BOOLEAN SUBALGEBRAS OF ORTHOALGEBRAS

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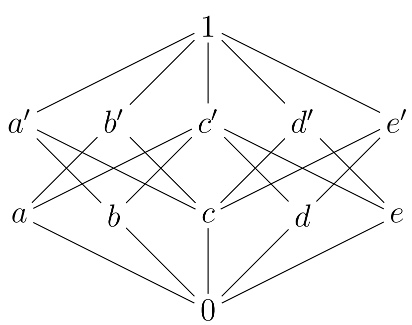
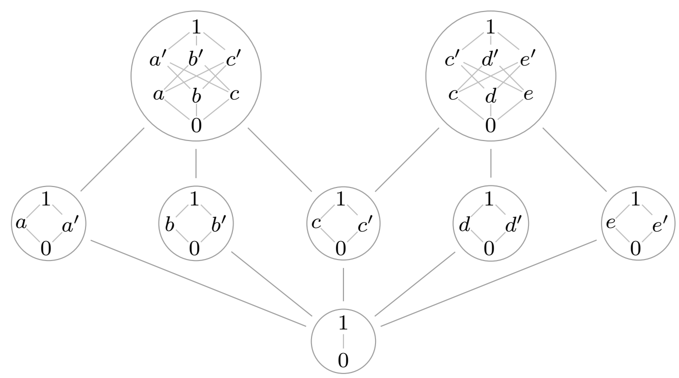
Abstract. We reconstruct orthoalgebras from their partially ordered set of Boolean subalgebras, and characterize partially ordered sets of this form, using a new notion of a direction. For Boolean algebras, this reconstruction is functorial, and nearly an equivalence.

1. Introduction

Orthoalgebras [5] are certain structures with a partially defined binary operation $\oplus$ called orthogonal sum, a unary operation $'$ called orthocomplementation, and constants $0, 1$. Examples are obtained by taking any Boolean algebra, orthomodular lattice, or orthomodular poset, and defining $\oplus$ to be the join of orthogonal pairs. Below at left is an orthoalgebra constructed from gluing together two Boolean algebras $\{0, a, b, c, a', b', c', 1\}$ and $\{0, c, d, e, c', d', e', 1\}$. The orthogonal sum $\oplus$ is the union of the orthogonal join operations of these Boolean algebras.





A *subalgebra* of an orthoalgebra is a subset that is closed under the operations. A subalgebra naturally forms an orthoalgebra. We say that a subalgebra is a *Boolean subalgebra* if it is isomorphic to an orthoalgebra obtained from a Boolean algebra by restricting its join operation to orthogonal elements. The diagram above at the right gives the Boolean subalgebras of the orthoalgebra at left, partially ordered by set inclusion.

It is known that every orthoalgebra is the union of its Boolean subalgebras, and that the operations of an orthoalgebra are determined by the operations of its Boolean subalgebras. Thus the diagram above at right that describes the Boolean subalgebras by listing their elements and operations determines the orthoalgebra at left. The first main result of this paper is that each non-trivial orthoalgebra can be reconstructed up to isomorphism from the order structure of its partially ordered set of Boolean subalgebras. Let us emphasize that here we forget everything about the elements of the partially ordered set except that they are Boolean subalgebras.

The main ingredient in this reconstruction is the new notion of a *direction*. Call an element of a poset *basic* if it is of height 1 or 0. Then the basic elements in the poset of Boolean subalgebras

Supported by EPSRC Fellowship EP/L002388/1.
The fourth author was supported by the Ministry of Education of the Czech Republic under Project RVO13000.

of an orthoalgebra are the subalgebras $\{0, a, a', 1\}$ given by some element a of the orthoalgebra. In each Boolean subalgebra B covering $\{0, a, a', 1\}$ in this poset of Boolean subalgebras, we can consider the subalgebra $\downarrow a \cup \uparrow a'$. This will be equal to either $\{0, a, a', 1\}$ or to B depending on which of a or a' is basic in B. A direction assigns a consistent choice of this to each cover of $\{0, a, a', 1\}$. If we assume that maximal Boolean subalgebras are not *small*, i.e. having 4 or fewer elements, each basic element $\{0, a, a', 1\}$ in this poset will have exactly 2 directions, and these serve the role of a and a' in an isomorphic copy of the given orthoalgebra built from the directions.

The second result of this paper is a characterization of the partially ordered sets that arise as those of Boolean subalgebras of an orthoalgebra. Certain obvious properties of the poset of Boolean subalgebras are abstracted to form what we call *orthodomains*, but these basic properties are not sufficient. The crucial ingredient is to have an orthodomain with enough directions. This is somewhat analogous to the characterization of lattices of open sets of topological spaces as frames with enough points. We then show that the orthodomains with enough directions whose structure is fundamentally determined at height 3 are the posets that are isomorphic to the Boolean subalgebra posets of orthoalgebras. As an interesting byproduct, if one wanted to describe an orthoalgebra by giving its poset of Boolean subalgebras, our results show that it is sufficient to give the poset of Boolean subalgebras of height 3 or less, in other words, the poset of its Boolean subalgebras having at most 16 elements, and one cannot do better.

The third result of this paper is an investigation of functorial aspects of the reconstruction. We have to make some exceptions because the Boolean algebra with 1 element and the Boolean algebra with 2 elements have the same partially ordered set of Boolean subalgebras. Similarly, the Boolean algebra with 4 elements has 2 automorphisms, whereas the lattice of its Boolean subalgebras has only 1 automorphism. We prove that the category of Boolean algebras is, up to these restrictions, equivalent to that of Boolean orthodomains. Extending these results to the orthoalgebra setting remains a work in progress.

These results fit in an established line of research. Sachs showed that every Boolean algebra with at least 4 elements is determined by its lattice of Boolean subalgebras [17]. Grätzer *et. al.* characterized the lattices that arise this way, and reconstructed the original Boolean algebra as a direct limit. Sachs' result extends to orthomodular lattices [8], and has versions relating C^*-algebras and von Neumann algebras to their posets of abelian subalgebras [3, 7] that are considered in the topos approach to quantum mechanics [10]; for a thorough account see [13]. Grätzer's limit approach was lifted to a more general setting in [9].

