The category of matroids

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Abstract The structure of the category of matroids and strong maps is investigated: it has coproducts and equalizers, but not products or coequalizers; there are functors from the categories of graphs and vector spaces, the latter being faithful and having a nearly full Kan extension; there is a functor to the category of geometric lattices, that is nearly full; there are various adjunctions and free constructions on subcategories, inducing a simplification monad; there are two orthogonal factorization systems; some, but not many, combinatorial constructions from matroid theory are functorial.

1 Introduction

Matroids are very general structures that capture the notion of independence. They were introduced by Whitney in 1935 as a common generalization of independence in linear algebra and independence in graph theory [24]. They were linked to projective geometry by Mac Lane shortly thereafter [17], and have found a great many applications in geometry, topology, combinatorial optimization, network theory, and coding theory since [23,19]. However, matroids have mostly been studied individually, and most authors do not consider maps between matroids. The only explicitly categorical work we are aware of is [3–6,16,22]. The purpose of this article is to survey the properties of the category of matroids.

Of course, speaking of "the" category of matroids entails a choice of morphisms. There are compelling reasons to choose so-called strong maps, which are a natural choice of structure-preserving functions:

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(Co)limit	present
Products	✗
Equalizers	✓
Pullbacks	✗
Coproducts	✓
Coequalizers	✗
Pushouts	✗
Exponentials	✗

Construction	functorial
Duality	✗
Addition	✓
Deletion	✓
Contraction	✓
Subobject	✓
Extension	✗
Truncation	✗
Erection	✗

Construction	monoidal
Union	✗
Intersection	✗
Half-dual union	✗
Intertwining	✗
Parallel connection	✓
Series connection	✓

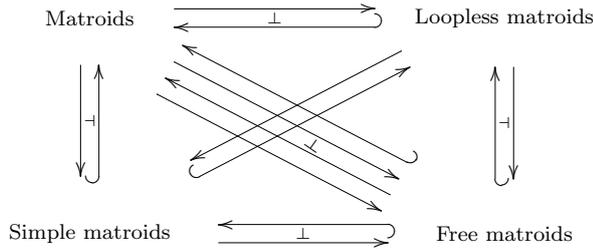**Fig. 1** Overview of the categorical nature of matroids

Fig. 2 Overview of adjunctions involving the category of matroids

- Subobjects then correspond to minors, a well-established notion in matroid theory that naturally models 'submatroids', as we will see in Section 3;
- The construction making matroids out of vector spaces is then functorial, and in fact nearly fully so, as we will show in Section 4.
- There is then an elegant orthogonal factorization system, as we will show in Section 7;

The resulting category has some limits and colimits, but not all, as we will show in Section 5, see Figure 1. It will also turn out that many combinatorial constructions from matroid theory are not even functorial, as we will show in Section 8, see Figure 1. Weaker choices of morphisms, such as the so-called weak maps and comaps [23], lead to less well-behaved categories.

To complete the overview of this article: Section 2 recalls the various equivalent definitions of the objects of the category that we will need, and Section 6 establishes adjunctions between various subcategories of well-studied types of matroids, showing that simplification is monadic. See Figure 2.

One feature of matroid theory that we leave for future work is duality: functoriality of this construction needs a choice of morphisms that stands to strong maps as relations stand to functions. Finally, generalized matroids called bimatroids [15] have a distinctly 2-categorical flavour to them, that we do not go in to here.

2 Objects

Matroids, the objects of our category of interest, can be defined in many equivalent ways. We list some that are used later, writing $\#X$ for the cardinality of a set X.

Although some of the theory of matroids goes through for infinite sets [8], that situation is much more intricate. Therefore we will only consider the finite case.

Definition 2.1 A *matroid M* consists of a finite *ground set* $|M|$ with, equivalently:

- a family of $\mathcal{I}$ of subsets of $|M|$, called the *independent sets*, satisfying:
    - nontrivial: the empty set is independent;
    - downward closed: if $I \in \mathcal{I}$ and $J \subseteq I$ then also $J \in \mathcal{I}$;
    - *exchange*: if $I, J \in \mathcal{I}$ and $\#I < \#J$, then $I \cup \{e\} \in \mathcal{I}$ for some $e \in J \setminus I$.

  a maximal independent set is called a *basis*, and their collection is denoted $\mathcal{B}$;
- a family $\mathcal{F}$ of subsets of $|M|$, called the *closed sets* or *flats*, satisfying:
    - nontrivial: $|M|$ itself is closed;
    - closed under intersection: if $F, G \in \mathcal{F}$ then also $F \cap G \in \mathcal{F}$;
    - *partitioning*: if $\{F_1, F_2, \ldots\}$ are the minimal flats properly containing a flat F, then $\{F_1 \setminus F, F_2 \setminus F, \ldots\}$ partitions $|M| \setminus F$.
- a *rank function* $\mathrm{rk}\colon 2^{|M|} \to \mathbb{N}$, satisfying:
    - bounded: $0 \leq r(X) \leq \#|M|$ for all $X \subseteq |M|$;
    - monotonic: if $X \subseteq Y \subseteq |M|$, then $\mathrm{rk}(X) \leq \mathrm{rk}(Y)$;
    - valuation: if $X, Y \subseteq |M|$, then $\mathrm{rk}(X \cup Y) + r(X \cap Y) \leq \mathrm{rk}(X) + \mathrm{rk}(Y)$.

To see how to turn the data of one of the equivalent definitions into the data of another, define the *closure* operation $\mathrm{cl}\colon 2^{|M|} \to 2^{|M|}$ by

$$\mathrm{cl}(X) = \big\{ x \in |M| \;\big|\; \mathrm{rk}(X \cup \{x\}) = \mathrm{rk}(X) \big\}.$$

A closed set or flat is then a subset of $|M|$ which equals its closure, and $\mathrm{rk}(X)$ is the size of the largest independent set contained in $X \subseteq |M|$. Maximal flats are also called *hyperplanes*; their collection is denoted by $\mathcal{H}$.

Some elements of a matroid are of particular interest.

Definition 2.2 A *loop* is an element of a matroid that is not contained in any independent set, or equivalently, an element that is contained in all flats. An *isthmus* is an element that is included in every basis. Nonloop elements of the same rank-1 flat are called *parallel*.

The following special types of matroids are of special interest in matroid theory.

Definition 2.3 A matroid is *pointed* when it has a distinguished loop, denoted $\bullet$ and called *the point*. A (pointed) matroid is:
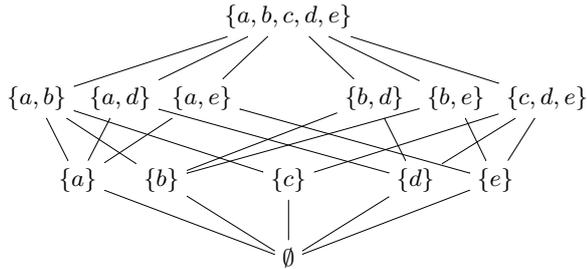
- *loopless* when it has no loops (other than the point);
- *simple* when it has no loops (other than the point) or parallel elements;
- *free* when every subset (not containing the point) is closed and independent.

Here are some typical examples.

Example 2.4 Any finite vector space V gives rise to a matroid $M(V)$, whose ground set is V, and whose independent sets are those subsets of V that are linearly independent; flats correspond to vector subspaces of V, the closure operation takes linear spans, and the rank function computes the dimension of the linear span. The matroid $M(V)$ is pointed when choosing the distinguished loop $\bullet$ to be the linearly dependent subset $\{0\}$. The pointed matroid $M(V)$ is free only when V is zero-dimensional or when $V = \mathbb{Z}_2$. (In fact, infinite vector spaces also satisfy the axioms for independence, but we will not consider the infinite setting here.)

Example 2.5 Any (undirected) graph G gives rise to a matroid $M(G)$, whose ground set consists of the edges, and where a subset is independent when it contains no cycles. Loops of $M(G)$ are precisely edges of G between a vertex and itself. Isthmuses are edges that are not contained in any cycle. Parallel elements of $M(G)$ are precisely parallel edges of G. We can point $M(G)$ by choosing a loop. The matroid $M(G)$ is simple when it has no loops and no parallel edges. It is free only when G has no cycles.

Example 2.6 We can specify a matroid M by giving (the Hasse diagram of) its partially ordered set $L(M)$ of flats. For example:

$$\{a, b, c, d, e\}$$

$$\{a,b\} \quad \{a,d\} \quad \{a,e\} \qquad \{b,d\} \quad \{b,e\} \quad \{c,d,e\}$$

$$\{a\} \quad \{b\} \quad \{c\} \quad \{d\} \quad \{e\}$$

$$\emptyset$$

is a matroid with ground set $\{a, b, c, d, e\}$. As we will see later, any geometric lattice L gives rise to a matroid $M(L)$ this way. The matroid $M(L)$ is simple. It is free only when L is a full powerset lattice.

3 Morphisms

Matroids form a category with strong maps between them.

Definition 3.1 A *strong map* from M to N is a function $f\colon |M| \to |N|$ such that the inverse image of any flat in N is a flat in M. Write **Matr** for the category of matroids and strong maps, and **LMatr**, **SMatr**, **FMatr** for the full subcategories of loopless, simple, and free matroids.

A strong map between pointed matroids is *pointed* when it sends the point to the point. Write **Matr$_\bullet$** for the category of pointed matroids and pointed strong maps, and **LMatr$_\bullet$**, **SMatr$_\bullet$**, **FMatr$_\bullet$** for the full subcategories of loopless, simple, and free matroids.

The following lemma shows that we may often switch between the pointed and non-pointed points of view.

Lemma 3.2 *The category* **Matr$_\bullet$** *is a reflective subcategory of* **Matr***: the inclusion* **Matr$_\bullet$** $\hookrightarrow$ **Matr** *has a left adjoint* $(-)_\bullet\colon$ **Matr** $\to$ **Matr$_\bullet$** *defined by*

$$M_\bullet = (|M| \cup \{\bullet\}, \mathcal{I})$$

$$f_\bullet(x) = \begin{cases} x & \text{if } x \in |M| \\ \bullet & \text{if } x = \bullet \end{cases}$$

but no right adjoint.

Proof Let M be a matroid, and observe that $M_\bullet$ is indeed a well-defined pointed matroid. Define a function $\eta_M \colon M \to M_\bullet$ by $x \mapsto x$.

$$
\begin{array}{ccc}
M & \xrightarrow{\ \eta_M\ } & M_\bullet \\
 & \searrow{\scriptstyle f} & \ \Big\downarrow{\scriptstyle \hat{f}} \\
 & & N
\end{array}
$$

If N is a pointed matroid, and $f \colon M \to N$ a strong map, there is a unique strong pointed map $\hat{f}$ making the above diagram commute, called the *transpose* of f, namely $\hat{f}(\bullet) = \bullet$ and $\hat{f}(x) = f(x)$ for $x \in |M|$. Thus $(-)_\bullet$ is left adjoint to the inclusion.

Suppose the functor R were a right adjoint. Consider the empty matroid O with $|O| = \emptyset$. For any pointed matroid N, there are no functions $N \to O$. But this contradicts the fact that there does exist a pointed strong map $N \to R(O)$, namely the map that sends every element of N to $\bullet$. □

It follows that the categories **Matr$_\bullet$** and **Matr** are not equivalent. In particular, **Matr$_\bullet$** has a zero object: the one-element matroid is both initial and terminal. This matroid is a terminal object in **Matr**, but not initial.

Next we investigate special kinds of morphisms, starting with monomorphisms.

Lemma 3.3 *A morphism of* **Matr** *or* **Matr$_\bullet$** *is monic if and only if it is injective.*

Proof Suppose $m \colon M \to N$ is an injective strong (pointed) map. If morphisms $f, g \colon P \to M$ satisfy $m \circ f = m \circ g$, then for all $x \in P$ clearly $m(f(x)) = m(g(x))$ and so $f(x) = g(x)$, whence $f = g$, making m monic.

Conversely, assume $m \colon M \to N$ is a monomorphism, and suppose $m(x) = m(y)$. Consider the free (pointed) matroid P with ground set consisting of the element $*$ (and the point $\bullet$). Then $f, g \colon P \to M$ defined by $f(*) = x$ and $g(*) = y$ are strong (pointed) maps satisfying $m \circ f = m \circ g$. Hence $f = g$, and so $x = y$. □

Lemma 3.4 *A morphism of* **Matr** *or* **Matr$_\bullet$** *is an isomorphism if and only if it is bijective and the direct image of any flat is a flat.*

Proof If $f \colon M \to N$ is an isomorphism, it is injective by Lemma 3.3. Also $N \ni y = f(f^{-1}(y))$, so f is surjective, and hence bijective. The direct image under f of a flat F equals the inverse image under f^{-1} of F, and therefore is a flat in N because f^{-1} is strong. The latter argument also establishes the converse. □

Let us recall some standard terminology from matroid theory.

