z^2, so normalisers are unique in any category where positive square roots are unique, when they exist (such as **FHilb**).

All special Frobenius algebras are symmetric. This turns out the be the case for dagger normalisable Frobenius algebras as well.

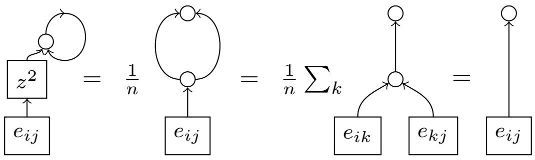
Proposition 2.7 *Normalisable dagger Frobenius algebras are symmetric.*

Proof Expand the counit.



Note that the step marked $(*)$ is just a diagram deformation: the two multiplication maps have traded places. This corresponds to cyclicity of the trace. □

The dagger Frobenius algebra $\mathbb{M}_n$ in **FHilb** is normalised by $z(a) = n^{-1/2}a$:

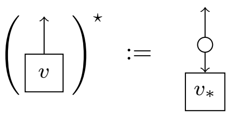


The point of normalisability is that the algebra $\mathbb{M}_m \oplus \mathbb{M}_n$ is also normalisable (but no longer special unless $m = n$), by the central map $z(a, b) = (m^{-1/2}a, n^{-1/2}b)$. Thus direct sums $\bigoplus_k \mathbb{M}_{n_k}$ of matrix algebras are normalisable dagger Frobenius algebras in **FHilb**. But these are precisely the finite-dimensional C*-algebras! This is a standard fact, see e.g. [16, Theorem III.1.1]. Recall that a finite-dimensional *C*-algebra* is a finite-dimensional algebra A equipped with an involution satisfying $\|a^*a\| = \|a\|^2$ (for some norm satisfying $\|ab\| \le \|a\|\|b\|$ that is then unique). The following theorem shows that this exhausts all examples of normalisable dagger Frobenius algebras in **FHilb**. Thus we may think of normalisable dagger Frobenius algebras in arbitrary categories as *abstract C*-algebras* (see also [37]).

We can show this directly by defining the C*-algebra structure in terms of the Frobenius algebra structure. First note that any Frobenius algebra fixes an isomorphism $A^* \cong A$ as follows:

These two maps are inverse because of snake identies from equation (1).

Theorem 2.8 *If* $(A, \ , \ , \ )$ *is a normalisable dagger Frobenius algebra in* **FHilb**, *then the following involution gives it the structure of a finite-dimensional C*-algebra:*



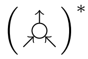
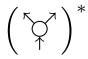
Conversely, up to isomorphism, all finite-dimensional C-algebras arise in this way.*

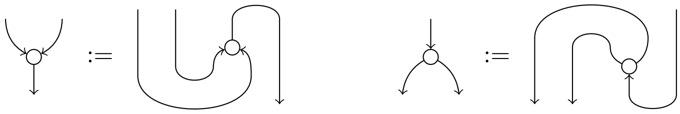
Proof Any dagger Frobenius algebra in **FHilb** is a C*-algebra under the involution above, and all finite-dimensional C*-algebras arise in this way [36], so it suffices to prove that any dagger Frobenius algebra $(A, \ , \ )$ in **FHilb** is normalisable. Since it is unitarily isomorphic to a C*-algebra of the form $\bigoplus_k \mathbb{M}_{n_k}$, there is an orthonormal basis $\{e_{ij}^{(k)} : 0 \leq i, j < n_k\}$ for A, in terms of which is defined as $e_{ij}^{(k)} \otimes e_{i'j'}^{(k')} \mapsto \delta_{kk'}\delta_{ji'}e_{ij'}^{(k)}$. Use this to compute $\mathrm{Tr}_A(\ )$ directly:

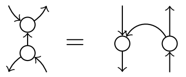
$$\begin{aligned}
\mathrm{Tr}_A(\ )(e_{ij}^{(k)}) &= \sum_{i'j'k'} \left(e_{i'j'}^{(k')}\right)^\dagger \ \left(e_{ij}^{(k)} \otimes e_{i'j'}^{(k')}\right) \\
&= \sum_{i'j'k'} \left(e_{i'j'}^{(k')}\right)^\dagger \delta_{kk'}\delta_{ji'}e_{ij'}^{(k)} \\
&= \sum_{j'} \left(e_{jj'}^{(k)}\right)^\dagger e_{ij'}^{(k)} \\
&= \sum_{j'} \delta_{ij} = n_k \delta_{ij}.
\end{aligned}$$

Also $(e_{ij}^{(k)}) = \delta_{ij}$. Therefore $e_{ij}^{(k)} \mapsto n_k^{-1/2} e_{ij}^{(k)}$ defines a normaliser: it is positive and invertible, satisfies $\mathrm{Tr}_A(\ ) \circ (\ )^2 = $, and acts by a constant scalar on each summand of A and so is central. □

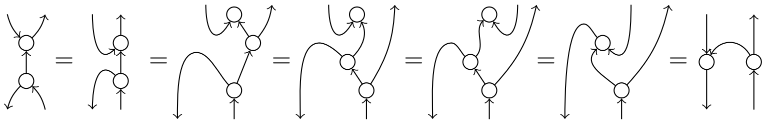
For future reference, we prove two lemmas about abstract C*-algebras, including an alternative form of the normalisability condition. As a matter of convention, we define the following shorthands:

We can use any such shorthand without ambiguity by stating that we always preserve the (cylic) ordering of inputs/outputs. That is, the left input of  will always be clockwise from the right input, and the right output of  will always be clockwise from the left output. This rule also applies to depictions of  and :

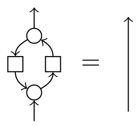


Lemma 2.9 *Any symmetric dagger Frobenius algebra satisfies*  .

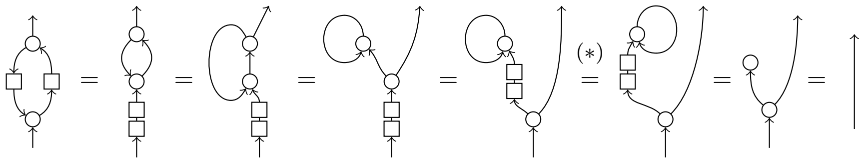
Proof Apply the Frobenius law and associativity.



The middle equation uses symmetry. □

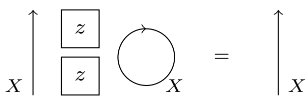
Lemma 2.10 *Any normalisable dagger Frobenius algebra satisfies*  .

Proof Use centrality of the normaliser, associativity, and unitality.



The marked equation follows from Proposition 2.4. □

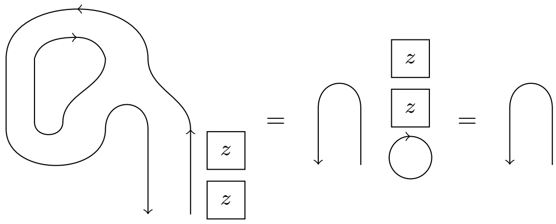
To end this section where it started, reconsider the algebra $\mathbb{M}_n$ in **FHilb**. It is isomorphic to $(\mathbb{C}^n)^* \otimes \mathbb{C}^n$ by $e_{ij} \mapsto \langle i| \otimes |j\rangle$, where $\{|0\rangle, \ldots, |n\rangle\}$ is any orthonormal basis of $\mathbb{C}^n$. As it turns out, this way of constructing C*-algebras works in the abstract, as long as the category is not too ill-behaved. To be precise, we call an object X in a dagger compact category *positive-dimensional* if there is a positive definite $z\colon I \to I$ satisfying



A dagger compact closed category is called positive-dimensional if all its objects are. All the categories we will consider are positive-dimensional.