There are two paths for yet more general structures than the orthoalgebras considered here, the partial Boolean algebras of Kochen and Specker [12], and effect algebras [4]. There are two non-isomorphic partial Boolean algebras having the same poset of Boolean subalgebras with one an orthoalgebra and the other not. Effect algebras which are not lattices even need not be unions of some "nice" maximal subalgebras, e.g., sub-MV-algebras [16]. These are fundamental barriers to lifting results to the partial Boolean algebra or effect algebra setting. A version of our results for effect algebras would be worth exploring, but would require a modification of the notion of a Boolean subalgebra as is seen by considering the 3-element effect algebra. Perhaps subalgebras that are effect algebra quotients of Boolean algebras may be a path.

This paper is organized in the following way. Section 2 starts by describing directions in the Boolean setting, and Section 3 introduces orthoalgebras and their Boolean subalgebras. In Section 4 directions are generalized to orthoalgebras and used to reconstruct an orthoalgebra A from its partially ordered set $\mathrm{BSub}(A)$ of Boolean subalgebras. Directions are used again in Section 5 to characterize the orthodomains that are isomorphic to $\mathrm{BSub}(A)$ for an orthoalgebra

A. Section 6 discusses the functoriality of these constructions in the Boolean setting, and Section 7 provides concluding remarks.

2. Subalgebras of Boolean algebras and their directions

We use standard terminology for partially ordered sets, as in *e.g.* [1]. In particular, for an element x of a partially ordered set X, denote its principal ideal and principal filter by

$$\downarrow x = \{w \in X \mid w \leq x\} \quad \text{and} \quad \uparrow x = \{y \in X \mid x \leq y\}.$$

Definition 2.1. Write $\mathrm{Sub}(B)$ for the set of Boolean subalgebras of a Boolean algebra B partially ordered by inclusion with $\perp$ its least element and $\top$ its largest element.

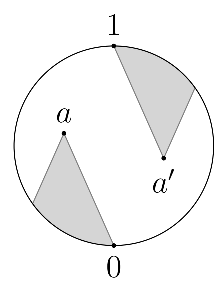
Since the intersection of Boolean subalgebras is a subalgebra, $\mathrm{Sub}(B)$ is a complete lattice, and since finitely generated Boolean algebras are finite, the compact elements of this lattice are the finite Boolean subalgebras of B. Here we recall that an element x of a partially ordered set X with directed joins is *compact* if $x \leq \bigvee Y$ for a directed subset $Y \subseteq X$ implies that $x \leq y$ for some $y \in Y$. Since every Boolean algebra is the union of its finite subalgebras, $\mathrm{Sub}(B)$ is an algebraic lattice. The algebraic lattices of the form $\mathrm{Sub}(B)$ were characterized by Grätzer *et. al.* [6] as follows. Here we recall that a partition lattice is a lattice that is isomorphic to the lattice of the partitions of a set.

Theorem 2.2. [6] *A poset X is isomorphic to* $\mathrm{Sub}(B)$ *for a Boolean algebra B if and only if:*

(1) *X is an algebraic lattice;*
(2) *$\downarrow x$ is a finite partition lattice for each compact element x of X.*

We call such lattices *Boolean domains.*

Definition 2.3. A subalgebra of a Boolean algebra is called a *(principal) ideal subalgebra* when it is of the form $I \cup I'$ for a (principal) ideal I, where $I' = \{a' \mid a \in I\}$.



Principal ideal subalgebras are of the form $\downarrow a \cup \uparrow a'$ for $a \in B$. They will play a central role throughout the paper. To describe their use, we begin with the order-theoretic characterization of ideal subalgebras given by Sachs [17].

Definition 2.4. An element x of a lattice is *dual modular* if $(x \vee y) \wedge z = x \vee (y \wedge z)$ for each z with $x \leq z$ and $(w \vee x) \wedge y = w \vee (x \wedge y)$ for each y with $w \leq y$.

Lemma 2.5. [17, Theorem 1] *The dual modular elements of* $\mathrm{Sub}(B)$ *are the ideal subalgebras.*

The least element $\perp$ of the Boolean domain $\mathrm{Sub}(B)$ is $\{0, 1\}$, the largest element $\top$ is B, and the atoms of $\mathrm{Sub}(B)$ are the elements $\{0, a, a', 1\}$ for $a \neq 0, 1$. Hence there is a bijection between complementary pairs $\{a, a'\}$ in B and elements of $\mathrm{Sub}(B)$ that are either $\perp$ or an atom.

Definition 2.6. Call an element of a poset with a least element *basic* if it is either an atom or the least element.

Lemma 2.7. *For a Boolean algebra B, the basic elements of* $\mathrm{Sub}(B)$ *that are dual modular are* $\{0, a, a', 1\}$ *where either a, a' is basic. In fact, they are principal ideal subalgebras.*

Proof. Follows immediately from Lemma 2.5. □

Our key definition is the following:

Definition 2.8. For B a Boolean algebra, we define the mapping $\varphi\colon B \to (\mathrm{Sub}(B))^2$ by

$$\varphi(a) = (\downarrow a \cup \uparrow a', \downarrow a' \cup \uparrow a)\,.$$

We call $\varphi(a)$ the *principal pair* corresponding to a.

We call a Boolean algebra *small* if it has at most 4 elements. Our aim is to show that if B is not small, than φ is one-to-one, and to characterize the range of φ in purely order-theoretical terms. This will allow us to reconstruct an isomorphic copy of B from the poset $\mathrm{Sub}(B)$. We formulate this for a general Boolean domain rather than a special case of $\mathrm{Sub}(B)$ for a Boolean algebra B, although all are isomorphic to such.

Definition 2.9. Let X be a Boolean domain. A *principal pair* of X is an ordered pair (y, z) of dual modular elements of X that satisfies one of the following conditions:

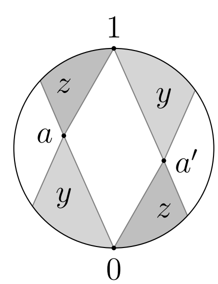
(1) $y = \top$, z is a basic element;
(2) $z = \top$, y is a basic element;
(3) $y \vee z = \top$ and $y \wedge z$ is a basic element which is not dual modular.