Definition 3.5 Let M be a matroid. The *deletion* of $Y \subseteq |M|$ from M is the matroid $M \setminus Y$ with ground set $|M| \setminus Y$ and rank function $X \mapsto \mathrm{rk}_M(X)$. The resulting matroid is said to be *embedded* in M, and the strong map $M \setminus Y \to M$ is called an *embedding*. The *contraction* of M by $Z \subseteq |M|$ is the matroid M/Z with ground set $|M| \setminus Z$ and rank function $X \mapsto \mathrm{rk}_M(X \cup Z) - \mathrm{rk}_M(Z)$. A *minor* of M is the matroid resulting from a sequence of deletions and contractions of M.

We can identify the (categorical) subobjects, that is, equivalence classes of monomorphisms $M \rightarrowtail N$, where two such monomorphisms are equivalent when there is an isomorphism $M \simeq M'$ making the triangle commute.

Lemma 3.6 *Subobjects of M in* **Matr** *are precisely minors of M.*

Proof See [23, Chapter 8]. □

Next we move to the dual notion of subobjects, (categorical) quotients: that is, equivalence classes of epimorphisms $M \twoheadrightarrow N$ where two such epimorphisms are equivalent when there is an isomorphism $N \simeq N'$ making the triangle commute.

Lemma 3.7 *A morphism of* **Matr** *or* **Matr**$_\bullet$ *is epic if and only if it is surjective.*

Proof Suppose $e\colon M \to N$ a surjective strong (pointed) map. If morphisms $f, g\colon N \to P$ satisfy $f \circ e = g \circ e$, then for all $y \in N$ there is $x \in M$ with $y = e(x)$, so that $f(y) = f(e(x)) = g(e(x)) = g(y)$, whence $f = g$, making e epic.

Conversely, assume $e\colon M \to N$ is an epimorphism, and suppose $y \in N$ is not in the image of e. Write P for the (pointed) matroid with ground set $\{0, 1, \bullet\}$ that is also its only flat. Define $f\colon N \to P$ by sending (the point to the point and) y to 0, and the rest to 1. Define $g\colon N \to P$ by sending (the point to the point and) everything (else) to 1. Both are well-defined (pointed) strong maps, and satisfy $f \circ e = g \circ e$. But then $f = g$, showing that y was in the image of e after all. □

Definition 3.8 A matroid Q is a *(matroid) quotient* of M if there exist a matroid N and some $X \subseteq |N|$ so that $M = N \setminus X$ and $Q = N/X$.

Hence by definition, (matroid) quotients are strong maps that are composed of a contraction after an embedding. The rest of this section proves that these are precisely the bijective strong maps, from which it follows by Lemma 3.7 that matroid quotients are not categorical quotients in the category of matroids. Theorem 4.8 in the next section shows that they do correspond to categorical quotients in a related category, that we now introduce.

The flats of a matroid M, when ordered by inclusion, form a geometric lattice $L(M)$, where the height of an element is the rank of the corresponding flat; and conversely, every geometric lattice is the lattice of some matroid [19, Theorem 1.7.5]. This gives another way to think about strong maps between matroids.

Lemma 3.9 *For $M, N \in$ **Matr** *and $f\colon |M| \to |N|$ the following are equivalent:*

(a) f is a strong map;
(b) $\mathrm{rk}(f(Y)) - \mathrm{rk}(f(X)) \leq \mathrm{rk}(Y) - \mathrm{rk}(X)$ for all $X \subseteq Y \subseteq |M|$;
(c) the function $L(f)\colon L(M) \to L(N)$ given by $X \mapsto \mathrm{cl}(f(X))$ preserves joins and sends elements of height 1 to elements of height 0 or 1.

Proof See [23, Propositions 8.1.3 and 8.1.6]. □

We can now derive that contractions, like embeddings, are strong maps.

Corollary 3.10 *If M is a matroid and $Z \subseteq |M|$, there is a strong map $M \to M/Z$.*

Proof For $A = X \cup Z$, it follows from Definition 2.1 that

$$\mathrm{rk}_M(A) + \mathrm{rk}_M(Y) \geq \mathrm{rk}_M(A \cup Y) + \mathrm{rk}_M(A \cap Y),$$

so $\mathrm{rk}_M(Y) - \mathrm{rk}_M(A \cap Y) \geq \mathrm{rk}_M(A \cup Y) - \mathrm{rk}_M(A)$, whence

$$\begin{aligned}
\mathrm{rk}_M(Y) - \mathrm{rk}_M(X) &\geq \mathrm{rk}_M(Y \cup Z) - \mathrm{rk}_M(X \cup Z) \\
&= \mathrm{rk}_M(Y \cup Z) - \mathrm{rk}_M(Z) - (\mathrm{rk}_M(X \cup Z) - \mathrm{rk}_M(Z)) \\
&= \mathrm{rk}_{M/Z}(Y) - \mathrm{rk}_{M/Z}(X).
\end{aligned}$$

Lemma 3.9 now establishes the result. □

Finally we establish that matroid quotients are precisely bijective strong maps.

Lemma 3.11 *A function $f: M \to N$ is a bijective strong map if and only if it factors as an embedding $M \to M \backslash X = Q$ followed by a contraction $Q = N/X \to N$.*

$$
\begin{array}{ccc}
 & Q & \\
{\scriptstyle embedding}\nearrow & & \searrow{\scriptstyle contraction} \\
M & \xrightarrow{\quad f \quad} & N
\end{array}
$$

Proof Sufficiency follows from Corollary 3.10, necessity is proven by Higgs [13]. □

4 Functors

This section considers functors into and out of the category of matroids from the categories of geometric lattices, vector spaces, and graphs. We will need the following general notions. Recall that the *essential image* of a functor $F: \mathbf{C} \to \mathbf{D}$ is the smallest subcategory of $\mathbf{D}$ that contains all morphisms of the form $F(f)$ and all isomorphisms to/from objects of the form $F(C)$.

Definition 4.1 A functor $F: \mathbf{C} \to \mathbf{D}$ is *nearly full* when any morphism g between objects in the essential image of F is of the form $g = F(f)$ for a morphism f in $\mathbf{C}$.

Any full functor is nearly full, but the converse is not true in general: the domain and codomain of g may not be of the form $F(M)$ for objects M in $\mathbf{C}$. Any essentially full functor in the sense of [12, Definition 5.10] is nearly full, and the converse holds precisely when the essential image of F is all of $\mathbf{D}$.

For the next notion, recall that the category of left actions on sets of a monoid M is monoidal [18, VII.4], so one can consider categories enriched in it. Any locally small monoidal category is an example via scalar multiplication [11, 1].

Definition 4.2 Let M be a monoid, and $\mathbf{C}$ a category enriched in left M-actions. A functor $F: \mathbf{C} \to \mathbf{D}$ is *nearly faithful* when $F(f) = F(g)$ implies $f = m \cdot g$ for some $m \in M$.

Any faithful functor is nearly faithful, but the converse it not true in general. Intuitively, a nearly faithful functor is 'faithful up to a scalar in M'.

4.1 Geometric lattices

We start with a functor from the category of matroids to the following category of geometric lattices, extending Example 2.6.

Definition 4.3 Write **GLat** for the category with geometric lattices as objects and as morphisms functions that preserve joins and that map every atom to either an atom or to the least element.

Note that morphisms in **GLat** are completely determined by their action on atoms, as in geometric lattices all elements are joins of atoms.

Proposition 4.4 *There is a functor $L\colon \mathbf{Matr} \to \mathbf{GLat}$ that sends a matroid M to its lattice $L(M)$ of flats ordered by inclusion, and sends a strong map $f\colon M \to N$ to the function $L(f)\colon L(M) \to L(N)$ given by $X \mapsto \mathrm{cl}(f(X))$.*

Proof Lemma 3.9 guarantees that $L(f)$ is a well-defined morphism in $\mathbf{GLat}$. If $X \in L(M)$ then $X = \mathrm{cl}(X)$, so L preserves identities. Finally, to see that L preserves composition, let $f\colon M \to N$ and $g\colon N \to P$ be strong maps. For $X \in L(M)$ then $\mathrm{cl}(f(M)) \in L(N)$, so because g is strong $\mathrm{cl}(g(f(X))) = \mathrm{cl}(g(\mathrm{cl}(f(X))))$. But $\mathrm{cl}(g(f(X))) = \mathrm{cl}(g(\mathrm{cl}(f(X))))$, so $L(g \circ f) = L(g) \circ L(f)$.   □

The functor L is surjective on objects, and injective on the objects of $\mathbf{SMatr}$, but not injective on objects in general.

Lemma 4.5 *The functor L is not faithful.*

Proof Write M for the matroid on the ground set $\{0, 1, 2\}$ with flats $\big\{\{0\}, \{0, 1, 2\}\big\}$. The function $f\colon M \to M$ given by $f(0) = 0$, $f(1) = 2$, and $f(2) = 1$ is a strong map. Now $L(f) = L(\mathrm{id}_M)$ both equal the identity function on $\mathcal{F}(M)$.   □

The functor L is not full. To see this, consider the unique matroid on ground set $\emptyset$. Any matroid M allows a unique morphism $L(M) \to L(\emptyset)$, but clearly there is no strong map $M \to \emptyset$ for nonempty M. This is the only obstruction to fullness in the following sense.

Proposition 4.6 *The functor L is nearly full; its restriction to $\mathbf{Matr}_\bullet$ is full.*

Proof Let M and N be matroids, and let $g\colon L(M) \to L(N)$ be a morphism in $\mathbf{GLat}$. First assume that M and N are pointed. Construct a function $f\colon M \to N$ as follows. For every rank-1 flat X in M for which $g(X)$ is the least element of $L(N)$, define $f(x) = \bullet$ for all $x \in X$. For every rank-1 flat X in M for which the flat $g(X)$ in N has rank 1, let f map all $x \in X$ that have not yet been accounted for to an element of $g(X)$ which is not a loop. Lemma 3.9 shows that f is a strong map, and by construction $L(f) = g$. Hence $L\colon \mathbf{Matr}_\bullet \to \mathbf{GLat}$ is full.

Finally, observe that in general $L(M) \simeq L(M_\bullet)$. It follows from the argument above that $L\colon \mathbf{Matr} \to \mathbf{GLat}$ is nearly full.   □

We will call a morphisms f is *lattice-preserving* when $L(f) = \mathrm{id}$. As promised in Section 3, we can now prove that matroid quotients are precisely categorical quotients in $\mathbf{GLat}$ via the functor L.

Lemma 4.7 *Epimorphisms in $\mathbf{GLat}$ are precisely the surjective morphisms.*

Proof That surjective maps are epimorphic is clear. The other direction follows by combining Lemma 3.4, Lemma 3.7, and Proposition 4.6.   □

Theorem 4.8 *The functor $L\colon \mathbf{Matr} \to \mathbf{GLat}$ maps matroid quotients to categorical quotients. The restricted functor from the subcategory of pointed matroid quotients to the subcategory of categorical quotients is not full but nearly full.*

Proof By Lemma 3.11, matroid quotients f are surjective functions. Hence $L(f)$ must also be surjective which are categorical quotients by Lemma 4.7.

Not every surjective morphism $g\colon L(M) \to L(N)$ in **GLat** has a matroid quotient f with $L(f) = g$: if N is the matroid with $1 + \#M$ loops and no other elements, then all morphisms into $L(N)$ are surjective, but there are no matroid quotients $M \to N$. Hence L is not full.

However, given a surjective map $g\colon L(M) \to L(N)$ in **GLat**, take M' to have $L(M') = L(M)$ with ground set the elements of $L(M)$ with $r(x) \leq 1$ (where the bottom element is a loop), and define N' by $L(N') = L(N)$ and by setting $F \subseteq |N'|$ to be a flat in N' when $F = g^{-1}(F) \cap \{x : x \in L(M), r(x) \leq 1\}$. Then there is clearly a matroid quotient $f\colon M' \to N'$ with $L(f) = g$, making L nearly full. □

There is also a functor in the opposite direction.

Proposition 4.9 *There is an embedding $S\colon \mathbf{GLat} \to \mathbf{Matr}_\bullet$ that:*

- *acts on objects G by letting $|S(G)|$ be the set of atoms of G together with a loop $\bullet$ (so that the flats of $S(G)$ form a lattice isomorphic to G);*
- *acts on morphisms $f\colon G \to H$ as follows: if f sends an atom of G to the least element of H, then $S(f)$ sends the corresponding element of $S(G)$ to $\bullet$; if f sends an atom A of G to an atom B of H, then $S(f)$ also sends A to B.*

Proof Proposition 4.6 guarantees that $S(f)$ is a legitimate morphism in **Matr**$_\bullet$. Clearly S preserves identities. Seeing that it also preserves composition comes down to noticing that for simple matroids M, a strong map $M \to N$ is completely defined by its action on $L(M)$, and that adding a loop does not change this.