Proposition 2.11 *In a positive-dimensional dagger compact category, every object of the form $H^* \otimes H$ carries a canonical normalisable dagger Frobenius algebra with the following multiplication and unit:*

Proof It follows immediately from compactness that this is a dagger Frobenius algebra. Positive-dimensionality provides a positive definite scalar z that satisfies $(z^2 \circ \mathrm{Tr}_X(1_X)) \otimes 1_X = 1_X$. Then:



Hence $1_{H^* \otimes H} \otimes z$ is a normaliser. □

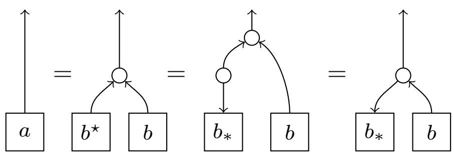
The abstract C*-algebra of the previous proposition is called an *abstract matrix algebra*, and is also denoted by $\mathcal{B}(H)$.

3 Abstract completely positive maps

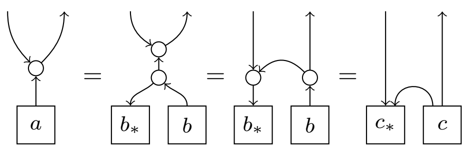
Having abstracted C*-algebras from **FHilb** to arbitrary categories, this section does the same for completely positive maps. This will lead to a fully abstract procedure, called the *CP*–construction*, that turns any dagger compact category (like **FHilb**) into the category of abstract C*-algebras and abstract completely positive maps.

First recall the definition of completely positive maps between C*-algebras. An element a of a C*-algebra A is *positive* when it is of the form $a = b^\star b$ for some $b \in A$. A linear function $f \colon A \to B$ between C*-algebras is *positive* when it takes positive elements to positive elements. It is *completely positive* when the function $f \otimes 1 \colon A \otimes \mathbb{M}_n \to B \otimes \mathbb{M}_n$ is positive for every natural number n. Completely positive maps form a large and well-studied class of transformations that send (possibly unnormalised) states of open systems to (possibly unnormalised) states, and hence account for dynamics [4,25,35]. There is some debate about whether other maps are in fact unphysical [3,26,33,38].

This definition translates to abstract C*-algebras as follows: an element $a \colon I \to A$ of an abstract C*-algebra (A, , ) is positive when $a =$  $(b^\star \otimes b)$ for some $b \colon I \to A$. Expanding definitions, we see that $a \colon I \to A$ is positive when

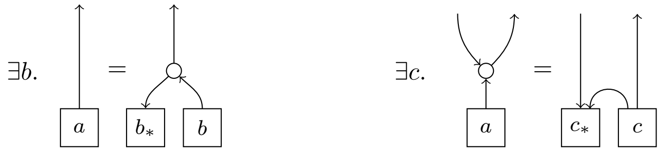


for some $b \colon I \to A$. By Lemma 2.10, this implies:



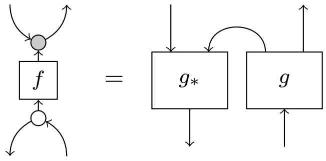
for some object X and $c\colon I \to X \otimes A$; the middle equation follows from Lemma 2.9.

In fact, for Hilbert spaces, the following two characterisations of positive element a are equivalent:



However, in other categories, the implication from left to right is strict. For this reason, we will take the weaker notion to define an abstract positive element.

This abstract description of positive elements generalises to maps $f\colon A \to B$ between abstract C*-algebras (A, ,) and (B, ,) as follows: there are an object X and a map $g\colon A \to X \otimes B$ satisfying



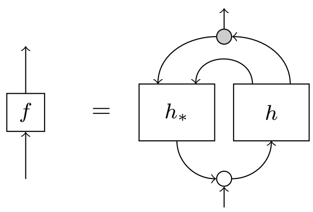
(2)

The *positive elements* of A are then precisely the maps $I \to A$ satisfying this condition. Equation (2) is called the *CP*–condition*. Proposition 3.4 below shows that this is precisely the right condition to capture complete positivity abstractly. But before that, the following lemma records that it indeed makes sense to take tensor products of abstract C*-algebras.

Lemma 3.1 *If* (A, , ,) *and* (B, , ,) *are normalisable dagger Frobenius algebras in a dagger compact category, then so is* ($A \otimes B$, , ,).

Proof All the required properties – associativity, unitality, the Frobenius law, and normalisability – follow easily from the graphical calculus for dagger compact categories. □

Incidentally, Lemmas 2.9 and 2.10 provide an alternative form of the CP*–condition that is sometimes more convenient: equation (2) holds if and only if



(3)

for some object X and morphism $h\colon A \to X \otimes B$.

Proposition 3.4 below shows that if a map $A \to B$ satisfies the CP*–condition (2), then its composition with another map $I \to A$ satisfying that condition still satisfies that condition. It is in fact easier to first prove the more general result that the CP*–condition is closed under composition, *i.e.* that maps satisfying (2) form a category. In fact, the rest of this section shows that if $\mathbf{V}$ is a dagger compact category, then so is the category of abstract C*-algebras in $\mathbf{V}$ and maps satisfying (2), that we now officially define.

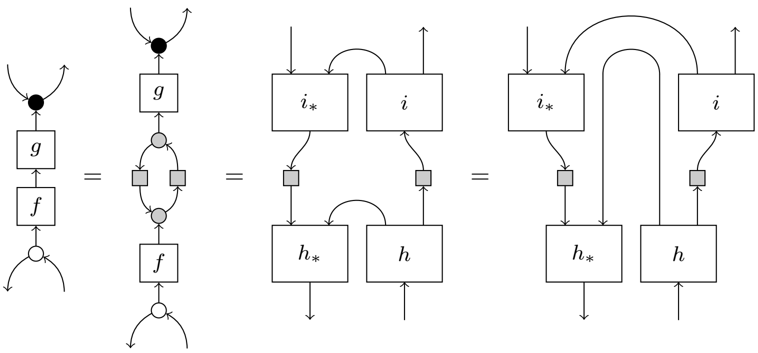
Definition 3.2 Given a dagger compact category **V**, we define the data for a new category $\mathbf{CP}^*[\mathbf{V}]$. Objects are normalizable dagger Frobenius algebras in **V**. Morphisms $(A, \ ) \to (B, \ )$ are morphisms $f\colon A \to B$ in **V** satisfying the CP*–condition (2).

The next theorem shows that $\mathbf{CP}^*[\mathbf{V}]$ is a well-defined category inheriting composition and identities from **V**. In fact, it also inherits tensor products from **V** by Lemma 3.1, and then becomes a dagger compact category.