We say that a principal pair (y, z) (as well as (z, y)) is a principal pair for the basic element x if $y \wedge z = x$. Write $\mathrm{Pp}(X)$ for the set of principal pairs of X.

Remark 2.10. Notice that in Definition 2.9 the element $y \wedge z = x$ is always basic and it is dual modular iff case (1) or (2) applies. Therefore, if x is a dual modular basic element, then $(x, \top)$ and $(\top, x)$ are the only principle pairs for x.

If $B = \{0, 1\}$, then $\bot = \top$ and $\varphi[B] = \mathrm{Pp}(\mathrm{Sub}(B)) = \{(\top, \top)\}$. If B has 4 elements, then $B = \{0, a, a', 1\} = \top$, $a \notin \{0, 1\}$, and $\varphi[B] = \mathrm{Pp}(\mathrm{Sub}(B)) = \{(\bot, \top), (\top, \bot), (\top, \top)\}$, where $(\top, \top) = \varphi(a) = \varphi(a')$. A principal pair (y, z) satisfies $y = z$ only if B is small and $y = z = \top$.

Proposition 2.11. *Let B be a Boolean algebra, $a \in B$, and $x = \{0, a, a', 1\}$ be the corresponding basic element of* $\mathrm{Sub}(B)$. *For the map φ of Definition 2.8, $\varphi(a)$ and $\varphi(a')$ are the only principle pairs for x, and if B is not small these are distinct. So the image $\varphi[B]$ is* $\mathrm{Pp}(\mathrm{Sub}(B))$, *and if B is not small then $\varphi\colon B \to \mathrm{Pp}(\mathrm{Sub}(B))$ is a bijection.*



Proof. If a is basic then $\varphi(a) = (x, \top)$, and if a' is basic then $\varphi(a) = (\top, x)$. In both cases, x is a dual modular basic element. So $\varphi(a), \varphi(a')$ are the two possible principal pairs for x. If B is not small, they are distinct (Remark 2.10).

Assume that a is not 0, 1, an atom, or a coatom. So there are b, c with $0 < b < a < c < 1$ (and B is not small). Then x is a basic element that is not dual modular. Let $y = \downarrow a \cup \uparrow a'$ and $z = \downarrow a' \cup \uparrow a$. It is clear that y, z are ideal subalgebras, and hence are dual modular. Also $y \wedge z = x$. For any $e \in B$ we have $e = (e \wedge a) \vee (e \wedge a')$. Since $e \wedge a \in y$ and $e \wedge a' \in z$, then e is in the subalgebra $y \vee z$ generated by y, z. So $y \vee z = \top$. Hence (y, z) and (z, y) are principal pairs for x. Since $b \in y$ and $b \notin z$, these principal pairs are distinct.

We now show these are the only principal pairs for x. Suppose that (v, w) is a principal pair for x. Since v, w are dual modular, they are ideal subalgebras. So $v = I \cup I'$ and $w = J \cup J'$ for ideals $I, J \subseteq B$. Now $x = v \wedge w$ gives

$$\{0, a, a', 1\} = (I \cap J) \cup (I \cap J') \cup (I' \cap J) \cup (I' \cap J')$$

It cannot be the case that $a \in I \cap J$ since then $b \in I \cap J$ because I and J are ideals, and similarly $a \notin I' \cap J'$. So one of a, a' belongs to $I \cap J'$ and the other to $I' \cap J$. Say $a \in I \cap J'$. Since J' is a filter, there cannot be an element of I other than 1 that is larger than a since it would belong to $I \cap J'$, and since I is an ideal and $a < c < 1$ it cannot be that $1 \in I$ since this would imply that $c \in I$. So a is the largest element of I, and similarly it is the least element of J'. So $(v, w) = (y, z)$. If $a \in I' \cap J$, then by symmetry $(v, w) = (z, y)$.

We have shown for any $a \in B$ that $\varphi(a), \varphi(a')$ are principal pairs for $x = \{0, a, a', 1\}$, are the only principle pairs for x, and that these are distinct if B is not small. Since every principal pair of $\mathrm{Sub}(B)$ is a principal pair for some basic element and all basic elements arise as $\{0, a, a', 1\}$ for some $a \in B$, this shows that φ is onto. If $a, b \in B$ and $\varphi(a) = \varphi(b)$, then $\varphi(a)$ and $\varphi(b)$ are principal pairs for the same basic element. Then if B is not small, $a = b$. □

As every Boolean domain is isomorphic to $\mathrm{Sub}(B)$ for some Boolean algebra B, and a Boolean domain with more than two elements is isomorphic to $\mathrm{Sub}(B)$ for some B that is not small, Proposition 2.11 and Remark 2.10 give the following:

Corollary 2.12. *If X is a Boolean domain, then each basic element has at most two principle pairs, and if X has more than two elements, each basic element has exactly two principle pairs. In the case where X has two or fewer elements, each element is basic, all elements other than $\top$ have two directions, and $\top$ has a single direction.*

How do we incorporate the Boolean algebra structure in our considerations? If $\varphi(a) = (y, z)$, then $\varphi(a') = (z, y)$. The partial ordering of $\mathrm{Pp}(\mathrm{Sub}(B))$ is more subtle. If $\varphi(a) = (y_1, z_1)$ and $\varphi(b) = (y_2, z_2)$, then $a \leq b$ implies $y_1 \leq y_2$, and $z_1 \geq z_2$. But these relationships can hold without $a \leq b$. If a is an atom of B, so $y_1 = \{0, a, a', 1\}$ is a dual modular atom of X, then $\varphi(a) = (y_1, \top)$ and $\varphi(a') = (\top, y_1)$, but of course $a \not\leq a'$. The following definition excludes this situation. Notice that this is the only situation requiring an exception. Suppose that $y_1 \leq y_2$, $z_1 \geq z_2$, and $a \not\leq b$. We can easily exclude the cases when $y_1 \wedge z_1$ is not a dual modular atom or when $y_1 = \top$. In the remaining case, $z_1 = \top$ and a is an atom. Thus $a \not\leq b$ means that a, b are orthogonal, i.e., $a \leq b'$. If $a < b'$, then $y_2 = \downarrow b \cup \uparrow b'$ does not contain a; a contradiction with $y_1 \leq y_2$. The case of $b = a'$ ($y_2 = z_1$, $z_2 = y_1$) is the only one which needs to be forbidden.