Faithfulness is a direct consequence of the fact that there is a one-to-one correspondence between atoms of the lattice and elements of the matroid's ground set; different mappings between atoms therefore give rise to differents functions between ground sets. Fullness follows from Lemma 3.9. Finally, S is injective on objects because its lattice of flats is unique to a matroid. □

Theorem 4.10 *There is a reflection $L \dashv S$.*

Proof When the codomain of a strong map f is a simple matroid with a single added loop and the action of f on rank-1 flats is known, then f is known; there is only one element in each rank-1 flat and the rank-0 flat that each element of the domain can map to. Hence $\mathbf{Matr}_\bullet(M, N) \simeq \mathbf{GLat}(L(M), L(N))$ for pointed matroids M and N if N is simple. Moreover, if N is simple then it is in the image of S (up to isomorphism), and by the definition of the action of S on objects we have $G \simeq L(S(G))$ for all geometric lattices G. Therefore $\mathbf{Matr}_\bullet(M, S(G)) \simeq \mathbf{GLat}(L(M), N)$, and this bijection is natural. Finally, note that the counit $L(S(G)) \to G$ is an isomorphism, making the adjunction into a reflection. □

Any adjunction gives rise to a monad, and in this case we recover the following standard matroid operation [19, Section 1.7].

Definition 4.11 A *simplification* si(M) of a matroid M is a matroid obtained by deleting all the loops and all but one element in each rank-1 flat.

Theorem 4.12 *Simplification is a monad* si$_\bullet = S \circ L\colon \mathbf{Matr}_\bullet \to \mathbf{Matr}_\bullet$*. Its category of Eilenberg-Moore algebras is (isomorphic to)* **SMatr**$_\bullet$*.*

Proof By definition $S(L(M))$ is a simplification of M. Because the monad unit $M \to S(L(M))$ sends each element to its closure, the unit law for Eilenberg-Moore algebras $\mathrm{si}(M) \to M$ implies that $\mathrm{cl}(x) = x$ in M; that is, M is simple, and the structure map has to be the identity. Hence morphisms of Eilenberg-Moore algebras just come down to strong maps.   □

4.2 Vector spaces

Next we extend Example 2.4 to a functor from the category $\mathbf{FVect}_k$ of finite vector spaces over k and linear maps to the category of matroids.

Proposition 4.13 *There is a functor $M \colon \mathbf{FVect}_k \to \mathbf{Matr}_\bullet$ that sends a vector space V to the matroid with ground set V whose independent sets are the linearly independent subsets, acting on morphisms as $M(f) = f$.*

Proof Since flats correspond to vector subspaces, $M(f)$ is indeed a strong map as the inverse image of a vector subspace is a vector subspace.   □

The functor M is faithful, but not full: for $V = k = \mathbb{Z}_4$, the function $f \colon V \to V$ given by $f(0) = 0$, $f(1) = 2$, $f(2) = 1$, $f(3) = 3$ is a strong map $M(V) \to M(V)$ but not linear.

Lemma 4.14 *The functor M does not preserve coproducts, so has no right adjoint.*

Proof As we will see in Section 5 below, coproducts of matroids have to satisfy $|M + N| = |M| + |N|$, but $|M(V \oplus W)| = V \oplus W \neq V \sqcup W = |M(V) + M(W)|$.   □

A matroid M is *representable* (over k) if there is a strong map $f \colon M \to M(V)$ for some vector space V (over k) such that a subset $X \subseteq |M|$ is indepent if and only if $f(X)$ is independent. In particular, to a matrix with entries in k we may associate a matroid whose ground set consists of the columns of the matrix, and where a subset is independent if the corresponding columns are linearly independent. The rank function of the matroid counts the rank of the matrix of selected columns. Every representable matroid arises this way. When the matroid N is represented by a matrix A, and $B \subseteq |M|$, we write $A_{[B]}$ for the ordered set of columns of A labelled by elements of B. Not all matroids are representable, so the functor M is not surjective on objects. Nor is it injective on objects: swapping the role of two collinear elements in a vector spaces results in the same matroid.

We now embark on altering the functor M to make it nearly full. Intuitively, we consider the above way to turn a matrix into a matroid, and remove some structure from the matrix.

Lemma 4.15 *If matrices A, B, C satisfy $A = CB$, and $I \subseteq J$ are sequences of columns of B, and $I' \subseteq J'$ are the corresponding sequences of columns of A, then*

$$\mathrm{rk}(A_{[J']}) - \mathrm{rk}(A_{[I']}) \leq \mathrm{rk}(B_{[J]}) - \mathrm{rk}(B_{[I]}).$$

Proof Take any column $j \in J \setminus I$ and the corresponding $j' \in J' \setminus I'$. Comparing $r = \mathrm{rk}(B_{[I]}) - \mathrm{rk}(A_{[I']})$ with $r' = \mathrm{rk}(B_{[I \cup \{j\}]}) - \mathrm{rk}(A_{[I' \cup \{i'\}]})$ gives three cases:

- $\mathrm{rk}(B_{[I \cup \{j\}]}) = \mathrm{rk}(B_{[I]})$ and $\mathrm{rk}(A_{[I' \cup \{j'\}]}) = \mathrm{rk}(A_{[I']})$, so $r' = r$.

- $\mathrm{rk}(B_{[I\cup\{j\}]}) = \mathrm{rk}(B_{[I]}) + 1$ and $\mathrm{rk}(A_{[I'\cup\{j'\}]}) = \mathrm{rk}(A_{[I']})$, so $r' = r+1$.
- $\mathrm{rk}(B_{[I\cup\{j\}]}) = \mathrm{rk}(B_{[I]}) + 1$ and $\mathrm{rk}(A_{[I'\cup\{j'\}]}) = \mathrm{rk}(A_{[I']}) + 1$, so $r' = r$.

In all cases $r \leq r'$, and so $\mathrm{rk}(A_{[I'\cup\{j'\}]}) - \mathrm{rk}(A_{[I']}) \leq \mathrm{rk}(B_{[I\cup\{j\}]}) - \mathrm{rk}(B_{[I]})$. The proof is completed by repeating for the other elements of $J \setminus I$. □

We will consider matrices as multisets of vectors. Recall that a *multisubset* of a set S is a function $j\colon S \to \mathbb{N}$, which is *finite* when $\mathrm{supp}(j) = j^{-1}(\mathbb{N}\setminus\{0\})$ is finite; a map between multisubsets $j \to j'$ is a function $\mathrm{supp}(j) \to \mathrm{supp}(j')$.

Definition 4.16 Write $\mathbf{MVect}_k$ for the category whose objects are finite multisubsets $j\colon V \to \mathbb{N}$ of some vector space V over k, and whose morphisms $(V, j) \to (V', j')$ are linear maps $V \to V'$ that restrict to $\mathrm{supp}(j) \to \mathrm{supp}(j')$.

There is a canonical inclusion $\mathbf{Vect}_k \to \mathbf{MVect}_k$.

Theorem 4.17 *There is a functor $M\colon \mathbf{MVect}_k \to \mathbf{Matr}$ sending (V, j) to the matroid with ground set having an element for each element of $\mathrm{supp}(j)$ where a subset is independent if and only if the corresponding multisubset of vectors in V is (a subset and) linearly independent. It makes the following diagram commute:*

$$
\begin{array}{ccc}
\mathbf{FVect}_k & \xrightarrow{\ M\ } & \\
\downarrow & \searrow & \mathbf{Matr} \\
\mathbf{MVect}_k & \xrightarrow[\ M\ ]{} &
\end{array}
$$

It is the left Kan extension of $M\colon \mathbf{Vect}_k \to \mathbf{Matr}$ along $\mathbf{Vect}_k \to \mathbf{MVect}_k$.

Proof It is clear that $M\colon \mathbf{MVect}_k \to \mathbf{Matr}$ is well-defined on objects and functorial. By construction the diagram commutes. To see that it is well-defined on morphisms, consider a morphism $(V, j) \to (V', j')$ of $\mathbf{MVect}_k$. Write $S = \mathrm{supp}(j)$, $S' = \mathrm{supp}(j')$, and denote the restriction by $f\colon S \to S'$. The matroid $M(f(S))$ has the same rank function as $M(S')$, because both count the rank of submatrices of $f(S)$ and T that are identical apart from possibly repeated columns. Lemma 3.9(b) implies that this map $M(f(S)) \to M(S')$ is strong. Therefore it suffices to prove that $M(S) \to M(f(S))$ is strong.

Choose bases for V, V', so we may regard all of f, V, V', S, S' as matrices. Let $I \subseteq J \subseteq |M(S)|$. By Lemma 4.15 then

$$
\begin{aligned}
& \mathrm{rk}_{M(S')}\big(M(f)(J)\big) - \mathrm{rk}_{M(S')}\big(M(f)(I)\big) \\
&= \mathrm{rk}_{M(f(S))}\big(M(f)(J)\big) - \mathrm{rk}_{M(f(S))}\big(M(f)(I)\big) \\
&= \mathrm{rk}\big((f(S))_{[M(f)(J)]}\big) - \mathrm{rk}\big((f(S))_{[M(f)(I)]}\big) \\
&\leq \mathrm{rk}(S_{[J]}) - \mathrm{rk}(S_{[I]}) \\
&= \mathrm{rk}_{M(S)}(J) - \mathrm{rk}_{M(S)}(I).
\end{aligned}
$$

Hence $\mathrm{rk}_{M(S')}\big(M(f)(J)\big) - \mathrm{rk}_{M(S')}\big(M(f)(I)\big) \leq \mathrm{rk}_{M(S)}(J) - \mathrm{rk}_{M(S)}(I)$, which by Lemma 3.9(b) implies that $M(f)$ is strong.

Finally, to see that this functor $\overline{M}\colon \mathbf{MVect}_k \to \mathbf{Matr}$ is a left Kan extension of $M\colon \mathbf{Vect}_k \to \mathbf{Matr}$ along $I\colon \mathbf{Vect}_k \to \mathbf{MVect}_k$, observe that for every $M'\colon \mathbf{MVect}_k \to \mathbf{Matr}$ and natural transformation $\alpha\colon M \Rightarrow M' \circ I$ there is a

unique natural transformation $\beta\colon \overline{M} \Rightarrow M'$ such that $\beta I = \alpha$. Uniqueness follows from the fact that any β satisfying $\beta_2 \circ \overline{M}(g) = M'(g) \circ \beta_1$ for all $g\colon 1 \to 2$ in $\mathbf{MVect}_k$ must act as the identity on elements (regarding repetitions as the same element). To show existence, it is easy to check that this β is a strong map.   □

To show that the functor $M\colon \mathbf{MVect}_k \to \mathbf{Matr}_\bullet$ is nearly full for the initial field $k = \mathbb{Q}$, we need to consider representability in Lemma 3.11.

Lemma 4.18 *A surjection $f\colon M \to N$ is a strong map if and only if it factors as a matroid quotient followed by a map τ with $L(\tau) = \mathrm{id}$. Moreover, in the setting of Lemma 3.11, if M and N are representable over k, then Q can be chosen representable over a field extension of k.*

Proof The first statement follows from the fact that every surjective strong map factors as a bijective strong map followed by a strong map τ that reflects and preserves representability (over k) [23, page 228]. Hence we may assume $|M| = |N|$ and ignore the map τ for the rest of the proof.

The independent sets of Q are the intersection of the sets of independent sets of two matroids representable over an extension of k [14], which is proven in steps:
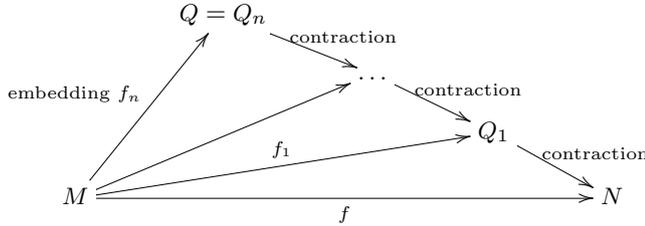
By induction it suffices to prove that Q_1 is representable over an extension of k.

Construction of Q_1. We first recall the construction of Q_1. Let S be a set of cardinality $n = \mathrm{rk}_N(N) - \mathrm{rk}_M(M)$. Define a matroid M_1 by $|M_1| = |M| \sqcup S$ and $\mathcal{I}_{M_1} = \{I_1 \cup I_2 \mid I_1 \in \mathcal{I}_P, I_2 \in 2^S\}$. This is clearly representable over k and hence over all its extensions. Define a matroid G by $|G| = |M| \sqcup S$ and $\mathcal{I}_G = \mathcal{I}_N$. Again this is clearly representable over k as it merely adds n loops, say with matrix D. Write f' for the map $M_1 \to G$ acting as f on $|M|$ and as the identity on S. We make M_2 with $|M_2| = |G|$ into a matroid by $\mathcal{I}_{M_2} = \{A \subseteq |G| \mid \mathrm{rk}_G(A) \geq |A| - 1\}$, which is well-defined as we will construct a matrix representing it.