Theorem 3.3 *If* **V** *is a dagger compact category,* $\mathbf{CP}^*[\mathbf{V}]$ *is again a well-defined dagger compact category.*

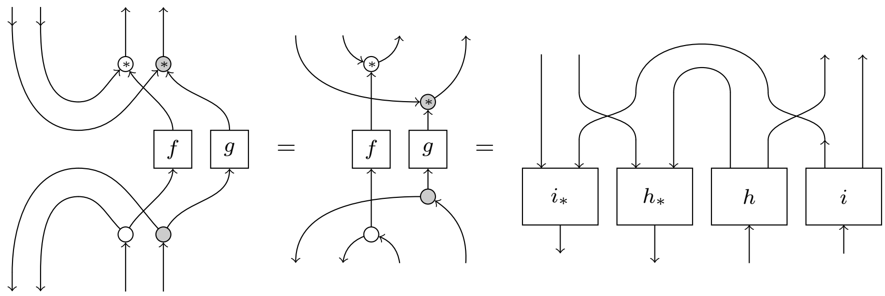
Proof Identity maps $1_A\colon (A, \ ) \to (A, \ )$ satisfy the $\mathbf{CP}^*$-condition by Lemma 2.9, where the role of g in equation (2) is played by .

Next, suppose $f\colon (A, \ ) \to (B, \ )$ and $g\colon (B, \ ) \to (C, \ )$ satisfy the CP*–condition. It then follows from Lemma 2.10 that their composition does, too.

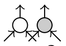
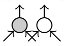


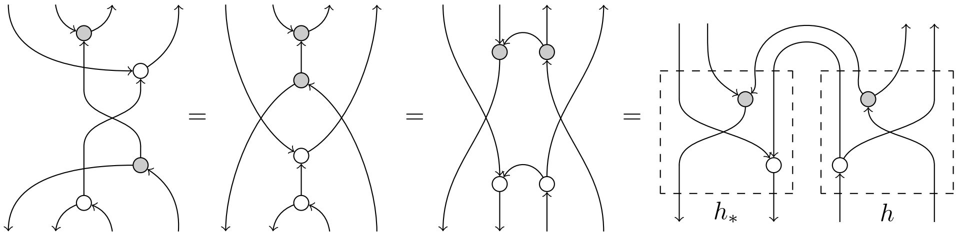
Thus $\mathbf{CP}^*[\mathbf{V}]$ is indeed a well-defined category.

Lemma 3.1 gives monoidal structure on the level of objects. Given a morphism $f\colon (A, \ ) \to (C, \ )$ with Kraus map h, and $g\colon (B, \ ) \to (D, \ )$ with Kraus map i, then $f \otimes g\colon (A \otimes B, \ ) \to (C \otimes D, \ )$ satisfies the CP*–condition:



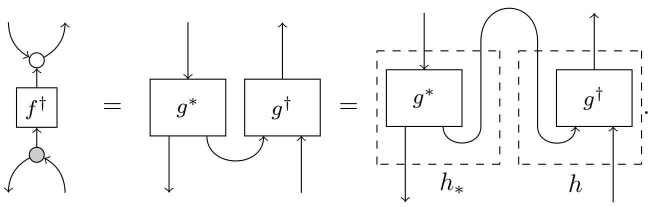
Note that (I, ρ_I), where $\rho_I\colon I \otimes I \to I$ is the coherence isomorphism of **V**, is a normalisable dagger Frobenius algebra by the coherence theorem. Using this definition of $\otimes$ and I, the coherence isomorphisms α, λ, and ρ from **V** trivially satisfy the CP*–condition. Thus $\mathbf{CP}^*[\mathbf{V}]$ is a monoidal category.

To show that $\mathbf{CP}^*[\mathbf{V}]$ inherits symmetry, it suffices to show that the swap map $\sigma_{A,B} \colon A \otimes B \to B \otimes A$ of $\mathbf{V}$ lifts to a morphism $\sigma_{A,B} \colon (A \otimes B,$  $) \to (B \otimes A,$  $)$ in $\mathbf{CP}^*[\mathbf{V}]$. This can be done with two applications of Lemma 2.9.

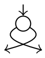


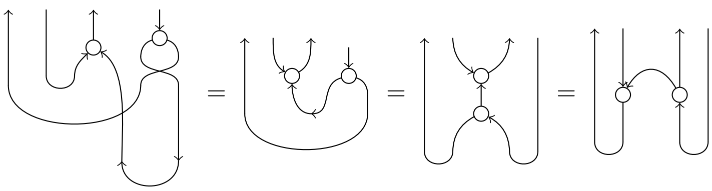
Thus $\mathbf{CP}^*[\mathbf{V}]$ is a symmetric monoidal category.

The category $\mathbf{CP}^*[\mathbf{V}]$ also inherits the dagger from $\mathbf{V}$. If $f \colon (A,$  $) \to (B,$  $)$ satisfies (2), then so too does $f^\dagger$: because  $\circ f^\dagger \circ$  $= \Big($  $\circ f \circ$  $\Big)^\dagger$,



Since the coherence isomorphisms of $\mathbf{CP}^*[\mathbf{V}]$ are those of $\mathbf{V}$, they are unitary, and thus $\mathbf{CP}^*[\mathbf{V}]$ is a dagger symmetric monoidal category.

Finally, for compactness, let $(A,$  $)$ be an object in $\mathbf{CP}^*[\mathbf{V}]$. Let A^* be a dual of A, with cap $\varepsilon_{A^*} \colon A^* \otimes A \to I$. If a Frobenius algebra is dagger normalisable, so too are the opposite algebra and the transposed algebra (*i.e.* the dual). Thus $(A^*,$  $)$ is a well-defined object of $\mathbf{CP}^*[\mathbf{V}]$. Now, $\varepsilon_{A^*} \colon (A,$  $) \otimes (A^*,$  $) \to I$ satisfies the CP*–condition, again by Lemma 2.9:



We have already showed that the dagger of a map satisfying the CP*–condition also satisfies the CP*–condition, so finally let $\eta_A = \varepsilon^\dagger_{A^*}$. We complete the proof by noting that σ, η, and ε are all defined with the same underlying maps as in $\mathbf{V}$, so the symmetry and snake equations are automatically satisfied. □

We have constructed a category whose objects are abstract C*-algebras, and we claim that the morphisms are abstract completely positive maps. The following proposition justifies that claim, by showing that maps satisfying the CP*–condition correspond exactly to maps that are completely positive in the usual sense, in that $f : A \to B$ applied to a positive (open) state preserves positivity.