Definition 2.13. Let X be a Boolean domain. Define a unary operation $'$ on $\mathrm{Pp}(X)$ by

$$(y, z)' = (z, y)\,.$$

Define a binary relation $\leq$ on $\mathrm{Pp}(X)$ by $(y_1, z_1) \leq (y_2, z_2)$ when $y_1 \leq y_2$, $z_1 \geq z_2$, and, additionally, if $y_1 \wedge z_1$ is a dual modular atom, then $(y_2, z_2) \neq (z_1, y_1)$.

Proposition 2.14. *For a Boolean algebra B that is not small, φ is an order isomorphism that preserves $'$.*

Proof. The complement $'$ of $\mathrm{Pp}(\mathrm{Sub}(B))$ commutes with φ.

Suppose $a, b \in B$, $a \leq b$. By Proposition 2.11, $\varphi(a)$ is a principal pair for $x = \{0, a, a', 1\}$. Observe $\downarrow a \cup \uparrow a' \subseteq \downarrow b \cup \uparrow b'$ and $\downarrow a' \cup \uparrow a \supseteq \downarrow b' \cup \uparrow b$. This suffices for $\varphi(a) \leq \varphi(b)$ unless x is a dual modular atom. If x is a dual modular atom, then, by Lemma 2.7, a or a' is an atom. Then $b \neq a'$, so $\varphi(b) \neq \varphi(a') = \varphi(a)'$, and hence $\varphi(a) \leq \varphi(b)$.

Finally, suppose $\varphi(a) \leq \varphi(b)$. We will show $a \leq b$ by contradiction; suppose $b' \not\leq a'$. Again $\downarrow a \cup \uparrow a' \subseteq \downarrow b \cup \uparrow b'$ and $\downarrow a' \cup \uparrow a \supseteq \downarrow b' \cup \uparrow b$. It follows that $a' \in \downarrow b$ and $b \in \downarrow a'$. So $a' \leq b$ and $b \leq a'$, giving $a' = b$. Since $\varphi(a) \leq \varphi(b) = \varphi(a)'$, the definition of $\leq$ implies that $x = \{0, a, a', 1\}$ cannot be a dual modular atom of $\mathrm{Sub}(B)$. Hence neither of a, a' is an atom of B. Since $a \not\leq b$ we have $a \neq 0$, so there is c such that $0 < c < a$. Then $c \in \downarrow a \cup \uparrow a'$, but $c \notin \downarrow a' \cup \uparrow a = \downarrow b \cup \uparrow b'$, a contradiction. □

For a Boolean algebra B that is small, φ preserves $\leq$ and $'$, but it is not a bijection, see the proof of Proposition 2.11.

Theorem 2.15. *For a Boolean algebra B with more than 4 elements, and a Boolean domain X with more than 2 elements:*

(1) $\mathrm{Sub}(B)$ *is a Boolean domain;*
(2) $\mathrm{Pp}(X)$ *is a Boolean algebra;*
(3) B *is isomorphic to* $\mathrm{Pp}(\mathrm{Sub}(B))$*;*
(4) X *is isomorphic to* $\mathrm{Sub}(\mathrm{Pp}(X))$.

Proof. Part (1) follows from Theorem 2.2 (even without any limitation of the number of elements of B). Part (3) follows from Proposition 2.14. Since X is a Boolean domain with more than 2 elements, Theorem 2.2 provides a Boolean algebra A with more than 4 elements with $X \simeq \mathrm{Sub}(A)$, so $\mathrm{Pp}(X) \simeq \mathrm{Pp}(\mathrm{Sub}(A))$, which is Boolean by (3), establishing part (2). To prove part (4), say $X \simeq \mathrm{Sub}(A)$ for a Boolean algebra A; then part (3) gives $\mathrm{Sub}(\mathrm{Pp}(X)) \simeq \mathrm{Sub}(\mathrm{Pp}(\mathrm{Sub}(A))) \simeq \mathrm{Sub}(A) \simeq X$. □

For a Boolean algebra B and $a \in B$, consider the Boolean subalgebras of B that contain a, and in each of these take the principle pair in its subalgebra lattice corresponding to a. While this is a more complex object, it leads to an alternative view of how principal pairs encode elements, and is the tool we use to extend matters to the orthoalgebra setting. For the following, we note that for $x = \{0, a, a', 1\}$, the upset $\uparrow x$ is the set of Boolean subalgebras of B that contain a. For $y \in \uparrow x$ we use the following notation.

$$\downarrow_y a = \{b \in y : b \leq a\} \quad \text{and} \quad \uparrow_y a = \{b \in y : a \leq b\}$$

Definition 2.16. For B a Boolean algebra, $a \in B$, and $x = \{0, a, a', 1\}$, let $d_a \colon \uparrow x \to (\mathrm{Sub}(B))^2$ be given by

$$d_a(y) = (\downarrow_y a \cup \uparrow_y a', \downarrow_y a' \cup \uparrow_y a).$$

Note that if y is a subalgebra of B, then the lattice of subalgebras $\mathrm{Sub}(y)$ of y is the interval $\downarrow y$ of $\mathrm{Sub}(B)$. Note also that the definition of d_a can be expanded to

$$d_a(y) = \big(y \wedge (\downarrow a \cup \uparrow a'), y \wedge (\downarrow a' \cup \uparrow a)\big).$$

The aim is as before — to characterize the mappings d_a order-theoretically and show that when B is not small that these are in bijective correspondence with B. Then we define structure on the collection of such mappings, and show that with respect to this structure, this bijective correspondence is an isomorphism.

Definition 2.17. For a Boolean domain X, a *direction* of X is a map $d\colon {\uparrow} x \to X^2$ for some basic element $x \in X$ such that for each $y, z \in {\uparrow} x$:

(1) $d(y)$ is a principal pair for x in the Boolean domain ${\downarrow} y$;
(2) if $y \leq z$ and $d(z) = (v, w)$, then $d(y) = (y \wedge v, y \wedge w)$.