Extend the field k with a transcendental λ_i for each $x_i \in |G|$. Extend D to a matrix D_+ over $k[\lambda_i]$ by adding one more row $\begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_{\#G} \end{pmatrix}$, and set $M_2' = M(D_+)$. By construction a subset $A \subseteq |M_2'| = |G|$ is independent in M_2' precisely when the corresponding columns of D_+ are independent, or equivalently $\mathrm{rk}(D_{+[A]}) = |A|$. As λ_i are algebraically independent from each other and from any other element of D_+, for A dependent in G we have $\mathrm{rk}(D_{+[A]}) = \mathrm{rk}(D_{[A]}) + 1$, and for A independent in G we have $\mathrm{rk}(D_{+[A]}) = \mathrm{rk}(D_{[A]})$. Because these are by definition the independent sets of the matroid M_2, we see that $M_2 = M(D_+)$ is representable over $k[\lambda_i]$. (The matroid M_2 is called the *first Higgs lift* of G.) Define $\mathcal{I}_{Q_1} = \mathcal{I}_{M_1} \cap \mathcal{I}_{M_2}$.

Well-definedness of Q_1. We will show that

$$\mathcal{F}(Q_1) = \{X \in \mathcal{F}(M_1) \mid \mathrm{rk}_{M_1}(X) = \mathrm{rk}_G(X) \text{ or } X = \sup_{f'(Y) = \mathrm{cl}_G(f'(X))}(Y)\},$$

which is a well-defined matroid with a strong map $M_1 \to Q_1$ [23, Exercise 7.20(a)]. We will derive $\mathcal{I}_{Q_1} = \mathcal{I}_{M_1} \cap \{I \cup \{q\} \mid I \in \mathcal{I}_G, q \in |G|\}$ from the above equation.

Observe that a strong map $M_1 \to Q_1$ implies $\mathcal{I}_{Q_1} \subseteq \mathcal{I}_{M_1}$. The independent sets of Q_1 are the minimal subsets I with $\mathrm{rk}_{Q_1}(I) = \mathrm{rk}_{K_1}(F)$, for $F \in \mathcal{F}(Q_1)$. It suffices to show that the flats of Q_1 are exactly those flats F of M_1 for which either $\mathrm{rk}_{Q_1}(F) = \mathrm{rk}_G(F)$ or $\mathrm{rk}_{Q_1}(F) = \mathrm{rk}_G(F) + 1$. If $F \in \{X \in \mathcal{F}(M_1) \mid \mathrm{rk}_{M_1}(X) = \mathrm{rk}_G(X)\}$ this is clear as $\mathrm{rk}_{M_1}(F) = \mathrm{rk}_G(F) = \mathrm{rk}_{Q_1}(F)$. It remains to show that $\mathrm{rk}_{Q_1}(F) = \mathrm{rk}_G(F) + 1$ when $\mathrm{rk}_{Q_1}(F) \neq \mathrm{rk}_G(F)$ and $F = \sup_{f'(Y)=\mathrm{cl}_G(f'(F))} Y$.

Since $\mathcal{I}_G \subseteq \mathcal{I}_{Q_1} \subseteq \mathcal{I}_{M_1}$, rank does not increase from M_1 to Q_1 and from Q_1 to G. Take $F_c = \mathrm{cl}_G(f'(F))$ and $F_0 = \inf_{f'(Y)=F_C} Y$, and pick $F' \in \mathcal{F}(M_1)$ with $F_0 \subset F' \subset F$. Now F' cannot be a flat of Q_1, as $\mathrm{rk}_{M_1}(F') > \mathrm{rk}_{M_1}(F_0) \geq \mathrm{rk}_G(F_0) = r_G(F_c) = r_G(F')$ and $F' \subset F = \sup_{f'(Y)=F_c} Y$. Therefore, only flats of at most two consecutive levels of $L(Q_1)$ can map to F_c. And since $\mathcal{I}_G \subseteq \mathcal{I}_{Q_1}$, there is always at least one $F_0 \in \mathcal{F}(Q_1)$ with $F_c = \mathrm{cl}_G(f'(F_0))$ and $\mathrm{rk}_{Q_1}(F_0) = \mathrm{rk}_G(F_c)$, namely $F_0 = \mathrm{cl}_{K_1}(I)$ for I independent in G. Hence $\mathrm{rk}_{Q_1}(F) = \mathrm{rk}_G(F) + 1$ and consequently $\mathcal{I}_{Q_1} = \mathcal{I}_{M_1} \cap \{I \cup \{q\} \mid I \in \mathcal{I}_G, q \in |G|\}$. Thus Q_1 is well-defined.

Representability of Q_1. Let M_1 be represented by the matrix C. Extend k with transcendentals μ_l for $l = 1, 2, \ldots, \mathrm{rk}(M_1)$, and multiply the lth row of C by μ_l. Now construct the matrix D'_+ from D as we constructed D_+ above, but now setting $\lambda'_i = \sum_l \mu_l C_{li}$ in place of λ_i in the new row. The elements of the new row are algebraically independent if and only if the corresponding columns of C are linearly independent. We will show that D'_+ in fact represents Q_1.

As before, if I is independent in G then it is also independent in D'_+. In addition, X dependent in G is independent in D'_+ if and only if $\mathrm{rk}_G(X) = \#X - 1$ and the new row of $D'_{+[X]}$ adds 1 to the rank, that is, if its elements $\{\lambda'_i\}$ are algebraically independent. But this happens exactly when X is independent in M_1! So the independent columns of D'_+ are

$$\mathcal{I}_G \cup \left(\mathcal{I}_{M_1} \cap \{X \mid \#X = \mathrm{rk}_G(X) + 1\}\right) = \mathcal{I}_{M_1} \cap \mathcal{I}_{M_2}.$$

Thus Q_1 is representable over $k[\lambda_j] \subseteq k[\mu_l]$. □

We can now prove our main result about the functor M, adapting [15].

Theorem 4.19 *The functor $M \colon \mathbf{MVect}_{\mathbb{Q}} \to \mathbf{Matr}_{\bullet}$ is nearly full.*

Proof A matroid representable over the finite fields $GF(3)$ and $GF(8)$ is also representable over $\mathbb{Q}$ and over all finite fields except possibly $GF(2)$ [19, Theorem 14.7.7]. Hence it suffices to prove that pointed strong maps $g \colon M(V) \to M(W)$ for $V, W \in \mathbf{MVect}_{\mathbb{Q}}$ are exactly those maps of the form $M(f)$ for a morphism $f \colon V' \to W'$ in $\mathbf{MVect}_{\mathbb{Q}}$ with $M(V) = M(V')$ and $M(W) = M(W')$.

First assume g is surjective. Factor it as in Lemma 4.18 and represent N as a matrix A over a field extension l of k. Multiply A by the matrix that has zeroes everywhere except the diagonal entries labelled by Q, where it has entries one. Now $B = CA$ represents N. Since matrices A and B represent objects of $\mathbf{MVect}_l$, the matrix C is a linear map between their respective spans.

If g is not surjective, precompose it with another embedding to give another matroid N' representable over k. Specifically, we may take $\mathcal{I}_{N'} = \{I_1 \cup I_2 \mid I_1 \in \mathcal{I}_N, I_2 \in \mathcal{I}_M\}$, let g' acts as g on $|N|$ and as the identity elsewhere [13]. □

The rest of this section considers the functor $L \circ M$ that turns a vector space into its lattice of vector subspaces, which is of interest to *e.g.* quantum logic.

Proposition 4.20 *The functor $L \circ Mat \colon \mathbf{FVect}_k \to \mathbf{GLat}$ is nearly faithful.*

Proof It suffices to show that two linear maps $f, g \colon V \to W$ give rise to the same lattice morphism $L(M(f)) = L(M(g))$ if and only if they are multiples of each other. For all flats X of $M(V)$ we have $\mathrm{cl}\big(M(f)(X)\big) = \mathrm{cl}\big(M(g)(X)\big)$. It follows from faithfulness of M that $f(X) = g(X)$ for every subspace X of V. Therefore $f(v) = w$ implies $g(v) = \beta_v w$ for some $\beta_v \in k$. If $v, v' \in V$ are not collinear then $f(v - v') = g(\beta v - \beta' v')$ for some $\beta, \beta' \in k$. So $f(v - v') = \beta' g(v - v') + (\beta - \beta')g(v)$. Because v, v' were not multiples of each other, $v - v'$ cannot be a multiple of v. So we must have $\beta = \beta'$, and $g = \beta f$. $\square$

The functor $L \circ M$ is not full. To see this, consider the $\mathbb{Z}_2$-vector space $V = \mathbb{Z}_2^2$, which has three 1-dimensional subspaces. Let $A = \{(0,0), (1,0)\}$, $B = \{(0,0), (0,1)\}$ and $C = \{(0,0), (1,1)\}$, and consider the lattice map $g \colon L(M(V)) \to L(M(V))$ that sends all atoms to B. There is only one way assignment $f \colon V \to V$ with $L(f) = g$, namely $f(0,0) = (0,0)$, $f(1,0) = (0,1)$, $f(0,1) = (0,1)$ and $f(1,1) = (0,1)$, but this function is not linear. Even if we allow infinite base fields, $L \circ M$ is not full. For example, take $V = \mathbb{R}^2$, map all the lines through the origin to one of the axes, and the origin to itself. This preserves the loop, atoms, and joins, because everything is mapped to the same subspace. But there is no linear map implementing this assignment.

4.3 Graphs

Lastly we briefly discuss functoriality of the construction of Example 2.5 turning an undirected graph into a matroid. We simplify the definition of undirected graph [7], as we need not to distinguish bands and loops.

Definition 4.21 Write **Graph** for the following category. Objects are undirected graphs: a set V of vertices, a set E of edges and source and a boundary map θ from E to the class of singleton and two-element subsets V. A morphism is a pair of maps $f \colon V \to V'$ and $g \colon E \to E'$ satisfying $f \circ \theta = \theta' \circ g$, and $\#(\theta'(e)) \leq \#(\theta(e))$.

To extend Example 2.5 to a functor **Graph** $\to$ **Matr**, we could restrict the category of graphs to only permits 'strong' morphisms of graphs, whose preimage preserves closed sets (here a set of edges is closed if the addition of an edge does not change the size of a spanning tree in the corresponding subgraph). There is some evidence that this choice of morphisms is useful for some applications of graph theory [21]. Alternatively, we could allow more functions than strong maps as morphisms between matroids. We must at least keep the restriction that loops map to loops. Or we could restrict both the domain and codomain. Let us write **Graph*** and **Matr*** for the chosen domain and codomain.

For a functor $M \colon \mathbf{Graph}^* \to \mathbf{Matr}^*$ to be of any practical use it should act as the identity on morphisms. It must then have the following properties:

– It cannot be surjective on objects. A matroid is of the form $M(G)$ precisely when it is *graphic*, and there exist non-graphic matroids.

– It cannot be injective on objects. Here are graphs $G_1 \not\cong G_2$ with $M(G_1) = M(G_2)$:
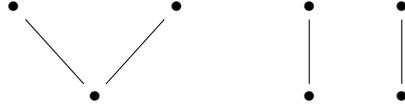
– It cannot be full. There are no maps $G_1 \to G_2$ that are surjective on edges, whereas $M(G_1)$ must have at least one morphism to itself (namely the identity).
– It cannot be faithful. Functions $G_2 \to G_1$ corresponding to the identity matroid map may act differently on vertices.

One could define functors from the category of graphs and strong maps to the category of matroids that assign more obscure matroids to graphs, that we briefly list, but none of them is surjective or injective on objects, nor likely full or faithful. *Bicircular matroids* [19, Section 12.1], *frame matroids* and *lift matroids* [26] reduce to the cycle matroid of Example 2.5. *Bond matroids* [19, Section 2.3], in the case of planar graphs, reduce to the cycle matroid of the dual graph. A *transversal matroid* [19, Section 1.6] is defined on the vertices of one side of a bipartite graph.

5 Limits and colimits

This section examines limits and colimits in **Matr** and its subcategories. We give proofs and counterexamples to accommodate different variations and to repair mistakes in the literature.

As mentioned in Section 3, the one-element matroid 1 where the element is a loop is a terminal object in all categories of matroids we consider.

Proposition 5.1 *The categories* **SMatr**, **SMatr.**, **LMatr**, **LMatr.**, **Matr**, *and* **Matr.** *do not have all products.*

Proof See [10, Proposition 5]. Consider the simple matroid M of rank 2 with $|M| = \{a, b, c, d\}$ and suppose $M \times M$ existed in **SMatr**. A pair $(x, y) \in |M| \times |M|$ induces two strong maps $x, y\colon 1 \to M$, and hence a unique tuple $(x, y)\colon 1 \to M \times M$ with $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$. Hence $M \times M$ has at least 16 points. The tuple of any permutation $\sigma\colon |M| \to |M|$ with the identity gives a strong map $M \to M \times M$, whose image $\{(a, \sigma(a)), (b, \sigma(b)), (c, \sigma(c)), (d, \sigma(d))\}$ must be a flat. But $\{(a, a), (b, b), (c, c), (d, d)\}$ and $\{(a, a), (b, b), (c, d), (d, c)\}$ cannot both be flats.

The same counterexample works for the other unpointed categories. Lemma 3.2 finally shows the pointed categories cannot have products. □

It follows that these categories do not have pullbacks or exponentials either.