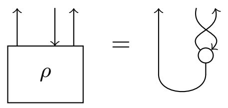
Proposition 3.4 *Let* $(A, ○)$ *and* $(B, ◍)$ *be normalisable dagger Frobenius algebras and* $f\colon A \to B$ *a morphism in a dagger compact category. The following are equivalent:*

(a) f *satisfies the CP*–condition* (2)*;*

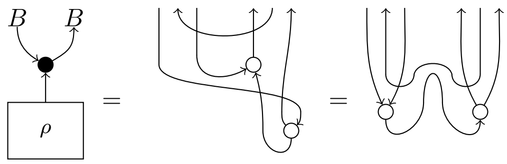
(b) $f \otimes 1_C$ *sends positive elements of* $(A, ○) \otimes (C, ●)$ *to positive elements of* $(B, ◍) \otimes (C, ●)$ *for all normalisable dagger Frobenius algebras* $(C, ●)$*;*

(c) $f \otimes 1_C$ *sends positive elements of* $(A, ○) \otimes (X^* \otimes X, ⌣⌢)$ *to positive elements of* $(B, ◍) \otimes (X^* \otimes X, ⌣⌢)$ *for all objects* X*.*

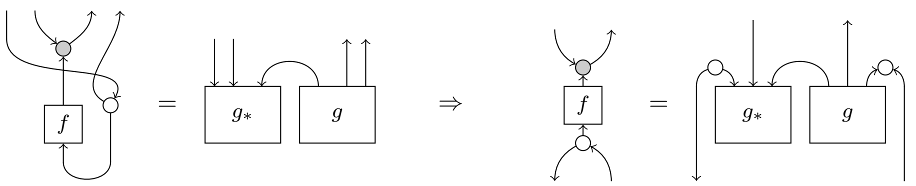
Proof For (a) ⇒ (b): if ρ is a positive element of $(A, ○) \otimes (C, ●)$, then it can be regarded as a morphism $\rho\colon I \to (A, ○) \otimes (C, ●)$ in $\mathbf{CP}^*[\mathbf{V}]$. It then follows from Theorem 3.3 that $(f \otimes 1_C) \circ \rho$ is also a morphism in $\mathbf{CP}^*[\mathbf{V}]$, so it must also be a positive element. The implication (b) ⇒ (c) is trivial. Finally, for (c) ⇒ (a): setting $X = A^*$, the following is a positive element of $(B, ●) := (A, ○) \otimes (X^* \otimes X, ⌣⌢)$.



Indeed, graphical rewriting using the Frobenius law and symmetry shows:



So, by assumption, $(f \otimes 1_{A^*}) \circ \rho$ is also a positive element. Applying white caps to both sides establishes that f satisfies the CP*–condition.



This finishes the proof. □

The previous proposition is a fully abstract version of Stinespring's dilation theorem [34], or rather (because our abstract C*-algebras are finite-dimensional) of Choi's theorem [6]. The morphism g in equation (2) therefore called a *Kraus map* for f; we emphasise that it is not unique. Traditional formulations in **FHilb** allow a sum of Kraus maps; this is expressed abstractly by the indexing object X in (2).

The abstract C*-algebras $(C, ●)$ and $(X^* \otimes X, ⌣⌢)$ in the previous proposition are called the ancillary system, or *ancilla*. In these terms, the previous proposition shows that the CP*–condition (2) characterises those maps that preserve positivity even when their input and output systems are regarded as open subsystems of larger systems. In fact, the previous proposition does slightly better than Choi's theorem, because the ancilla can be an arbitrary abstract C*-algebra instead of just an abstract matrix algebra.

Because of the way we have modeled the definition of $\mathbf{CP}^*[\mathbf{V}]$ after the case of **FHilb**, the category $\mathbf{CP}^*[\mathbf{FHilb}]$ is indeed that of (concrete) finite-dimensional C*-algebras and completely positive maps, as the following proposition records.

Proposition 3.5 $\mathbf{CP}^*[\mathbf{FHilb}]$ *is equivalent to the category of finite-dimensional C*-algebras and completely positive maps.*

Proof Define a functor E from $\mathbf{CP}^*[\mathbf{FHilb}]$ to the category of finite-dimensional C*-algebras and completely positive maps, acting on objects as in Theorem 2.8 and as the identity on morphisms. This functor is then essentially surjective on objects by that theorem. Furthermore, Proposition 3.4 shows that $E(f)$ is a completely positive map between concrete C*-algebras if and only if f satisfies the CP*–condition. This makes E a well-defined functor that is full. It is faithful by construction, and hence it is an equivalence of categories. □

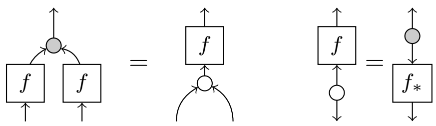
Remark 3.6 We have employed complex Hilbert spaces. It is natural to wonder about performing the CP*–construction on real finite-dimensional Hilbert spaces.

On the level of objects, Theorem 2.8 still goes through: abstract C*-algebras in the category of real finite-dimensional Hilbert spaces correspond to so-called finite-dimensional *real C*-algebras* (see [23]). However, these need not be direct sums of complex matrix algebras; rather, they are direct sums of algebras of matrices over the real numbers, complex numbers, or over the quaternions [23, Theorem 5.7.1].

On the level of morphisms, Proposition 3.4 still holds. However, in the real case these morphisms do not give all completely positive maps [29, Theorem 4.3]. The underlying issue is that there are more positive elements in real C*-algebras than those of the form a^*a.

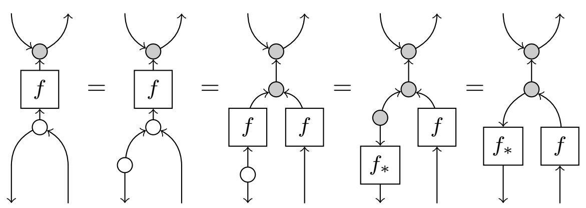
In the concrete case, *-homomorphisms between C*-algebras are automatically completely positive. We conclude this section by proving this holds fully abstractly, providing an easy way to show that some maps are morphisms in $\mathbf{CP}^*[\mathbf{V}]$.

Definition 3.7 If $(A, \,)$ and $(B, \,)$ are dagger normalisable Frobenius algebras, a morphism $f\colon A \to B$ is called a **-homomorphism* when it satisfies the following equations.



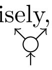
Lemma 3.8 *Let* $(A, \,)$ *and* $(B, \,)$ *be dagger normalisable Frobenius algebras in a dagger compact category* $\mathbf{V}$. *If* $f\colon A \to B$ *is a *-homomorphism, then it is a well-defined morphism in* $\mathbf{CP}^*[\mathbf{V}]$.

Proof Graphical manipulation shows the following.



Hence this morphism is a composition of $f^* \otimes f$ and  $\circ$ . Both morphisms are completely positive, *i.e.* of the form of the right-hand side of equation (2): the former by construction, the latter by Lemma 2.9. Therefore f is also completely positive by Theorem 3.3, and hence a morphism in $\mathbf{CP}^*[\mathbf{V}]$. □

4 Completely classical systems and completely quantum systems

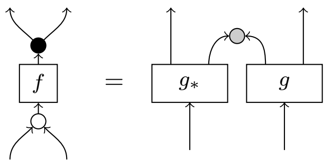
As discussed in Section 2, commutative abstract C*-algebras (A, ) in **FHilb** correspond to orthogonal bases of A. More precisely, the basis vectors are the *copyable points*, *i.e.* morphisms $p\colon I \to A$ that satisfy  $\circ\, p = p \otimes p$. Expanding arbitrary vectors in this basis, one can show that the normalised positive elements of A are precisely those vectors with positive coefficients summing to 1. Thus, normalised positive elements of a commutative abstract C*-algebra may be regarded as probability distributions over its copyable points. That is, we may think of commutative abstract C*-algebras as "completely classical" systems.[2]

On the other hand, Section 2 showed that abstract matrix algebras can be regarded as "completely quantum" systems: their states have no probabilistical mixing aspect at all. In general, abstract C*-algebras are combinations of "completely classical" and "completely quantum" parts. This section focuses on these two extreme cases. It proves that the CP*–construction subsumes earlier constructions that remained separate: the Stoch–construction into its "completely classical" part [12], and the so-called CPM–construction into its "completely quantum" part [30, 10, 5]. Thus the CP*–construction combines the two, and places classical and quantum systems and channels on an equal footing in a single category.