We say d is a *direction for* x, and write $\mathrm{Dir}(X)$ for the set of directions of X.

Proposition 2.18. *Let X be a Boolean domain.*

(1) *Each direction d of X determines a principal pair $d(\top)$ of X and vice versa.*
(2) *If d is a direction for x and $x < y$, then $d(y)$ determines $d(\top)$ and hence d.*
(3) *For each principal pair (u, v) of X there is a unique direction d with $d(\top) = (u, v)$.*

In particular, there is a bijection $\gamma\colon \mathrm{Dir}(X) \to \mathrm{Pp}(X)$ with $\gamma(d) = d(\top)$.

Proof. (1) This is clear from the definition of a direction. (2) Assume first that $x = \bot$. Then the principal pairs for x in X are $(\top, \bot)$ and $(\bot, \top)$. If $d(\top) = (\top, \bot)$, then the definition of a direction gives $d(y) = (y, \bot)$, and if $d(\top) = (\bot, \top)$, then $d(y) = (\bot, y)$. Since $y \neq \bot$, $d(y)$ determines $d(\top)$. Suppose that x is an atom of X. Then, since $x < y$, ${\downarrow} y$ has more than two elements. So by Theorem 2.15 the components of the principal pair $d(y)$ for x in ${\downarrow} y$ are different. If $d(\top) = (u, v)$, then $d(y) = (y \wedge u, y \wedge v)$, and if $d(\top) = (v, u)$, then $d(y) = (y \wedge v, y \wedge u)$. Thus $d(y)$ determines $d(\top)$. (3) Since every Boolean domain is isomorphic to $\mathrm{Sub}(B)$ for some Boolean algebra B, we may assume $X = \mathrm{Sub}(B)$. Suppose (u, v) is a principal pair for a basic element x of X. By Theorem 2.15 there is $a \in B$ with $x = \{0, a, a', 1\}$ and $\varphi(a) = (u, v)$. So $u = {\downarrow} a \cup {\uparrow} a'$ and $v = {\downarrow} a' \cup {\uparrow} a$. Define $d\colon {\uparrow} x \to X^2$ by $d(y) = ({\downarrow_y} a \cup {\uparrow_y} a', {\downarrow_y} a' \cup {\uparrow_y} a)$. It is easily seen that d is a direction with $d(\top) = (u, v)$. Its uniqueness follows from (2). □

Proposition 2.18 allows us to count the number of directions for a basic element using the known number of principal pairs for it.

Corollary 2.19. *Let X be a Boolean domain. If $x \neq \top$ is a basic element of X, then there are exactly two directions for x. If $\top$ is a basic element of X (so X has at most two elements), then there is exactly one direction for $\top$.*

The bijection of Proposition 2.18 can be used to define a unary operation $'$ and binary relation $\leq$ on $\mathrm{Dir}(X)$ so that $\mathrm{Pp}(X)$ is isomorphic to $\mathrm{Dir}(X)$. For a direction d, we have that d' is the direction with the same domain and if $d(y) = (u, v)$ then $d'(y) = d(y)' = (v, u)$. For directions d, e, we have $d \leq e$ iff the principal pairs $d(\top)$ and $e(\top)$ satisfy $d(\top) \leq e(\top)$. The following corollary of Theorem 2.15 is then immediate.

Corollary 2.20. *For a Boolean algebra B with more than 4 elements, and a Boolean domain X with more than 2 elements:*

(1) $\mathrm{Sub}(B)$ *is a Boolean domain;*
(2) $\mathrm{Dir}(X)$ *is a Boolean algebra;*
(3) *B is isomorphic to* $\mathrm{Dir}(\mathrm{Sub}(B))$*;*
(4) *X is isomorphic to* $\mathrm{Sub}(\mathrm{Dir}(X))$.

We conclude this section with an alternate view of the reconstruction of a Boolean algebra B from its Boolean domain $X = \mathrm{Sub}(B)$. Let $a \in B$ and $x = \{0, a, a', 1\}$. We consider the case when $x \neq \bot$. For each cover y of x we have that the 4-element Boolean algebra x is a subalgebra of the 8-element Boolean algebra y. The element $a \in x$ can either embed as an atom in y, or as a coatom in y. In the first case $(\downarrow_y a \cup \uparrow_y a', \downarrow_y a' \cup \uparrow_y a)$ is (x, y), and in the second, it is (y, x). If we use $\downarrow$ for (x, y) and $\uparrow$ for (y, x), a direction d of X for the basic element x assigns to each cover y of x the value $d(y) = \downarrow$ or $d(y) = \uparrow$ describing how x is embedded. This assignment of $\downarrow$ and $\uparrow$ to the covers of x must be done in a way that is consistent with d being a direction, and for each x there are only two possibilities, one obtained from the other by interchanging $\downarrow$ and $\uparrow$ for each cover. Virtually identical remarks hold when $x = \bot$, except that we consider embedding a 2-element Boolean algebra into 4-element ones.

Example 2.21. Consider the power set $B = \mathcal{P}(\{1, 2, 3, 4\})$ of $\{1, 2, 3, 4\}$. Its poset X of Boolean subalgebras is given in Figure 1.