Proposition 5.2 *The categories* **SMatr**, **SMatr.**, **LMatr**, **LMatr.**, **Matr**, *and* **Matr.** *do not have all pushouts.*

Proof See [10, Proposition 7]. Consider the simple matroid M with six elements where every pair forms a hyperplane. Let M_1 be the extension of M by one element so that two of the hyperplanes are triples, and let M_2 be the extension of M by one element so that three of the hyperplanes are triples, two of which coincide with

M_1. Suppose the pushout P of the embeddings $f_1\colon M_1 \to M$ and $f_2\colon M_2 \to M$ existed. Then the image of M colimit cone P' cannot be of rank 3, because then M_1 and M_2 would be embedded in P with overlap M. The flats $\{a, b\}$ and $\{d, e\}$ meet in a point of P which is not in the flat $\mathrm{cl}\{e, f\}$ of M_1 but is in the flat $\mathrm{cl}\{e, f\}$ in M_2. Therefore the image of M in P has rank at most 1. But there exist different choices P', such that no rank-1 flat has a strong map to both choices P'.

The same counterexample works for the other unpointed categories, and $M_\bullet$ works for the pointed versions. $\square$

The empty matroid is an initial object in all unpointed categories of matroids we consider, whereas the one-element matroid is initial in all pointed categories of matroids we consider.

Proposition 5.3 *The categories* **SMatr**, **SMatr$_\bullet$**, **LMatr**, **LMatr$_\bullet$**, **Matr**, *and* **Matr$_\bullet$** *have coproducts.*

Proof See [9] and also [10, Proposition 4]. The coproduct $M + N$ has ground set $|M + N| = |M| \sqcup |N|$ and flats $\{F \cup G \mid F \in \mathcal{F}_M, G \in \mathcal{F}_N\}$, where we identify $\bullet_M$ and $\bullet_N$ if need be. It is easy to see that if M and N are pointed simple/loopless, then so is $M + N$. The coprojections are the inclusions $M \to M + N$ and $N \to M + N$, and it is clear that there is a unique strong cotuple $M + N \to P$ of strong maps $M \to P$ and $N \to P$. $\square$

Going back to Section 4.3, recall that the coproduct in **Graph** is given by disjoint union. Taking strong morphisms in does not change this, that is, the inclusion **Graph**$^* \to$ **Graph** preserves and reflects coproducts. Therefore the functor $M\colon$ **Graph**$^* \to$ **Matr** must preserve coproducts.

Proposition 5.4 *The categories* **SMatr**, **SMatr$_\bullet$**, **LMatr**, **LMatr$_\bullet$**, **Matr**, *and* **Matr$_\bullet$** *have equalizers.*

Proof See also [3, Theorem 53]. The equalizer of $f, g\colon M \to N$ is the inclusion of the matroid E with $|E| = \{x \in |M| \mid f(x) = g(x)\}$ and $\mathcal{F}_E = \{F \cap |E| \mid F \in \mathcal{F}_M\}$. This is a well-defined matroid, that clearly satisfies the universal property. Finally, if M is simple/loopless, then so is E. $\square$

Proposition 5.5 *The categories* **SMatr**, **SMatr$_\bullet$**, **LMatr**, **LMatr$_\bullet$**, **Matr**, *and* **Matr$_\bullet$** *do not have coequalizers.*

Proof By Lemma 3.2 it suffices to consider the pointed categories. Consider the following objects and morphisms:

$$|M| = |N| = \{\bullet, 1, 2, 3, 4\},$$
$$\mathcal{H}_M = \mathcal{H}_N = \big\{\{\bullet, 1, 2\}, \{\bullet, 1, 3\}, \{\bullet, 2, 3\}, \{\bullet, 1, 4\}, \{\bullet, 2, 4\}, \{\bullet, 3, 4\}\big\},$$
$$f = \mathrm{id},$$
$$g = \big(\bullet \mapsto \bullet, 1 \mapsto 2, 2 \mapsto 1, 3 \mapsto 3, 4 \mapsto 4\big).$$

If the coequalizer $c\colon N \to C$ of $f, g\colon M \to N$ existed, then [3, Theorem 54][1] its ground set must be (in bijection with) the quotient $|N|/\!\sim$ by the equivalence relation generated by $f(x) \sim g(x)$ for all $x \in |N|$, and c must map x to its equivalence

---

[1] That proof is invalid, as the map $y \mapsto [y]$ need not be strong. However, it does hold for the choices in our proof below; that is, there are strong maps from N to options (1) and (3).

class $[x]$. Explicitly, $|C| = \{\bullet, [12], [3], [4]\}$.

$$M \underset{g}{\overset{f}{\rightrightarrows}} N \xrightarrow{c} C$$
$$\downarrow k$$
$$\searrow^{c'} \quad C'$$

Each flat G of C must be of the form $[F] = \{[x] \mid x \in F\}$ for some flat F of N, because $G = c(c^{-1}(G)) = [c^{-1}(G)]$. In fact each flat F of C must satisfy $F = [F]$ in N. Thus the only flats which may be contained in C are

$$\{\bullet\}, \{\bullet, [3]\}, \{\bullet, [4]\}, \{\bullet, [12]\}, \{\bullet, [3], [4]\}, \{\bullet, [12], [3], [4]\}.$$

That leaves only three possibilities for a well-defined geometric lattice $\mathcal{F}$:

$$\big\{\{\bullet\}, \{\bullet, [12], [3], [4]\}\big\}, \tag{1}$$
$$\big\{\{\bullet\}, \{\bullet, [12]\}, \{\bullet, [3], [4]\}, \{\bullet, [12], [3], [4]\}\big\}, \text{ or} \tag{2}$$
$$\big\{\{\bullet\}, \{\bullet, [12]\}, \{\bullet, [3]\}, \{\bullet, [4]\}, \{\bullet, [12], [3], [4]\}\big\}. \tag{3}$$

Option (1) fails when we set C' to have the same ground set as C with flats (2), because then the inverse image under k of any rank-1 flat is not a flat. Option (2) fails when we give $|C'| = \{[\bullet 4], [12], [3]\}$ flats $\big\{\{[\bullet 4]\}, \{[\bullet 4], [12], [3]\}\big\}$, because then the inverse image under k of the rank-0 flat is not a flat. Option (3) fails when we endow $|C'| = \{[\bullet 34], [12]\}$ with flats $\big\{\{[\bullet 34]\}, \{[\bullet 34], [12]\}\big\}$, because then the inverse image under k of the rank-0 flat is not a flat. $\qquad\square$

Lemma 5.6 *Every contraction is a coequalizer in* **Matr**$_\bullet$, *but not conversely.*

Proof Suppose $c\colon N \to N/Z$ is a contraction with $c(z) = \bullet$ for $Z \subseteq |N|$. Let M be the free matroid on $|M| = |N|$. Define $f, g\colon M \to N$ by $f(x) = x$, and

$$g(x) = \begin{cases} x, & \text{if } x \notin Z, \\ \bullet, & \text{if } x \in Z. \end{cases}$$

Then c is a coequalizer of f and g.

Conversely, keeping f the same but letting g send all nonloop elements to the same nonloop element results in a coequalizer that is not a contraction. $\qquad\square$

Finally we consider **FMatr** and **FMatr**$_\bullet$.

Proposition 5.7 *The category* **FMatr**$_{(\bullet)}$ *is isomorphic to the category* **Set**$_{(\bullet)}$ *of (pointed) finite sets and (pointed) functions, and hence has finite (co)limits.*

Proof Every free matroid is completely determined by its ground set, and every function between ground sets of free matroids is a strong map. (Co)completeness of **Set** follow immediately; for **Set**$_\bullet$ see [2, 28.9.5]. $\qquad\square$

6 Adjunctions

We have seen various classes of matroids: all matroids, simple matroids, free matroids, and loopless matroids. In this section we study free and cofree constructions translating between these classes. Proposition 5.7 already showed that free matroids over sets exist, and are precisely what we have been calling free matroids. We now consider whether the inclusions

$$\mathbf{FMatr}_{(\bullet)} \hookrightarrow \mathbf{SMatr}_{(\bullet)} \hookrightarrow \mathbf{LMatr}_{(\bullet)} \hookrightarrow \mathbf{Matr}_{(\bullet)}$$

have adjoints. We first focus on the pointed categories. Because these categories do not have many colimits, we cannot invoke the adjoint functor theorem.

6.1 Free pointed matroids

We start with right adjoints of functors out of the category of free pointed matroids.

Theorem 6.1 *The category* $\mathbf{FMatr}_\bullet$ *is a coreflective subcategory of* $\mathbf{LMatr}_\bullet$: *the inclusion* $\mathbf{FMatr}_\bullet \hookrightarrow \mathbf{LMatr}_\bullet$ *has a right adjoint* F *defined by*

$$|F(M)| = |M|, \qquad \mathcal{F}_{F(M)} = 2^{|M|} \setminus 2^{|M| \setminus \{\bullet\}}, \qquad F(f) = f.$$

Proof See [3, Theorem 86] and [5,4]. □

The above functor F extends to right adjoints of the inclusions $\mathbf{FMatr}_\bullet \hookrightarrow \mathbf{Matr}_\bullet$ and $\mathbf{FMatr}_\bullet \hookrightarrow \mathbf{SMatr}_\bullet$. We now examine whether the functor F itself has a right adjoint for each of those three cases.

Proposition 6.2 *The functors* $F \colon \mathbf{SMatr}_\bullet \to \mathbf{FMatr}_\bullet$ *and* $F \colon \mathbf{LMatr}_\bullet \to \mathbf{FMatr}_\bullet$ *have no right adjoints.*[2]

Proof Suppose V were a right adjoint. Let N be the matroid with flats $\{\{\bullet\}, \{\bullet, e\}\}$, and let M be the matroid with flats $\{\{\bullet\}, \{\bullet, a\}, \{\bullet, b\}, \{\bullet, c\}, \{\bullet, a, b, c\}\}$. There are 8 morphisms $F(M) \to N$. Now count the morphisms $M \to V(N)$. For $\mathbf{SMatr}_\bullet$:

- If $\#V(N) = 2$, it has flats $\{\{\bullet\}, \{\bullet, e\}\}$: there are 5 morphisms;
- If $\#V(N) = 3$, it has flats $\{\{\bullet\}, \{\bullet, e_1\}, \{\bullet, e_2\}, \{\bullet, e_1, e_2\}\}$, and 11 morphisms;
- If $\#V(N) > 3$, there are at least 11 morphisms.

For $\mathbf{LMatr}_\bullet$ we have to consider one extra case: $\mathcal{F}_{V(N)} = \{\{\bullet\}, \{\bullet, e_1, e_2\}\}$, giving 21 morphisms $M \to V(N)$. □

The above proof method, of counting morphisms between clever choices of matroids, will be used often and abbreviated in this section.

Theorem 6.3 *The functor* $F \colon \mathbf{Matr}_\bullet \to \mathbf{FMatr}_\bullet$ *has adjoints* $F \dashv V \dashv H$ *given by*

$$\mathcal{F}_{V(M)} = \{|M|\}, \qquad\qquad V(f) = f,$$
$$H(M) = F(M'), \qquad\qquad H(f) = \mathrm{id}.$$

where M' *is got from* M *by deleting all but loops. The functor* H *has no right adjoint.*

---

[2] The purported right adjoint in [3, Theorem 126] fails for every function mapping $1 < n < \#M$ elements to $\bullet$.

Proof Start with the first right adjoint. For every pointed matroid M, every free pointed matroid N, and every strong map $f: M \to V(N)$, there must exist a strong map $\eta_M: M \to V(F(M))$ and a unique strong map $\hat{f}: F(M) \to N$ satisfying $f = V(\hat{f}) \circ \eta_M$. Set $\eta_M(x) = x$ for all $x \in |D|$, and $\hat{f} = f$.

For the second right adjoint, take the unit $\eta_M: M \to H(V(M))$ to be the function $\eta_M(x) = x$, and the transpose $\hat{f}: V(M) \to N$ to equal $f: M \to H(N)$.

Finally, suppose B were right adjoint to H. Let M be a free matroid with at least 2 elements. There is only one morphism $H(M) \to N$, so there can be only one morphism $M \to B(N)$, regardless of N. So $B: \mathbf{FMatr}_\bullet \to \mathbf{Matr}_\bullet$ must be the constant functor 1. But this cannot be right adjoint to H: if M and N have more than one loop, there is more than one morphism $H(M) \to N$, but only one morphism $M \to B(N)$. $\square$

Our functors out of the category of free pointed matroids have no left adjoints.