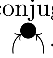
4.1 Completely classical systems

First, recall the Stoch–construction [12]. Like the CP*–construction of the previous section, it turns a dagger compact category $\mathbf{V}$ into a new one, $\mathbf{Stoch}[\mathbf{V}]$. It will turn out that it is precisely the full subcategory of $\mathbf{CP}^*[\mathbf{V}]$ consisting of commutative abstract C*-algebras, and that we may regard it as the subcategory of classical channels.

Objects of $\mathbf{Stoch}[\mathbf{V}]$ are commutative normalisable dagger Frobenius algebras. Morphisms (A, ) $\to$ (B, ) in $\mathbf{Stoch}[\mathbf{V}]$ are morphisms $f\colon A \to B$ in $\mathbf{V}$ with



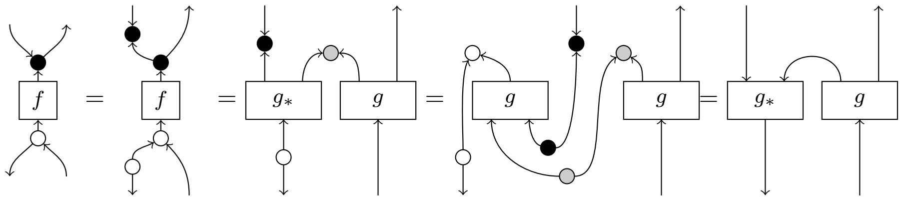
(4)

for some commutative normalisable dagger Frobenius algebra (X, ) and a morphism $g\colon A \to X \otimes B$ in $\mathbf{V}$. Here, the conjugation g_* is taken with respect to the caps and cups induced by , , and .

[2] Commutativity might be too strong a notion of "completely classical" system in the abstract. A weaker notion of broadcastability, that coincides with commutativity in **FHilb**, seems more reasonable. Subsequent work will investigate such more operational notions of classicality.

Theorem 4.1 *For a dagger compact category* **V***, the category* **Stoch**[**V**] *is isomorphic to the full subcategory of* $\mathbf{CP}^*[\mathbf{V}]$ *consisting of all commutative normalisable dagger Frobenius algebras.*

Proof We show that (4) implies (2).



The converse holds since the dualisers , , and , are always invertible. ◻

The following corollary justifies thinking of **Stoch**[**V**] as a category of classical channels. We call a morphism $f\colon (A, \;) \to (B, \;)$ in $\mathbf{CP}^*[\mathbf{V}]$ *normalised* if it preserves counits: $\circ f =$. Recall that a *stochastic map* between finite-dimensional Hilbert spaces is a matrix with positive real entries whose every column sums to one.

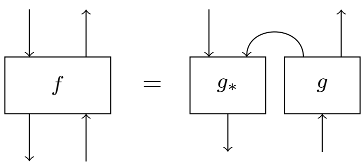
Corollary 4.2 *Normalised morphisms in* **Stoch**[**FHilb**] *correspond to stochastic maps between finite-dimensional Hilbert spaces.*

Proof Combine Theorem 4.1, Proposition 3.5 and [21, 3.2.3 and 2.1.3]. ◻

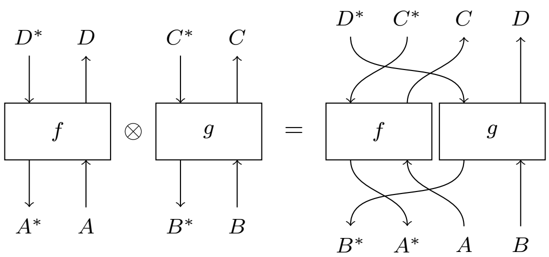
4.2 Completely quantum systems

First, we recall the CPM–construction [30,10]. Like the CP*–construction of the previous section, it turns a dagger compact category **V** into a new one, **CPM**[**V**]. It will turn out that it is precisely the full subcategory of $\mathbf{CP}^*[\mathbf{V}]$ consisting of abstract matrix algebras $\mathcal{B}(H) = (H^* \otimes H, \;)$, that are simply identified with H, and that we may regard it as the subcategory of quantum channels.

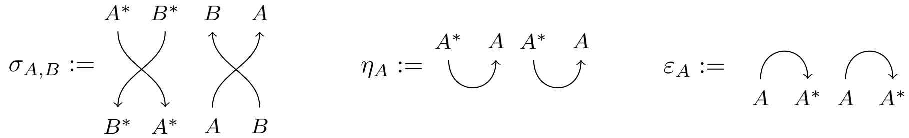
Objects of **CPM**[**V**] are the same as those of **V**, and morphisms $f\colon A \to B$ in **CPM**[**V**] are morphisms $f\colon A^* \otimes A \to B^* \otimes B$ in **V** for which there exist an object X and a morphism $g\colon A \to X \otimes B$ satisfying:



Composition, identity maps, and $\otimes$ on objects of **CPM**[**V**] are as in **V**. The tensor product is defined on morphisms of **CPM**[**V**] as follows:



CPM[**V**] inherits symmetry and compact structure from **V**, only "doubled".

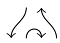


The following theorem proves that **CPM**[**V**] embeds in **CP***[**V**], preserving all structure. To formulate that embedding, recall that a functor F is dagger symmetric monoidal if it comes with natural unitary isomorphisms $\varphi_{A,B}\colon F(A\otimes B)\to F(A)\otimes F(B)$ satisfying $\varphi_{I,A}=\varphi_{A,I}=1_A$ and

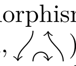
$$\begin{array}{ccc} F(A\otimes B\otimes C) & \xrightarrow{\varphi_{A,B\otimes B}} & F(A)\otimes F(B\otimes C) \\ {\scriptstyle \varphi_{A\otimes B,C}}\downarrow & & \downarrow{\scriptstyle 1_{F(A)}\otimes\varphi_{B,C}} \\ F(A\otimes B)\otimes F(C) & \xrightarrow[\varphi_{A,B}\otimes 1_{F(C)}]{} & F(A)\otimes F(B)\otimes F(C) \end{array} \qquad \begin{array}{ccc} F(A\otimes B) & \xrightarrow{\varphi_{A,B}} & F(A)\otimes F(B) \\ {\scriptstyle F(\sigma_{A,B})}\downarrow & & \downarrow{\scriptstyle \sigma_{F(A),F(B)}} \\ F(B\otimes A) & \xrightarrow[\sigma_{B,A}]{} & F(B)\otimes F(A) \end{array}$$

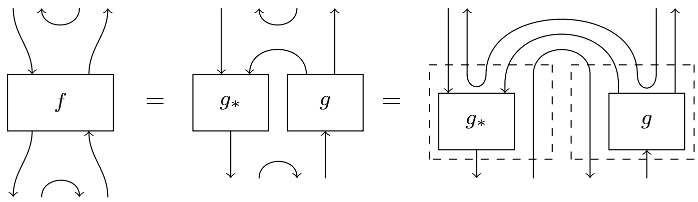
For simplicity, we have assumed that the categories involved are strict monoidal.