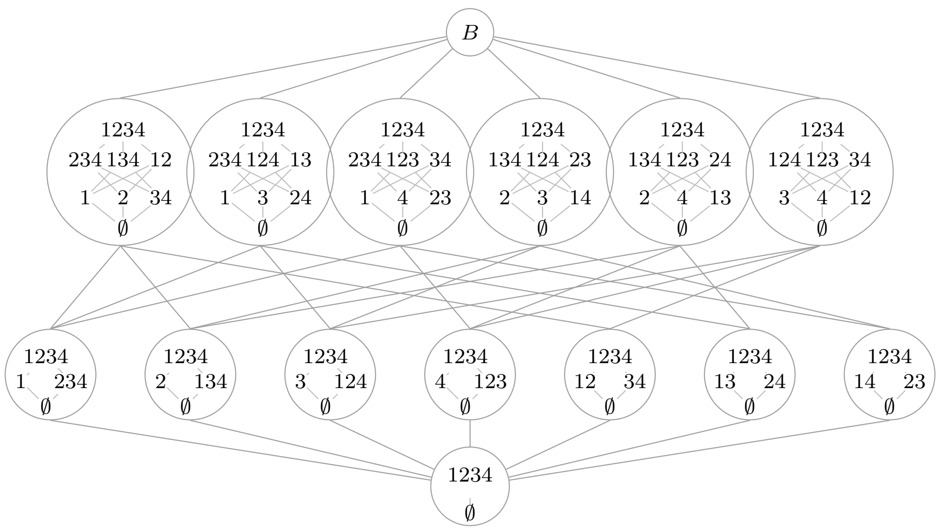


FIGURE 1. The poset of subalgebras of a 16-element Boolean algebra

We describe the directions of X corresponding to the elements $a = \{1\}$ and $b = \{1, 2\}$ in Figure 2 by indicating their values $\downarrow$ or $\uparrow$ on the covers of the basic elements corresponding to these elements.

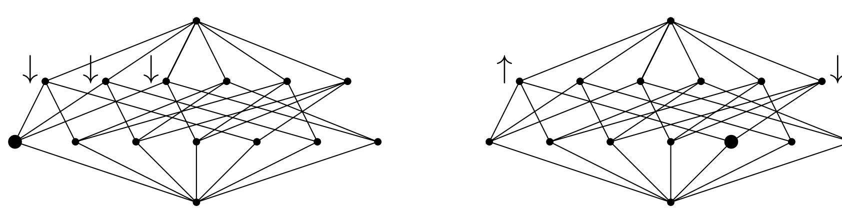


FIGURE 2. Direction for a distinguished atom in a 16-element Boolean algebra

We note that the upper covers of a basic element will usually be assigned a mixture of values of $\downarrow$ and $\uparrow$, a matter we return to in greater detail when we consider orthoalgebras.

3. Orthoalgebras

This section briefly recalls the basics of orthoalgebras and their subalgebras [5].

Definition 3.1. An *orthoalgebra* is a set A, together with a partial binary operation $\oplus$ with domain of definition $\perp$, a unary operation $'$, and constants $0, 1$, satisfying:

(1) $\oplus$ is commutative and associative in the usual sense for partial operations;
(2) a' is the unique element with $a \oplus a'$ defined and equal to 1;
(3) $a \oplus a$ is defined if and only if $a = 0$.

An orthoalgebra is *Boolean* when it arises from a Boolean algebra by restricting the join to pairs of orthogonal elements.

Any orthoalgebra is partially ordered by $a \leq c$ if $a \perp b$ and $a \oplus b = c$ for some b. An orthoalgebra is Boolean if and only if this is the partial ordering of a Boolean algebra.

Definition 3.2. Let A be an orthoalgebra. A subset $S \subseteq A$ is a *subalgebra* if:

(1) $0, 1 \in S$;
(2) $a \in S \Rightarrow a' \in S$;
(3) if $a, b \in S$ and $a \perp b$ then $a \oplus b \in S$.

A subalgebra that is a Boolean orthoalgebra is a *Boolean subalgebra*.

Proposition 3.3. *An orthoalgebra A is Boolean if and only if every finite subset $S \subseteq A$ is contained in $\{\bigoplus E \mid E \subseteq F\}$ for some orthogonal join F.*

Proof. First assume A is a Boolean orthoalgebra, say it is the restriction of a Boolean algebra B. Any finite subset $S \subseteq A$ is contained in a finite subalgebra C of B because finitely generated subalgebras of Boolean algebras are finite. Let F be the set of atoms of C. This is a jointly orthogonal set in A. For each $b \in B$ let $E = \{x \in F \mid x \leq b\}$. Now, because b is the join of B in E, we have $b = \bigoplus E$.

For the converse, suppose that every finite subset $S \subseteq A$ is contained in $\{\bigoplus E \mid E \subseteq F\}$ for some orthogonal join F. First assume that A is finite. I nthis case, there is a jointly orthogonal family F such that every element of A equals $\bigoplus E$ for some $E \subseteq F$. Clearly $\bigoplus F = 1$, and if F contains 0, we may remove 0 from F to get a jointly orthogonal family with the same properties. So we may choose F to be a *partition of unity* of L, in the sense that $0 \notin F$, F is jointly orthogonal, and $\bigoplus F = 1$. Since $A = \{\bigoplus E \mid E \subseteq F\}$, it is isomorphic to the orthoalgebra induced by the Boolean algebra $\mathcal{P}(F)$, and hence A is Boolean.

Now consider the case where A is infinite. For each partition of unity F of A, let B_F be the subalgebra of A generated by F. Explicitly, $B_F = \{\bigoplus E \mid E \subseteq F\}$, and in particular each B_F is a finite Boolean orthoalgebra. By hypothesis, each finite subset of A is contained in B_F for some partition of unity F. If F_1 and F_2 are partitions of unity, then $B_{F_1} \cup B_{F_2}$ is a finite subset of A, hence $B_{F_1} \cup B_{F_2}$ is contained in B_F for some partition of unity F. Thus

$$\{B_F \mid F \text{ is a finite partition of unity for } A\}$$

is an up-directed family of subalgebras of A. Furthermore, each finitely generated subalgebra of A is contained in some member of this family. But then the union of this family is all of A. Hence A is Boolean. □

A *block* of an orthoalgebra is a maximal Boolean subalgebra. A block is *small* when it has 4 or fewer elements. Write $\mathrm{BSub}(A)$ for the set of Boolean subalgebras of A partially ordered by inclusion.

When an orthoalgebra A has more than 2 elements, all of its small blocks have 4 elements. In this case small blocks are also known as horizontal summands. By removing a small block from such A, we mean removing the two elements of the block that are not $0, 1$. Except when A has only small blocks, removing the small blocks from A leaves an orthoalgebra A^* without small blocks, and A can be recovered from A^* by taking the horizontal sum of A^* and an appropriate number of 4-element Boolean algebras.