Proposition 6.4 *The inclusions* $\mathbf{FMatr}_\bullet \hookrightarrow \mathbf{SMatr}_\bullet$, $\mathbf{FMatr}_\bullet \hookrightarrow \mathbf{LMatr}_\bullet$, *and* $\mathbf{FMatr}_\bullet \hookrightarrow \mathbf{Matr}_\bullet$ *have no left adjoint.*

Proof Suppose K were a left adjoint. Let M be the matroid with flats

$$\big\{\{\bullet\}, \{\bullet, a\}, \{\bullet, b\}, \{\bullet, c\}, \{\bullet, d\},$$
$$\{\bullet, a, b\}, \{\bullet, a, c\}, \{\bullet, a, d\}, \{\bullet, b, c\}, \{\bullet, b, d\}, \{\bullet, c, d\}, \{\bullet, a, b, c, d\}\big\}.$$

Count the morphisms $K(M) \to N$ and $M \to N$:

- If $\#K(M) \le 3$, pick $\#N = 2$ for fewer maps $K(M) \to N$ than $M \to N$.
- If $\#K(M) = 4$, pick $\#N = 4$. Since the unit η of the adjunction cannot map nonloops to loops, it must map nonloops $a \ne b$ of M to the same nonloop of $K(M)$. But this prevents surjective $f: M \to N$ mapping a, b to the same nonloop from having a mate $\hat{f}: K(M) \to N$.

The case $\#K(M) \ge 5$ reduces to the previous ones by noting that the submatroid $\eta(M) \subseteq K(M)$ has at most 4 elements. $\square$

6.2 Simple pointed matroids

Next, we turn to the inclusion of simple matroids into larger categories.

Proposition 6.5 *The category* $\mathbf{SMatr}_\bullet$ *is a reflective subcategory of* $\mathbf{Matr}_\bullet$: *the inclusion* $\mathbf{SMatr}_\bullet \hookrightarrow \mathbf{Matr}_\bullet$ *has a left adjoint* $\mathrm{si}_\bullet$. *The functor* $\mathrm{si}_\bullet$ *has no left adjoint.*

Proof The first statement follows directly from Theorem 4.12. For the second, suppose $K \dashv \mathrm{si}_\bullet$. Take M to be the pointed matroid with two parallel elements a, b and the loop $\bullet$, and write F_1 for its rank-1 flat. Take S to be the pointed simple matroid with elements e and $\bullet$. Let f map e to F_1. Its transpose $\hat{f}$ must map some element e' of $K(S)$ to either a or b. But if $e' \mapsto a$ satisfies $f = \mathrm{si}_\bullet(\hat{f}) \circ \eta$ then so does $e' \mapsto b$, and vice versa. Therefore, $\hat{f}$ cannot be unique. $\square$

Proposition 6.6 *The inclusion* $\mathbf{SMatr}_\bullet \hookrightarrow \mathbf{Matr}_\bullet$ *has no right adjoint.*

Proof If $R\colon \mathbf{Matr}_\bullet \to \mathbf{SMatr}_\bullet$ were a right adjoint, there would be a natural isomorphism $F \cong F \circ R\colon \mathbf{Matr}_\bullet \to \mathbf{FMatr}_\bullet$, whence R must (1) preserve cardinality and (2) reflect surjectivity. Now write S for the simple matroid with flats $\big\{\{\bullet\}, \{\bullet, 1\}, \{\bullet, 2\}, \{\bullet, 3\}, \{\bullet, 1, 2, 3\}\big\}$. By property (1), $R(S)$ has 4 elements, so its family of flats must be either that of S or

$$\big\{\{\bullet\}, \{\bullet, 1\}, \{\bullet, 2\}, \{\bullet, 3\}, \{\bullet, 1, 2\}, \{\bullet, 2, 3\}, \{\bullet, 1, 3\}, \{\bullet, 1, 2, 3\}\big\}.$$

Let M be the matroid with flats $\big\{\{\bullet\}, \{\bullet, 1\}, \{\bullet, 2, 3\}, \{\bullet, 1, 2, 3\}\big\}$. The family of flats of $R(M)$ must be either that of S or

$$\big\{\{\bullet\}, \{\bullet, 1\}, \{\bullet, 2\}, \{\bullet, 3\}, \{\bullet, 1, 2\}, \{\bullet, 2, 3\}, \{\bullet, 1, 3\}, \{\bullet, 1, 2, 3\}\big\}.$$

The latter case gives 25 strong maps $S \to M$ but only 14 strong maps $S \to R(M)$. In the former case the transpose $\hat{f}$ of any map f must be surjective by property (2), but there are no surjective strong maps $S \to M$. $\qquad\square$

6.3 Loopless pointed matroids

Let us consider adjoints of the inclusion of loopless matroids into larger categories.

Theorem 6.7 *The category* $\mathbf{LMatr}_\bullet$ *is a reflective subcategory of* $\mathbf{Matr}_\bullet$*: the inclusion* $\mathbf{LMatr}_\bullet \hookrightarrow \mathbf{Matr}_\bullet$ *has a left adjoint J that deletes every loop except $\bullet$ from objects and acts on morphisms $f\colon M \to N$ as*

$$J(f)(e) = \begin{cases} f(e) & \text{if } f(e) \in J(N) \\ \bullet & \text{if } f(e) \notin J(N) \end{cases}$$

The functor $J\colon \mathbf{Matr}_\bullet \to \mathbf{LMatr}_\bullet$ *has no left adjoint.*

Proof For the unit $\eta_M\colon M \to J(M)$ we may take the strong map that sends every loop to $\bullet$ and every other element to itself. Then a morphisms $f\colon M \to N$ correspond bijectively to $\hat{f} = J(f)$ satisfying $\hat{f} \circ \eta_M = f$.

Suppose $G \dashv J$. Then G cannot be the constant functor $\bullet$; pick an object K and $\bullet \neq e \in |G(K)|$. Let M be the matroid with loops $*$ and $\bullet$. Then strong maps $f\colon K \to J(M)$ must map every element of K to $\bullet$, and any function $G(K) \to M$ is strong. But altering a transpose of f with $e \mapsto *$ into $e \mapsto \bullet$ or vice versa still gives a transpose, so transposes are not unique. $\qquad\square$

Theorem 6.8 *The inclusion* $\mathbf{LMatr}_\bullet \hookrightarrow \mathbf{Matr}_\bullet$ *has no right adjoint.*

Proof If N were a right adjoint, there would be a natural isomorphism $F \cong F \circ N\colon \mathbf{Matr}_\bullet \to \mathbf{FMatr}_\bullet$, whence N must (1) preserve cardinality and (2) reflect surjectivity. Let D be the matroid with flats $\big\{\{\bullet\}, \{\bullet, c\}, \{\bullet, a, b\}, \{\bullet, a, b, c\}\big\}$, and M the matroid with flats $\big\{\{\bullet, *\}, \{\bullet, *, e\}\big\}$. Now $\#N(D) = 4$ and $\#N(M) = 3$ by property (1), leaving two possible choices for $\mathcal{F}_{N(M)}$. The first choice is $\big\{\{\bullet\}, \{\bullet, *\}, \{\bullet, e\}, \{\bullet, *, e\}\big\}$; then there are 9 strong maps $D \to N(M)$ but 15 strong maps $D \to M$. The second choice is $\big\{\{\bullet\}, \{\bullet, *, e\}\big\}$; by property (2), epimorphisms $D \to N(M)$ correspond to epimorphisms $D \to M$, but there are 10 of the former and 3 of the latter. Either choice gives a contradiction. $\qquad\square$

The following theorem summarizes all adjunctions in the pointed case.

Theorem 6.9 *The inclusions have the following adjunctions:*
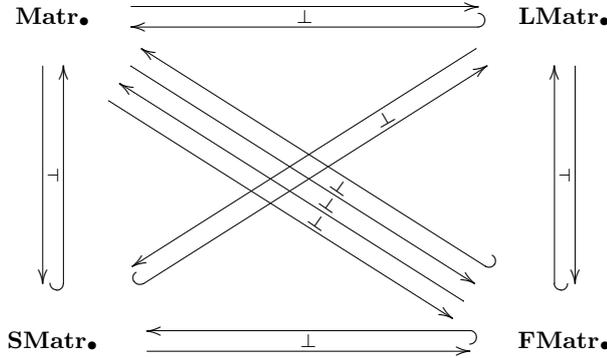
The functors in the above diagram have no adjoints other than those indicated.

Proof Collate the previous results in this section.   □

6.4 Unpointed categories

We would like to have a translation principle between pointed and unpointed versions of our matroid categories. Unfortunately, the inclusion **Matr.** ↪ **Matr** only has a left adjoint and no right adjoint, as proven in Lemma 3.2 above. So our results for the pointed categories do not necessarily translate into the unpointed versions, and we have to reason directly for the (non)existence of adjoints.

Proposition 6.10 *The category* **FMatr** *is a coreflective subcategory of* **LMatr**: *the inclusion* **FMatr** ↪ **LMatr** *has a right adjoint F defined by:*

$$|F(M)| = |M|, \qquad \mathcal{F}_{F(M)} = 2^{|M|}, \qquad F(f) = f.$$

It extends to right adjoints of the inclusions **FMatr** ↪ **Matr** *and* **FMatr** ↪ **SMatr**.

Proof Simply take the unit $\eta_M \colon M \to F(M)$ to be the identity, and $\hat{f} = f$.   □

The functor $F \colon$ **Matr** → **FMatr** has a right adjoint $V \colon$ **FMatr** → **Matr**, which in turn has a right adjoint $H \colon$ **Matr** → **FMatr**, both defined as in Theorem 6.3, and H has no right adjoint; we omit the proofs.

Proposition 6.11 *The functor* $F \colon$ **SMatr** → **FMatr** *has no right adjoint.*

Proof Suppose V were a right adjoint. Let M be the free matroid on m elements, and let D be the matroid with flats $\{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b, c\}\}$. There are m^3 strong maps $F(D) \to M$. If $V(M)$ contains D as a submatroid, then there are at least 9 maps $D \to V(M)$, because there are 9 maps $D \to D$. Otherwise, for the map f to be strong, every element of D must map to the same element of $V(M)$, giving $\#V(M)$ many maps $D \to V(M)$, so $\#V(M) = m^3$. Now take D to be the simple matroid with $|M| = \{e\}$. There are m maps $F(D) \to M$, but m^3 maps $D \to V(M)$, and in general $m \neq m^3$.   □

Proposition 6.12 *The functor $F\colon \mathbf{LMatr} \to \mathbf{FMatr}$ has a right adjoint U given by:*

$$|U(M)| = |M|, \qquad \mathcal{F}_{U(M)} = \{\emptyset, |M|\}, \qquad U(f) = f.$$

The functor U has no right adjoint.

Proof Take the unit to be the identity and $\hat{f} = f$ to establish $F \dashv U$. Suppose $U \dashv G$. Take D to be the free matroid on 2 elements, so $U(D)$ is the matroid with 2 parallel elements. Taking $M = D$ gives exactly 2 morphisms $U(D) \to M$, whereas there are $\#G(M)^2$ morphisms $D \to G(M)$. □

Proposition 6.13 *None of the inclusions $\mathbf{FMatr} \hookrightarrow \mathbf{SMatr}$, $\mathbf{FMatr} \hookrightarrow \mathbf{LMatr}$, and $\mathbf{FMatr} \hookrightarrow \mathbf{Matr}$ have a left adjoints.*

Proof We treat the first inclusion as the other two are similar. Suppose K were a left adjoint. Let N be the free matroid on 2 elements, and let M be the matroid with flats $\{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a,b\}, \{a,c\}, \{a,d\}, \{b,c\}, \{b,d\}, \{c,d\}, \{a,b,c,d\}\}$. There are 8 strong maps $M \to N$, and $2^{\#K(M)}$ strong maps $K(M) \to N$, so $\#K(M) = 3$.

As in Proposition 6.4, the unit η has to map $a \neq b \in |M|$ to the same element of $K(M)$ and each of the other two elements of M to the two remaining elements of $K(M)$. Picking N with 3 elements gives a contradiction as in that Proposition. □

The inclusions $\mathbf{SMatr} \hookrightarrow \mathbf{Matr}$ and $\mathbf{LMatr} \hookrightarrow \mathbf{Matr}$ have no left adjoint K: if M has at least one loop then there are no morphisms $M \to K(M)$ at all.

Corollary 6.14 *The category $\mathbf{SMatr}$ is a reflective subcategory of $\mathbf{LMatr}$: the inclusion has a left adjoint* si *as in Lemma 4.12. The functor* si *has no left adjoint.*

Proof The first statement follows from Remark 4.4 as before. The second statement is proven just like in Proposition 6.5. □

Proposition 6.15 *The inclusions $\mathbf{SMatr} \hookrightarrow \mathbf{Matr}$ and $\mathbf{SMatr} \hookrightarrow \mathbf{LMatr}$ have no right adjoint.*

Proof If $R\colon \mathbf{Matr} \to \mathbf{SMatr}$ were a right adjoint, it would satisfy properties (1) and (2) as in Proposition 6.6. Let S be the matroid with flats $\{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2,3\}\}$, and let M be the matroid with flats $\{\emptyset, \{1\}, \{2,3\}, \{1,2,3\}\}$. The family of flats of $R(M)$ must be either that of S or

$$\{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}, \{1,2,3\}\}.$$

In either case, the number of strong maps $S \to R(M)$ does not equal the number of strong maps $S \to M$. □

Theorem 6.16 *The inclusion $\mathbf{LMatr} \hookrightarrow \mathbf{Matr}$ has no right adjoint.*

Proof If N were a right adjoint, it would satisfy properties (1) and (2) as in Theorem 6.8. Let D be the matroid with flats $\{\emptyset, \{c\}, \{a,b\}, \{a,b,c\}\}$, and M the matroid with flats $\{\{a,b\}, \{a,b,c\}\}$. Now $\#N(D) = \#N(M) = 3$ by property (1), giving four choices for $\mathcal{F}_{N(M)}$. First, $\{\emptyset, \{a,b,c\}\}$ gives 27 strong maps $D \to N(M)$. Second, $\{\emptyset, \{c\}, \{a,b\}, \{a,b,c\}\}$ gives 15 strong maps $D \to N(M)$. Third, $\{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b,c\}\}$ gives 9 strong maps $D \to N(M)$. Fourth and last, $\{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}, \{a,b,c\}\}$ gives 9 strong maps $D \to N(M)$. As there are 15 strong maps $D \to M$, it must be the second option. But if D has flats $\{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b,c\}\}$, there are 9 strong maps $D \to N(M)$. □

The following theorem summarizes all adjunctions in the unpointed case.