Theorem 4.3 *If* **V** *is a positive-dimensional dagger compact category,*

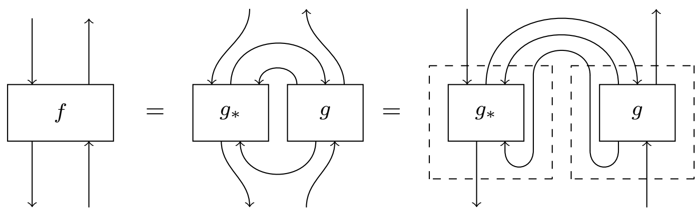
$\mathcal{B}(A)=(A^*\otimes A,$ $) \qquad \mathcal{B}(f)=f$

defines a functor $\mathcal{B}\colon$ **CPM**[**V**] $\to$ **CP***[**V**] *that is full, faithful, and dagger symmetric monoidal.*

Proof First of all, $\mathcal{B}$ is well-defined, because a morphism $f\colon A^*\otimes A\to B^*\otimes B$ in **V** determines a morphism $A\to B$ in **CPM**[**V**] precisely when if it determines a morphism $(A^*\otimes A,$ $)\to(B^*\otimes B,$ $)$ in **CP***[**V**]. Indeed, if f is a morphism in **CPM**[**V**], it also satisfies the CP*–condition:



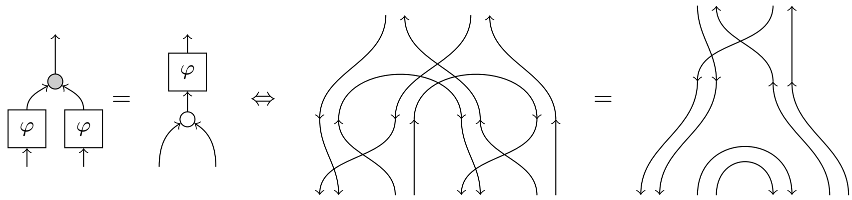
Conversely, if f is in **CP***[**V**], then it is also in **CPM**[**V**]:

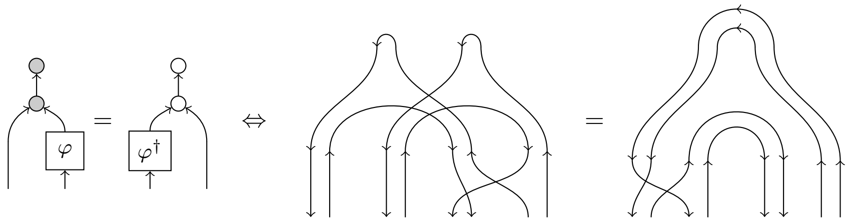


Composition is defined identically in **CPM**[**V**] and **CP***[**V**], so $\mathcal{B}$ is functorial, full, and faithful.

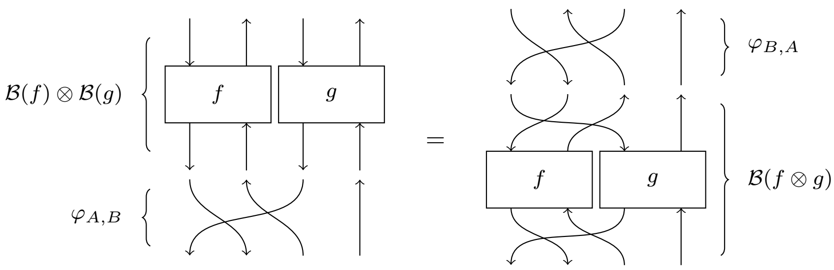
Define $\varphi_{A,B}\colon\mathcal{B}(A\otimes B)\to\mathcal{B}(A)\otimes\mathcal{B}(B)$ as the following "reshuffling map".

To verify that this defines a morphism in $\mathbf{CP}^*[\mathbf{V}]$, it suffices to show that it is a *-homomorphism by Lemma 3.8.

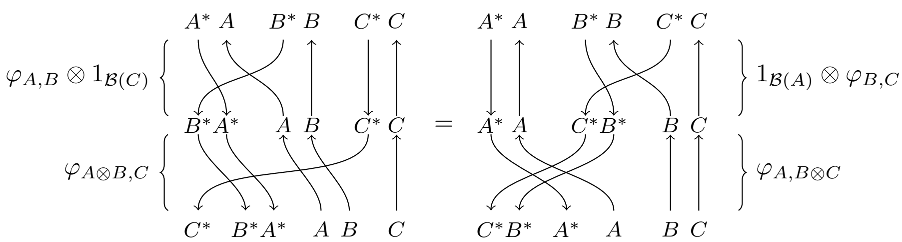




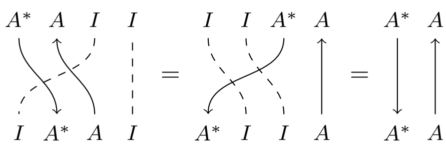
Next, we show naturality of φ:



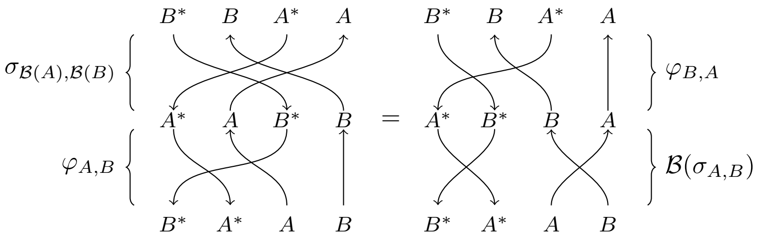
The last thing that remains to be shown is coherence for φ with respect to the symmetric monoidal structure. For associativity:



As for the unit equations:



Finally, as for symmetry:



Thus $\mathcal{B}$ is a full, faithful, dagger symmetric monoidal functor. □

As a consequence of the previous theorem and Proposition 3.5, the category **CPM**[**FHilb**] is equivalent to the category of matrix algebras and completely positive maps. This justifies thinking of the "completely quantum" part of $\mathbf{CP}^*[\mathbf{V}]$ as a category of quantum channels.

The category **CPM**[**FHilb**] is strictly smaller than the category $\mathbf{CP}^*[\mathbf{FHilb}]$ of all finite-dimensional C*-algebras and completely positive maps. That is, the embedding $\mathcal{B}$ of the previous theorem does not extend to an equivalence of categories: for example, the finite-dimensional C*-algebra $A = \mathbb{M}_1 \oplus \mathbb{M}_2$ cannot be isomorphic to a matrix algebra $\mathbb{M}_n$ because $\dim(A) = 1^2 + 2^2 = 5 \neq n^2 = \dim(\mathbb{M}_n)$.

In analogy to the case $\mathbf{V} = \mathbf{FHilb}$, it stands to reason to regard objects H of $\mathbf{V}$ as systems whose state space consists of pure states, and objects $\mathcal{B}(H)$ of $\mathbf{CP}^*[\mathbf{V}]$ as systems whose state space consists of mixed states. So one might think that the "pure" category $\mathbf{V}$ should embed into the "mixed" category $\mathbf{CP}^*[\mathbf{V}]$. The following corollary shows that this is indeed the case.