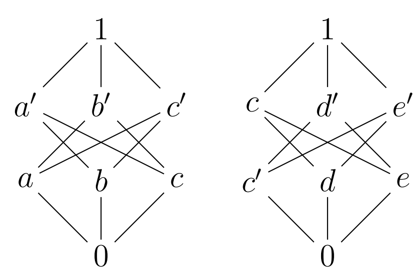
We collect in the following remark motivation for why orthoalgebras are a natural choice of ambient structure to reconstruct from Boolean subalgebras.

Remark 3.4. Each element a of an orthoalgebra belongs to the Boolean subalgebra $\{0, a, a', 1\}$. Thus any orthoalgebra *pastes together* a family of Boolean orthoalgebras. More generally, call a family $\mathcal{F}$ of Boolean orthoalgebras *compatible* [2, 1.7] if for each $B, C \in \mathcal{F}$:

(1) B and C have the same 0 and 1;
(2) If $a \in B \cap C$, then a' in B equals a' in C;
(3) for $a, b \in B \cap C$, $a \oplus b$ exists in B iff it exists in C, and when defined they are equal.

Any compatible family gives rise to a structure $(A, \oplus, ', 0, 1)$ by union. A structure $(A, \oplus, ', 0, 1)$ that arises this way is called a *weak orthostructure*, extending [2]. This general setup includes orthoalgebras as well as partial Boolean algebras [12].

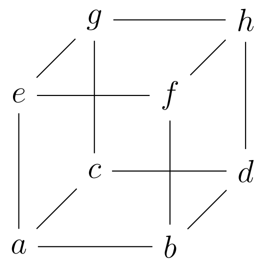
A Boolean subalgebra of a weak orthostructure A is a subset $B \subseteq A$ that is closed under $0, 1, ', \oplus$ and forms a Boolean orthoalgebra. One might hope to reconstruct A from its poset $\mathrm{BSub}(A)$ of Boolean subalgebras, but this is impossible: the partially ordered set $\mathrm{BSub}(A)$ in the introduction is not only induced by the orthoalgebra A in the introduction, but it is also isomorphic to $\mathrm{BSub}(D)$ for the weak orthostructure D obtained by taking two 8-element Boolean algebras that intersect in a 4-element Boolean algebra $\{0, c, c', 1\}$ where c is an atom of one of the 8-element Boolean algebras and a coatom of the other 8-element Boolean algebra. This D is not only a weak orthostructure, but is a partial Boolean algebra. This structure D is not an orthoalgebra, and cannot be depicted via a Hasse diagram.



4. Orthodomains and directions

This section abstracts basic properties of $\mathrm{BSub}(A)$ for orthoalgebras A into a notion of orthodomain. We generalize directions from Boolean domains to directions on orthodomains, and show that an orthoalgebra A can be reconstructed from the directions on its orthodomain $\mathrm{BSub}(A)$. First, an example that exhibits some counterintuitive behavior in $\mathrm{BSub}(A)$.

Example 4.1. The *Fraser cube* is the orthoalgebra A displayed in the diagram. The four vertices of each face are the atoms of a 16-element Boolean subalgebra of A.



Consider the element $a \oplus b$. Its orthocomplement in the Boolean algebra corresponding to the bottom face is $c \oplus d$, and its orthocomplement in the Boolean algebra corresponding to the front face is $e \oplus f$. Thus $c \oplus d = e \oplus f$. Similarly, the intersection of the Boolean subalgebras for the top and bottom of the cube consists of $0, a \oplus b$, $b \oplus d$, $c \oplus d$, $a \oplus c$, and 1. Thus the intersection of two Boolean subalgebras need not be Boolean. This implies that in $\mathrm{BSub}(A)$, two elements need not have a meet, and two elements that have an upper bound need not have a least upper bound, in contrast to the situation for Boolean domains and posets of Boolean subalgebras of orthomodular posets.

Definition 4.2. Write $\lessdot$ for the covering relation in a partially ordered set: $x \lessdot z$ means $x < z$ and there is no y with $x < y < z$.

Definition 4.3. An *orthodomain* is a partially ordered set X with least element $\perp$ such that:

(1) every directed subset of X has a join;
(2) X is atomistic and the atoms are compact;
(3) each principal ideal $\downarrow x$ is a Boolean domain;
(4) if x, y are distinct atoms and $x, y \lessdot w$, then $x \vee y = w$.

Lemma 4.4. *Each element of an orthodomain lies beneath a maximal element.*

Proof. Let X be an orthodomain and $x \in X$. Zorn's lemma produces a maximal directed set containing $x \in X$. Taking the join of this maximal directed set provides a maximal element of X above x. □

We next examine condition (4) more closely.

Definition 4.5. Atoms x, y of an orthodomain are called *near* if they are distinct, their join exists and covers x and y. Equivalently, by condition (4): x and y are near if they are distinct and have an upper bound of height 2.

The following property, similar to the exchange property of geometry, will be key.

Proposition 4.6 (Exchange property)**.** *If x, y are near atoms of an orthodomain with $x \vee y = w$, then there is exactly one atom z that is distinct from x, y and with $z \lessdot w$. Further, any two of x, y, z are near.*

Proof. By nearness, $x \vee y = w$ exists and covers x and y, and by the definition of an orthodomain, $\downarrow w$ is a Boolean domain. Since the top of this Boolean domain covers an atom in it, the Boolean domain w must be isomorphic to the subalgebra lattice of an 8-element Boolean algebra. Then $\downarrow w$ must have 3 distinct atoms, so there is a third atom z distinct from x, y with $z \lessdot w$. Then $x, y, z \lessdot w$. It follows from the definition of orthodomain that w is the join of any two of x, y, z, hence any two of x, y, z are near. □

Definition 4.7. For an orthodomain X, Let N be the partial binary operation on X which, for two near atoms x, y, assigns the third near atom z from Proposition 4.6, thus $\mathrm{N}(x, y) = z$.

Proposition 4.8. *If A is an orthoalgebra,* $\mathrm{BSub}(A)$ *is an orthodomain where directed joins are given by unions.*

Proof. Let $S \subseteq \mathrm{BSub}(A)$ be a directed family. By directedness, $B = \bigcup S$ is closed under $\oplus, ', 0, 1$, and is hence is a subalgebra. Also by directedness, Proposition 3.3 shows that B is Boolean. Thus $\mathrm{BSub}(A)$ has directed joins given by union.