Theorem 6.17 *The inclusions have the following adjunctions:*
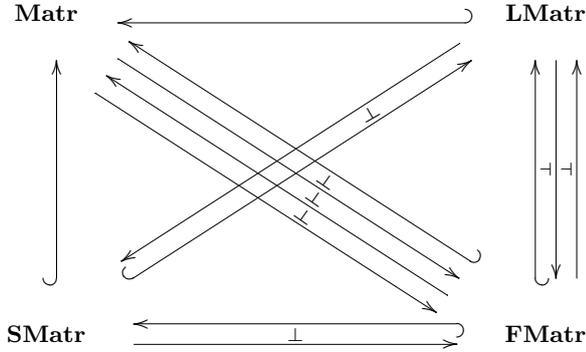
$$
\begin{array}{ccc}
\textbf{Matr} & & \textbf{LMatr} \\
& & \\
\textbf{SMatr} & & \textbf{FMatr}
\end{array}
$$

The functors in the above diagram have no adjoints other than those indicated.

Proof Collate the previous results in this subsection.   □

7 Factorization

In this section we study how morphisms between matroids can be factored into easier classes of strong maps. Let us first recall the basic definition [2].

Definition 7.1 A *weak factorization system* in a category consists of two classes of morphisms $\mathcal{L}$ and $\mathcal{R}$ such that:

– every morphism f factors as $f = r \circ l$ for some $l \in \mathcal{L}$ and $r \in \mathcal{R}$;
– both $\mathcal{L}$ and $\mathcal{R}$ contain all isomorphisms;
– if $l, l' \in \mathcal{L}$, $r, r' \in \mathcal{R}$, and arbitrary morphism f, g make the following diagram commute, then there is *fill-in h* making both squares commute:

$$
\begin{array}{ccccc}
M & \xrightarrow{\ l\ } & I & \xrightarrow{\ r\ } & N \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle g} \\
M' & \xrightarrow[\ l'\ ]{} & I' & \xrightarrow[\ r'\ ]{} & N'
\end{array}
\tag{4}
$$

In an *orthogonal factorization system* the fill-in h is additionally unique.

The standard example of an orthogonal factorization system is that every function between sets factors as an epimorphism followed by a monomorphism. The category of matroids has a very similar orthogonal factorization system.

Lemma 7.2 *The category* **Matr**$_\bullet$ *has an orthogonal factorization system where* $\mathcal{L}$ *consists of epimorphisms and* $\mathcal{R}$ *consists of embeddings.*

Proof The fill-in is the restriction of g to the image of l, which is a strong map.   □

Epimorphisms in **Matr$_\bullet$** can be further decomposed into a quotient followed by a lattice-preserving map [23, page 228]. This would give another orthogonal factorization system with $\mathcal{L}$ quotients and and $\mathcal{R}$ lattice-preserving maps followed by embeddings, except that the fill-in h, which has to be the function $l' \circ f \circ l^{-1}$ by Lemma 3.11, need not be a strong map.

Finally, any quotient can be decomposed into an embedding followed by a contraction, as we have seen in Lemma 4.18. Again this does not give a weak factorization system, but the matroid Q of Lemma 4.18 is the minimal matroid through which a quotient factors like this [14].

Theorem 7.3 *The category* **Matr$_\bullet$** *has an orthogonal factorization system where $\mathcal{L}$ consists of lattice-preserving maps and $\mathcal{R}$ consists of maps that are injective on elements of each rank-1 flat.*

Proof Given a morphism $f\colon M \to N$, a matroid I is uniquely specified by saying that $L(I) = L(M)$, that I has the same rank-0 flat as N, and that the nonloop elements of each rank-1 flat F_i are copies of the elements of the elements of $f(F_i)$ (including extra copies of the loops). Now f decomposes as $M \xrightarrow{l} I \xrightarrow{r} N$, where l is the lattice-preserving map that acts as f on elements when ignoring indices i, and r is the map with $L(r) = L(f)$ that sends each element of I to the unindexed version of the element in N. Then l is strong because $L(l) = \mathrm{id}_{L(M)}$, and r is strong because $L(r) = L(f)$. By definition l is lattice-preserving, and r is obviously injective on elements of each rank-1 flat. Both $\mathcal{L}$ and $\mathcal{R}$ contain all isomorphisms by Lemma 3.4.

For the fill-in h in (4), take the morphisms with $L(h) = L(f)$ that acts on g on elements when ignoring indices. This is by construction the unique strong map that makes both squares commute, as we conclude by considering what h does on lattices and on elements.   □

Remark 7.4 The category **Matr$_\bullet$** has a double factorization system [20]: every morphism decomposes as a lattice-preserving map followed by an epimorphism injective on elements of each rank-1 flat followed by an embedding. This would give a *Quillen model structure*, whose *fibrations* are the maps that distinguish between parallel elements (*i.e.* are injective on elements of each rank-1 flat), whose *cofibrations* are the epimorphisms, and whose *weak equivalences* are the monomorphisms f such that $L(f)$ preserves rank, except that g need not be a weak equivalence when f and $g \circ f$ are.

Proof As embeddings are by construction injective, the orthogonal factorization systems of Lemma 7.2 and Theorem 7.3 are comparable in the sense of [20, Proposition 2.6], and hence form a double factorization system [20, Theorem 2.7]. The middle class of morphisms are those that are both epic and injective on rank-1 flats, that is, strong maps that are bijective when restricted to rank-1 flats. As for the purported Quillen model structure, the weak equivalences are the strong maps that factor as a lattice-preserving map followed by an embedding, which are precisely the strong maps as in the statement, but [20, Theorem 3.10.1b] is not satisfied. For an explicit counterexample, take $\mathcal{F}(M_1) = \{\{\bullet\}\}, \mathcal{F}(M_2) = \{\{\bullet\}, \{\bullet, *\}\}, \mathcal{F}(M_3) = \{\{\bullet, *\}\}$ and $f : M_1 \to M_2$, $g : M_2 \to M_3$ mapping each element to itself; f and

$g \circ f$ are clearly embeddings whereas g cannot be expressed as a lattice-preserving map followed by an embedding).   □

Clearly factorization systems in $\mathbf{Matr_\bullet}$ are strongly linked to decomposing morphisms in $\mathbf{GLat}$, where we now turn our attention. Call a morphism $f\colon A \to B$ in $\mathbf{GLat}$ a *contraction* when its restriction to the interval $[\bigvee f^{-1}(0_B), 1_A]$ is an isomorphism; call $g\colon A \to B$ an *embedding* when its corestriction to the interval $[0_B, g(1_A)]$ is an isomorphism. This is in line with the definitions of contraction and embedding for matroids: $L(f)$ is a contraction/embedding if and only if f is a contraction/embedding.

Lemma 7.5 *The category $\mathbf{GLat}$ has a weak factorization system where $\mathcal{L}$ consists of embeddings and $\mathcal{R}$ consists of contractions; it is not an orthogonal factorization system.*

Proof Every surjective map $f\colon M \to N$ in $\mathbf{Matr_\bullet}$ factorizes as $f = \tau \circ r \circ l$, where l is an embedding, r is a contraction and τ is a lattice-preserving map. Then $L(f) = L(\tau \circ r \circ l) = L(\tau) \circ L(r) \circ L(l) = L(r) \circ L(l)$ with $L(l) \in \mathcal{L}$ and $L(r) \in \mathcal{R}$. The middle object I is the lattice of the nth Higgs lift of $S(N)$ towards $S(M + F_n)$ along f_+, where n is the *nullity* $\mathrm{rk}_M(1_M) - \mathrm{rk}_N(f(1_M))$ of f, where F_n is the geometric lattice whose height and width are both n, and where f_+ coincides with f on the atoms of M and sends atoms of F_n to 0_N. Explicitly, $L(I)$ is the sublattice $\{X \mid X = \bigvee f^{-1}(f(X))\} \cup \{X \mid \mathrm{rk}(f(X)) = \mathrm{rk}(X)\}$ of $L(M)$, the canonical embedding $L(r)\colon L(M) \to L(I)$ corestricts to the identity, and the canonical contraction $L(l)\colon L(I) \to L(N)$ acts as f_+. All of this extends to maps that are not surjective on flats [13]. For the fill-in (4), define $h\colon I \to I'$ as the morphism that sends every atom a of I that is also in $L(A)$ to $f(a)$, and all other atoms to $0_{I'}$.   □

The rest of this section shows that if $\mathbf{GLat}$ has an orthogonal factorization system, then it must induce a factorization system in $\mathbf{Matr_\bullet}$.

Proposition 7.6 *For every $N \in \mathbf{Matr_\bullet}$, the functor $L^N\colon \mathbf{Matr_\bullet}/N \to \mathbf{GLat}/L(N)$ between slice categories has a right adjoint R^N that is full.*

Proof For an object in $l\colon G \to L(N)$ in $\mathbf{GLat}/L(N)$, where G has atoms a_i, define $R_l^N(G)$ to be the matroid with flats G and the same rank-0 flat as N where the nonloop elements of each rank-1 flat F_i with $L(F_i) = a_i$ are copies, indexed by i, of the elements of $L^{-1}(l(a_i))$, including extra copies of the loops. Then $R^N(l)\colon R_l^N(G) \to N$ is the map with $L(R^N(l)) = l$ that sends each element in $R_l^N(G)$ to the unindexed version of the element in N.

For a morphism $f\colon l_1 \to l_2$ in $\mathbf{GLat}/L(N)$, define $R^N(f)\colon R_{l_1}^N(G) \to R_{l_2}^N(G)$ to the map with $L(R^N(f)) = f$ that acts as the identity on elements. Since $R^N(l_1) \circ R^N(f) = R^N(l_2)$, this is a legitimate morphism in $\mathbf{Matr_\bullet}/N$, and it clearly respects identities and composition, giving a well-defined functor R^N.

Moreover, we claim that the there uniquely exist morphisms η_f and $\hat{h}$ making the following diagrams commute for every $f\colon M \to N$ in $\mathbf{Matr_\bullet}$, every $g\colon K \to$

$L(M)$ in **GLat**, and every strong map $h\colon M \to \mathrm{dom}(R^N(g))$ with $R^N(g) \circ h = f$:
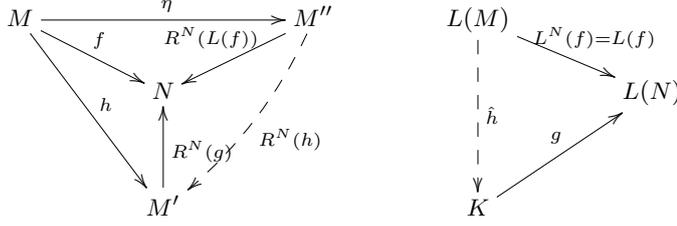
Namely, take η to be the lattice-preserving map that acts as f on elements when ignoring indices. In the left diagram, the upper and right triangles then commute by construction. By the left triangle, the map h acts as f on elements when ignoring indices. Because both paths along the outer triangle act as f on elements, and both act as h on the lattice, the outer triangle commutes. Since η is lattice-preserving, there can exist at most one $\hat{h}$ that makes the large triangle commute; namely $L(h)$.

Finally, to see that R^N is full: within each rank-1 flat, the strong maps forming the objects of **Matr**$_\bullet/N$ are one-to-one. Therefore, fixing the lattice maps that form the objects of **GLat**$/L(N)$ constrains the morphisms of **Matr**$_\bullet/N$ to identities on elements.   □

Corollary 7.7 *Any orthogonal factorisation system* $(\mathcal{L}, \mathcal{R})$ *in* **GLat** *induces an orthogonal factorization system* $(L^{-1}(\mathcal{L}), \mathcal{R}')$ *in* **Matr**$_\bullet$.

Proof Apply [25] to the previous proposition.   □

8 Constructions

This final section examines functoriality of various operations of matroid theory.

We start with one of the most prominent ones: the *dual* M^* of a matroid M has the same ground set, but the bases of M^* are complements of bases of M. One might hope that taking duals is functorial on matroids and strong maps, but that is not the case.