Corollary 4.4 *If* $\mathbf{V}$ *is a dagger compact category,*

$$A \mapsto \mathcal{B}(A) \qquad f \mapsto f^* \otimes f$$

defines a dagger symmetric monoidal functor $\mathbf{V} \to \mathbf{CP}^*[\mathbf{V}]$*.*

Proof Combine the previous theorem with [30, Theorem 4.20]. □

There are no meaningful functors in the opposite directions. A construction $\mathbf{CP}^*[\mathbf{V}] \to \mathbf{V}$ would model decoherence, which cannot be a structure preserving functor. More precisely, the functor $\mathbf{V} \to \mathbf{CP}^*[\mathbf{V}]$ does not have any adjoints, because it does not preserve (co)limits: $\mathcal{B}(H \oplus K) \ncong \mathcal{B}(H) \oplus \mathcal{B}(K)$ for nontrivial Hilbert spaces H and K. Similarly, a functor $\mathbf{CP}^*[\mathbf{V}] \to \mathbf{CPM}[\mathbf{V}]$ would need to coherently turn an (abstract) C*-algebra into an (abstract) matrix algebra. Again, it cannot be an adjoint because it cannot preserve (co)limits.

5 Nonstandard models

So far, we have abstracted classical and quantum systems and channels from the category **FHilb** to arbitrary dagger compact categories $\mathbf{V}$. Now it is high time to see some other examples. This section considers three: the category of sets and relations, the category of matrices with positive entries, and the category of relations with values in a cancellative quantale. We will see that abstract C*-algebras in these categories turn out to be important well-known structures, that are nevertheless quite different from concrete C*-algebras.

5.1 Relations

First, recall the category **Rel**. Its objects are sets, and morphisms $A \to B$ are relations $R \subseteq A \times B$. The composition of $R\colon A \to B$ and $S\colon B \to C$ is given by

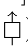
$$S \circ R = \{(a, c) \in A \times C \mid \exists b \in B\colon (a, b) \in R, (b, c) \in S\},$$

and $\{(a, a) \mid a \in A\}$ is the identity on A. Cartesian product makes **Rel** into a compact category. Finally, it becomes a dagger compact category by

$$R^\dagger = \{(b, a) \mid (a, b) \in R\}.$$

We start by investigating the objects of $\mathbf{CP}^*[\mathbf{Rel}]$. This immediately shows that these nonstandard abstract C*-algebras are quite different from C*-algebras (in **FHilb**): they are precisely groupoids. Recall that a *groupoid* is a category whose morphisms are all invertible [24].

Proposition 5.1 *Normalisable dagger Frobenius algebras in* **Rel** *are (in one-to-one correspondence with) groupoids.*

Proof By [18, Theorem 7], it suffices to show that normalisability implies speciality in **Rel**. Let (A, , , ) be a normalisable dagger Frobenius algebra in **Rel**. Then the normaliser  is an isomorphism. In **Rel**, this means  $= \{(a, z(a)) \mid a \in A\}$ for a bijection $z\colon A \to A$. But  is also positive, and hence self-adjoint. Since all isomorphisms in **Rel** are unitary, z equals its own inverse. Therefore  $\circ$  $= 1_A$, that is, (A, ) is special. □

Explicitly, the set $A = \mathrm{Mor}(\mathbf{G})$ of morphisms of a groupoid **G** becomes an abstract C*-algebra in **Rel** under

 $= \{((g, f), g \circ f) \mid f$ and g are composable morphisms in $\mathbf{G}\}$,

 $= \{(*, 1_a) \mid a$ is an object of $\mathbf{G}\}$.

The proof of the previous proposition illustrates that we may take  $= 1_A$, but that normalisers of dagger Frobenius algebras are not unique.

Next, we determine the morphisms of $\mathbf{CP}^*[\mathbf{Rel}]$.

Definition 5.2 A relation $R \subseteq \mathrm{Mor}(\mathbf{G}) \times \mathrm{Mor}(\mathbf{H})$ between groupoids **G** and **H** *repects inverses* when $(g, h) \in R$ implies $(g^{-1}, h^{-1}) \in R$ and $(1_{\mathrm{dom}(g)}, 1_{\mathrm{dom}(h)}) \in R$.

Proposition 5.3 *The category* $\mathbf{CP}^*[\mathbf{Rel}]$ *is isomorphic to the category of groupoids and relations respecting inverses.*

Proof Unfolding definitions shows that a morphism $R \subseteq (A \times A) \times (B \times B)$ in **Rel** is completely positive, *i.e.* is of the form of the right-hand side of equation (2), precisely when

$$((a, a'), (b, b') \in R \implies ((a', a), (b', b) \in R,\ ((a, a), (b, b)) \in R. \qquad (*)$$

If **G** and **H** are groupoids, corresponding to Frobenius algebras (G, ) and (H, ), and $R \subseteq G \times H$, then

 $= \{((g, g'), g^{-1} \circ g') \in G^3 \mid g^{-1} \text{ and } g' \text{ are composable}\},$

 $\circ R \circ$  $= \{((g, g'), (h, h')) \in G^2 \times H^2 \mid g^{-1} \text{ and } g' \text{ are composable}\},$ $h^{-1} \text{ and } h' \text{ are composable}\},$ $(g^{-1} \circ g', h^{-1} \circ h') \in R\}.$

Substituting this into (∗) translates precisely into R respecting inverses. □

Next we investigate "completely quantum" objects in $\mathbf{CP}^*[\mathbf{Rel}]$. Recall that a category is *indiscrete* when there is precisely one morphism between each two objects. Indiscrete categories are automatically groupoids.

Proposition 5.4 *The objects in* $\mathbf{CP}^*[\mathbf{Rel}]$ *that are isomorphic to* $\mathcal{B}(A)$ *for some set* A *are (in one-to-one correspondence with) indiscrete groupoids.*

Proof By definition, $\mathcal{B}(A)$ corresponds to a groupoid whose set of morphisms is $A \times A$, and whose composition is given by

$$(b_2, b_1) \circ (a_2, a_1) = \begin{cases} (b_2, a_1) & \text{if } b_1 = a_2, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

We deduce that the identity morphisms of $\mathcal{B}(A)$ are the pairs (a_2, a_1) with $a_2 = a_1$. So objects of $\mathcal{B}(A)$ just correspond to elements of A. Similarly, we find that the morphism (a_2, a_1) has domain a_1 and codomain a_2. Hence (a_2, a_1) is the unique morphism $a_1 \to a_2$ in $\mathcal{B}(A)$. □

In other words, the essential image of the embedding $\mathcal{B} \colon \mathbf{CPM}[\mathbf{Rel}] \to \mathbf{CP}^*[\mathbf{Rel}]$ is the full subcategory of $\mathbf{CP}^*[\mathbf{Rel}]$ consisting of indiscrete groupoids.

There are many more connections between the theory of groupoids and abstract C*-algebras. For example, projections in an abstract C*-algebra in **Rel** are precisely the connected components of its corresponding groupoid [11, Lemma 22].

5.2 Positive matrices

To conclude this section, we consider categories that are in some sense between the categories **FHilb** and **Rel**; the former can be thought of as involving matrices over the complex numbers, whereas the latter can be thought of as involving matrices over the two element set. We will consider matrices ranging over other domains.

We start with the category $\mathbf{Mat}(\mathbb{R}_{\geq 0})$. Its objects are natural numbers, and a morphism $m \to n$ is an m-by-n matrix whose entries are nonnegative real numbers, *i.e.* elements of $[0, \infty)$. Composition is matrix multiplication, and identity matrices give identity morphisms. Tensor product acts as multiplication on objects, and as Kronecker product on morphisms.

We will determine the objects of $\mathbf{CP}^*[\mathbf{Mat}(\mathbb{R}_{\geq 0})]$ by reducing to $\mathbf{CP}^*[\mathbf{FHilb}]$. There is an obvious dagger symmetric monoidal functor $\mathbf{Mat}(\mathbb{R}_{\geq 0}) \to \mathbf{FHilb}$, sending n to $\mathbb{C}^n$ with its canonical basis. Hence a normalisable dagger Frobenius algebra (n, , , ) in $\mathbf{Mat}(\mathbb{R}_{\geq 0})$ also defines a C*-algebra structure on $\mathbb{C}^n$. Recall that any finite-dimensional C*-algebra A can be written in standard form as $A \cong \bigoplus_k \mathbb{M}_{n_k}$.

Definition 5.5 Write $E_n = \{e_{ij} \mid i, j = 1, \ldots, n\}$ for the standard basis of $\mathbb{M}_n$. By the *matrix* of a linear map $f\colon \mathbb{M}_m \to \mathbb{M}_n$, we mean the function $F\colon E_m \times E_n \to \mathbb{C}$ given by the entries $F(e_{ij}, e_{kl}) = \langle e_{kl} \mid f\big(e_{ij}\big)\rangle = \mathrm{Tr}(e_{lk} f(e_{ij}))$. This definition extends to linear maps $f\colon \bigoplus_k \mathbb{M}_{m_k} \to \bigoplus_l \mathbb{M}_{n_l}$ between finite-dimensional C*-algebras in standard form. We say that f is *really positive* when its matrix F has entries in $\mathbb{R}_{\geq 0}$. If f is completely positive and really positive, we call it *really completely positive.*

Proposition 5.6 *The category* $\mathbf{CP}^*[\mathbf{Mat}(\mathbb{R}_{\geq 0})]$ *is isomorphic to the category of finite-dimensional C*-algebras in standard form and really completely positive maps.*

Proof The fact that the functor $\mathbf{Mat}(\mathbb{R}_{\geq 0}) \to \mathbf{FHilb}$ is dagger symmetric monoidal and faithful implies that the induced functor $\mathbf{CP}^*[\mathbf{Mat}(\mathbb{R}_{\geq 0})] \to \mathbf{CP}^*[\mathbf{FHilb}]$ is also dagger symmetric monoidal and faithful. It is full by construction, and injective on objects. Hence it suffices to show that it is surjective on objects. First, observe that the structure maps , , and  of the C*-algebra $\mathbb{M}_n$ are really completely positive. The matrices $M\colon E_n^3 \to \mathbb{R}_{\geq 0}$ for multiplication, $U\colon E_n \to \mathbb{R}_{\geq 0}$ for the unit, and $Z\colon E_n^2 \to \mathbb{R}_{\geq 0}$ then take the form

$$\begin{aligned} M(e_{ij}, e_{kl}, e_{pq}) &= \delta_{jk}\delta_{ip}\delta_{lq}, \\ U(e_{ij}) &= \delta_{ii}, \\ N(e_{ij}) &= 1/\sqrt{n}. \end{aligned}$$

Hence $\mathbb{M}_n$ is in the image of the functor $\mathbf{CP}^*[\mathbf{Mat}(\mathbb{R}_{\geq 0})] \to \mathbf{CP}^*[\mathbf{FHilb}]$. Because $\mathbf{CP}^*[\mathbf{Mat}(\mathbb{R}_{\geq 0})]$ has biproducts, C*-algebras in standard form are reached, too. □

Finally, let us consider matrices with entries ranging over other sets of positive numbers, such as the unit interval $[0, 1]$. To be precise, we will consider the category $\mathbf{Mat}(Q)$, where Q is a cancellative commutative quantale. Recall that a *quantale* is a partial order $(Q, \leq)$ that has suprema of arbitrary subsets, together with a commutative multiplication $(Q, \cdot, 1)$ satisfying

$$x \cdot (\bigvee_i y_i) = \bigvee_i x \cdot y_i.$$

It is cancellative when $x \cdot y = x \cdot z$ implies $y = z$ or $x = 0$, where $0 = \bigvee \emptyset$. For more information we refer to [28]. The extended nonnegative real numbers $[0, \infty]$ form an example under the usual ordering and multiplication, as does the unit interval $[0, 1]$. Another example is the Boolean algebra $\{0, 1\}$ under the usual ordering and multiplication.

The category $\mathbf{Mat}(Q)$ has sets as objects, morphisms $A \to B$ are Q-valued matrices, *i.e.* functions $A \times B \to Q$. Composition of $R\colon A \to B$ and $S\colon B \to C$ is

$$S \circ R(a, c) = \bigvee_b R(a, b) \cdot S(b, c).$$

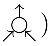
Cartesian product and matrix transpose makes this into a dagger compact category, very much like **Rel**. In fact, notice that $\mathbf{Mat}(\{0, 1\}) = \mathbf{Rel}$.[3]

[3] Notice also that $\mathbb{R}_{\geq 0}$ is not a quantale under its usual ordering.

Lemma 5.7 *Any normalisable dagger Frobenius algebra in* $\mathbf{Mat}(Q)$ *induces a groupoid.*

Proof Let $(A,$ , , $)$ be a normalisable dagger Frobenius algebra in $\mathbf{Mat}(Q)$. Notice that there is a (unique) homomorphism $f\colon Q \to \{0,1\}$ of quantales such that $f(x) = 0$ if and only if $x = 0$, namely

$$f(x) = \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{otherwise.} \end{cases}$$

It induces a dagger symmetric monoidal functor $f^*\colon \mathbf{Mat}(Q) \to \mathbf{Rel}$; see also [2, Section 5.2]. Therefore $G := A$ becomes a dagger Frobenius algebra in $\mathbf{Rel}$ with multiplication $f^*($$)$ and unit $f^*($$)$. Moreover, $f^*($$) = f^*($$) \circ f^*($$)^2$ by normalisability. But, as in the proof of Proposition 5.1, $f^*($$)$ is a positive isomorphism in $\mathbf{Rel}$, and so $f^*($$)^2 = 1_G$. At this point Lemma 2.10 guarantees that $(G, f^*($$), f^*($$), 1_G)$ is a normalisable dagger Frobenius algebra in $\mathbf{Rel}$, which corresponds to a groupoid by Proposition 5.1. □

The previous lemma shows that if we "collapse" the matrix $M\colon G^3 \to Q$ of multiplication to $M\colon G^3 \to \{0,1\}$, it becomes the multiplication table of a groupoid. Similarly, the matrix $U\colon G \to Q$ becomes the set of identities of that groupoid. The only freedom left is what nonzero elements of Q to place in the nonzero entries of these matrices. It is easy to obtain several constraints on these values [2, Section 5.2]. However, in general, $\mathbf{CP}^*[\mathbf{Mat}(Q)]$ does not seem to correspond to a familiar category such as $\mathbf{CP}^*[\mathbf{Rel}]$. We refrain from explicating it further, but note that it does provide a nonstandard model that lends itself to easy calculation, for example to find counterexamples.

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