Remark 8.1 There are matroids M, N with strong maps $M \to N$ but no strong maps $M^* \to N^*$ or $N^* \to M^*$, so taking duals is not functorial on **Matr**. (To see this, take $\mathcal{F}_M = \{\emptyset, \{a, b, c\}\}$ and $\mathcal{F}_N = \{\{*\}\}$; then there is only one map $M \to N$ and no maps $N \to M$, but there are maps both $M^* \to N^*$ and $N^* \to M^*$.)

Taking duals is functorial on the subcategory of **Matr** of surjective strong maps between matroids of equal cardinality, since a quotient $q\colon M \to N$ does induce a strong map $q^*\colon N^* \to M^*$.

We already saw in Lemma 3.2 that adding loops is a functorial process. We now prove that the same holds for adding isthmuses.

Proposition 8.2 *There is an endofunctor* $A\colon$ **Matr** $\to$ **Matr** *that acts on objects as* $\mathcal{B}_{A(M)} = \{B \cup \{*\} \mid B \in \mathcal{B}_M\}$, *and on morphisms as* $A(f)(*) = *$ *and* $A(f)(x) = f(x)$.

Proof By construction $*$ is an isthmus in $A(M)$. Since $A = (-) + D$, where D is the free matroid on $\{*\}$, the assignment A is clearly functorial.   □

Next we consider the operations of deletion and contraction. As defined in Definition 3.5, they are not functorial. To see this, suppose $f\colon M \to N$ maps $m \mapsto n$ and $g\colon L \to M$ maps $l \mapsto m$, where l, m, n are all nonloops; if m is among the elements by which we contract or delete but the elements l and n are not, then the composite morphism $f \circ g$ cannot canonically be mapped to any strong map, either covariant or contravariant. However, these operations become functorial when we change the category to ensure the deleted/contracted elements in M are exactly those that map to deleted/contracted elements of N.

Proposition 8.3 *Write* $\mathbf{Matr}_*$ *for the category whose objects are pairs* (M, Z) *of* $M \in \mathbf{Matr}$ *and* $Z \subseteq |M|$, *and whose morphisms* $(M, Z) \to (M', Z')$ *are strong maps* $M \to M'$ *where* Z *is the preimage of* Z'. *There are functors:*

$$C\colon \mathbf{Matr}_* \to \mathbf{Matr} \qquad\qquad D\colon \mathbf{Matr}_* \to \mathbf{Matr}$$
$$(M, Z) \mapsto M/Z \qquad\qquad\qquad (M, Z) \mapsto M\backslash Z$$
$$f \mapsto f \qquad\qquad\qquad\qquad f \mapsto f$$

Proof The operations on objects are clearly well-defined, and identity and composition are clearly respected. We need only show that the operations on morphisms are well-defined. For $X \subseteq Y \subseteq |M| \setminus Z$:

$$
\begin{aligned}
\mathrm{rk}_{C(M')}(f(Y)) - \mathrm{rk}_{C(M')}(f(X)) &= \mathrm{rk}_{M'}(f(Y) \cup Z') - \mathrm{rk}_{M'}(f(X) \cup Z') \\
&= \mathrm{rk}_{M'}(f(Y \cup Z)) - \mathrm{rk}_{M'}(f(X \cup Z)) \\
&\leq \mathrm{rk}_M(Y \cup Z) - \mathrm{rk}_M(X \cup Z) \\
&= \mathrm{rk}_{C(M)}(Y) - \mathrm{rk}_{C(M)}(X),
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{rk}_{D(M')}(f(Y)) - \mathrm{rk}_{D(M')}(f(X)) &= \mathrm{rk}_{M'}(f(Y)) - \mathrm{rk}_{M'}(f(X)) \\
&\leq \mathrm{rk}_M(Y) - \mathrm{rk}_M(X) \\
&= \mathrm{rk}_{D(M)}(Y) - \mathrm{rk}_{D(M)}(X).
\end{aligned}
$$

The result now follows from Lemma 3.9. $\square$

We may implement a series of n deletions and contractions by employing the category $\mathbf{Matr}_{*n}$, whose objects are $(M, Z_1, \ldots, Z_n)$ where the sets $Z_i \subseteq |M|$ are disjoint, and whose morphisms $(M, Z_1, \ldots, Z_n) \to (M', Z'_1, \ldots, Z'_n)$ are strong maps $f\colon M \to M'$ such that Z_i is the preimage of Z'_i for all i. Then we can define functors $C, D\colon \mathbf{Matr}_{*n+1} \to \mathbf{Matr}_{*n}$. The composition of all these functors produces a minor, so taking subobjects in $\mathbf{Matr}$ is functorial.

Theorem 8.4 *Deletion* $D\colon \mathbf{Matr}_{*n+1} \to \mathbf{Matr}_{*n}$ *is right adjoint to the inclusion* $\mathbf{Matr}_{*n} \to \mathbf{Matr}_{*n+1}$ *given by* $(M, Z_1, \ldots, Z_n) \mapsto (M, Z_1, \ldots, Z_n, \emptyset)$ *and* $f \mapsto f$.

Proof Take the unit η to be the identity, and the transpose $\hat{f}\colon (M, Z_1, \ldots, Z_n, \emptyset) \to (M', Z'_1, \ldots, Z'_{n+1})$ to have the same underlying function as $f\colon (M, Z_1, \ldots, Z_n) \to (M', Z'_1, \ldots, Z'_n)$. $\square$

The following matroid operation of extension turns out not to be functorial.

Definition 8.5 ([23], Proposition 7.3.3) The *free extension* of a matroid M by p is defined as the matroid $X(M)$ with $|X(M)| = |M| \cup \{p\}$ and flats

$$\big\{ K \in \mathcal{F}_M \setminus \{|M|\} \big\} \cup \big\{ K \cup \{p\} \mid K \in \mathcal{F}_M \setminus \mathcal{H}(M) \big\}.$$

Remark 8.6 Let M be the free matroid on $\{a, b\}$ and N the free matroid on $\{a, b, c, d\}$. Let $f \colon M \to N$ be the strong map $f = \{a \mapsto a, b \mapsto b\}$. Then $\mathcal{F}_{X(M)} = \big\{ \emptyset, \{a\}, \{b\}, \{d\}, \{a, b, d\} \big\}$ and

$$\mathcal{F}_{X(N)} = \big\{ \emptyset, \{a\}, \{b\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c, d\} \big\}.$$

There are no strong maps $X(M) \to X(N)$ that agree with f. Hence $X(f)$ cannot be canonically defined in a way that respects identities, and the free extension cannot be functorial.

It follows that the matroid operation of *truncation* (contraction by p after free extension by p) cannot be functorial, because it is equivalent to a free extension by p after the addition of an isthmus p which is functorial by Proposition 8.2. A similar counterexample (omitted here) shows that the dual operations of the free extension and the truncation, namely the *free coextension* and the Higgs *lift*, are not functorial either.

Finally we consider *erection*, the inverse matroid operation of truncation. The erections of a matroid M form a lattice based on a certain ordering [23, Chapter 7.5], whose top element $E(M)$ is the so-called *free erection* and whose bottom element is M itself; by convention $E(M) = M$ if M has no erections at all. This operation turns out to be not functorial.

Remark 8.7 Let M, N be the erectible matroids with the following sets of flats:

$$
\begin{aligned}
\mathcal{F}(M) = \big\{ &\emptyset, \\
&\{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \\
&\{0, 1\}, \{0, 2\}, \{0, 3\}, \{0, 4\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3, 4\}, \\
&\{0, 1, 2, 3, 4\} \big\} \\
\mathcal{F}(N) = \big\{ &\emptyset, \{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{0, 1, 2, 3, 4\} \big\}.
\end{aligned}
$$

Then the identity function is a strong map $f \colon M \to N$. But there are no strong maps $E(M) \to E(N)$ that agree with f, so erection cannot be functorial.

So far we have considered operations that input a single matroid. The rest of this section considers operations that combine two matroids into a new one, and discusses whether they give monoidal structure on the category **Matr** of **Matr**$_\bullet$. For example, by Proposition 5.3, the coproduct gives one monoidal structure.

The following operations from in matroid theory do not give monoidal structure: the *sum* or *union* $M \cup N$ is the matroid whose independent sets are the unions of the independent sets of its constituents; the *product* or *intersection* of $M \cap N$ is the matroid $(M^* \cup N^*)^*$; the *half-dual sum* is the matroid $M^* \cup N^*$.

Remark 8.8 Unions do not give monoidal structure: there are matroids A, B, C, D and maps $A \to C$ and $B \to D$ such that there are no maps $A \cup B \to C \cup D$.

$$\mathcal{F}_A = \mathcal{F}_B = \big\{\{\bullet\}, \{\bullet, 0, 1, 2\}\big\},$$

$$\mathcal{F}_C = \big\{\{f\}, \{f, a\}, \{f, b\}, \{f, c\}, \{f, d\}, \{f, e\},$$
$$\{f, a, b\}, \{f, a, c\}, \{f, a, d\}, \{f, a, e\}, \{f, e, b\}, \{f, e, c\}, \{f, e, d\},$$
$$\{f, a, e, b\}, \{f, a, e, c\}, \{f, a, e, d\}, \{f, a, e, b, c, d\}\big\},$$

$$\mathcal{F}_D = \big\{\{e\}, \{e, a\}, \{e, b\}, \{e, c\}, \{e, d\}, \{e, f\},$$
$$\{e, a, b\}, \{e, a, c\}, \{e, a, d\}, \{e, a, f\}, \{e, b, c\},$$
$$\{e, b, d\}, \{e, b, f\}, \{e, c, d\}, \{e, c, f\}, \{e, d, f\},$$
$$\{e, f, a, b\}, \{e, f, a, c\}, \{e, f, a, d\}, \{e, f, b, c\}, \{e, f, b, d\}, \{e, f, c, d\}, \{e, a, b, c, d\},$$
$$\{e, a, b, c, d, f\}\big\}$$

Then $\mathcal{F}_{A \cup B} = \big\{\{\bullet\}, \{\bullet, 0\}, \{\bullet, 1\}, \{\bullet, 2\}, \{\bullet, 0, 1, 2\}\big\}$ and $C \cup D$ is the free matroid on $\{a, b, c, d, e, f\}$. Hence $A \cup B$ has a loop but $C \cup D$ does not, so there can be no maps $A \cup B \to C \cup D$, whereas there are maps $A \to C$ and $B \to D$.

Remark 8.9 Intersections do not give monoidal structure: there are strong maps $A \to C$ and $B \to D$ such that there are no maps $A \cap B \to C \cap D$.

$$\mathcal{F}_A = \mathcal{F}_B = \big\{\emptyset, \{1, 2\}\big\}, \qquad \mathcal{F}_C = \mathcal{F}_D = \big\{\emptyset, \{x\}\big\}.$$

Then $\mathcal{F}_{A \cap B} = \big\{\{1, 2\}\big\}$ and $\mathcal{F}_{C \cap D} = \big\{\emptyset, \{x\}\big\}$. Hence there is a map $1, 2 \mapsto x$ in each homset $\mathbf{Matr}(A, C)$ and $\mathbf{Matr}(B, D)$, but there can be no map $A \cap B \to C \cap D$ because there is no loop in $C \cap D$ to receive the loops of $A \cap B$.

Example 8.10 Half-dual unions do not give monoidal structure: there are maps $A \to C$ and $B \to D$ such that there are no maps $A \cup B^* \to C \cup D^*$.

$$\mathcal{F}_A = \big\{\{0\}\big\}, \qquad \mathcal{F}_B = \big\{\emptyset, \{0\}\big\}, \qquad \mathcal{F}_C = \mathcal{F}_D = \big\{\{*\}\big\}.$$

Then $\mathcal{F}_{A \cup B^*} = \big\{\{0\}\big\}$ and $\mathcal{F}_{C \cup D^*} = \big\{\emptyset, \{*\}\big\}$. Once again, the homsets $\mathbf{Matr}(A, C)$ and $\mathbf{Matr}(B, D)$ each have to contain at least the map sending everything to the loop, whereas $A \cup B^*$ has a loop that cannot be mapped to anything in $C \cup D^*$.

The *intertwining* of two matroids, defined as a minor-minimal matroid that contains them both as minors, is not a monoidal product either, as it is not always unique up to isomorphism.

We end on a positive note, by showing that altering the category slightly allows for two more monoidal structures. Write $\mathbf{Matr}_\times$ for the category of matroids with a distinguished element and strong maps preserving the distinguished element. The *parallel connection $M \| N$* is the coproduct in this category [23]. Explicitly, the ground set of $M \| N$ is the disjoint union of $|M|$ and $|N|$, the distinguished elements are identified, and the flats of $M \| N$ are the unions of flats in M and in N. This is similar to the coproduct in $\mathbf{Matr}$, except that the distinguished element need not be a loop.

Finally, the *series connection MN* is defined dually to the parallel connection: $(M^* \| N^*)^*$, and gives another monoidal structure on $\mathbf{Matr}_\times$. This monoidal structure is not a product, but remarkably enough it is affine, in the sense that there are always maps $MN \to M$ and $MN \to N$. Neither of the parallel connection and the series connection distributes over each other.

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