The atoms of $\mathrm{BSub}(A)$ are the Boolean subalgebras $\{0, a, a', 1\}$ where $a \neq 0, 1$. Since directed joins are given by unions, it follows that the compact elements of $\mathrm{BSub}(A)$ are exactly the finite Boolean subalgebras, and hence every atom is compact. Any $B \in \mathrm{BSub}(A)$ is the union and hence join of the atoms beneath it, making $\mathrm{BSub}(A)$ atomistic. Finally, for a Boolean orthoalgebra B, its Boolean subalgebras are exactly its subalgebras that are Boolean, so (3) holds in $\mathrm{BSub}(A)$.

For (4), suppose $x, y \in \mathrm{BSub}(A)$ be distinct atoms with $x, y \lessdot w$. Say $x = \{0, a, a', 1\}$ and $y = \{0, b, b', 1\}$ with $0, a, a', b, b', 1$ all distinct. Since w covers an atom and $\downarrow w$ is a Boolean domain, it is an 8-element Boolean subalgebra of A containing x, y. So $w = \{0, a, a', b, b', c, c', 1\}$ is an 8-element Boolean subalgebra of A for some $c \in A$. One of a, a' is an atom of w, as is one of b, b', and one of c, c'. We may assume that a, b, c are atoms. Then $a \oplus b = c'$ in w, and so $a \oplus b = c'$ in A. Then if v is a Boolean subalgebra of A that contains x, y, we have $a, b \in v$, hence $a \oplus b = c' \in v$. Thus $w = \{0, a, a', b, b', c, c', 1\} \subseteq v$, and $x \vee y = w$. Thus $\mathrm{BSub}(A)$ is an orthodomain. □

We now begin the task of reconstructing an orthoalgebra A from its orthodomain $\mathrm{BSub}(A)$. The idea is to extend the directions used in the Boolean case to the orthoalgebra setting. The reader should consult Definitions 2.16 and 2.17.

Definition 4.9. Let A be an orthoalgebra, $a \in A$, and $x = \{0, a, a', 1\}$. Define the *direction corresponding to a* to be the map $d_a \colon \uparrow x \to (\mathrm{BSub}(A))^2$ given by

$$d_a(y) = (\downarrow_y a \cup \uparrow_y a', \downarrow_y a' \cup \uparrow_y a).$$

We seek an order-theoretic description in terms of an orthodomain X of the mappings d_a. These are again called directions, since when restricted to the setting of Boolean domains, these are the directions given in Definition 2.17.

Definition 4.10. A *direction* for a basic element x of an orthodomain X is a map $d \colon \uparrow x \to X^2$ such that for each $y, z \in \uparrow x$:

(1) $d(y)$ is a principal pair for x in the Boolean domain $\downarrow y$;
(2) if $y \leq z$ and $d(z) = (v, w)$, then $d(y) = (y \wedge v, y \wedge w)$;
(3) if $x \lessdot y, z$ and $d(y) = (x, y)$, $d(z) = (z, x)$, then $y \vee z$ exists and $y, z \lessdot y \vee z$.

Write $\mathrm{Dir}(X)$ for the set of directions for basic elements of X.

Condition (3) of Definition 4.10 looks strange, but its effect will become clear in the proof of Proposition 4.12.

Note that if d is a direction for some basic element x, then x can be determined from the partial mapping d as the least element of its domain.

Proposition 4.11. *Let d be a direction for a basic element x of an orthodomain X.*

(1) *for any $x \leq y \leq z$, the value of $d(y)$ is determined by $d(z)$;*

(2) *for any* $x < y \leq z$, *the value of* $d(z)$ *is determined by* $d(y)$.

Proof. (1) This is immediate from the definition of direction. For (2), let $v, w \in {\downarrow} z$ be such that (v, w) and (w, v) are the two principal pairs for x in ${\downarrow} z$, so $d(z) = (v, w)$ or $d(z) = (w, v)$. In the first case, $d(y) = (y \wedge v, y \wedge w)$, in the second $d(y) = (y \wedge w, y \wedge v)$. We claim that $y \wedge v \neq y \wedge w$, so $d(y)$ determines $d(z)$. To see this, Definition 2.9 gives that $v \wedge w = x$. So if $y \wedge v = y \wedge w$, then $x = y \wedge v \wedge w = y \wedge v = y \wedge w$, but $d(y) = (x, x)$ contradicts $d(y)$ being a principal pair for x in ${\downarrow} y$ (Remark 2.10). □

We will show that the directions of an orthodomain BSub(A) form an orthoalgebra A, but we will see that not all orthodomains are of the form BSub(A) for an orthoalgebra A. So our task in the orthodomain setting is more complex than in the Boolean domain setting. To make our path efficient, we next prove the following result that is valid for any orthodomain.

Proposition 4.12. *A basic element* x *of an orthodomain* X *has at most two directions.*

Proof. If x is maximal, (x, x) is the only principal pair for x in ${\downarrow} x$, so there is only one direction for x in X. Suppose that x is not maximal. By Lemma 4.4 and the definition of a direction, any direction d for x is determined by its value on the maximal elements $w > x$. For any such w, the value $d(w)$ is a principal pair for x in ${\downarrow} w$, so by Corollary 2.12 can take two values.

Suppose there are three distinct directions d_1, d_2, d_3 for x. Choose any maximal $w > x$. Then two of d_1, d_2, d_3 must agree at w, say d_1 and d_2. Since $d_1 \neq d_2$, there is a maximal v with $d_1(v) \neq d_2(v)$. Choose y, z with $x \lessdot y \leq w$ and $x \lessdot z \leq v$. Since $d_1(w) = d_2(w)$ and $y \leq w$, we have $d_1(y) = d_2(y)$, and since $d_1(v) \neq d_2(v)$, we have $d_1(z) \neq d_2(z)$.

As x is basic and $x \lessdot y$, either $x = \bot$ and y is an atom, or x is an atom and y has height 2. In either case the principal pairs for x in ${\downarrow} y$ are (x, y) and (y, x), and similarly the principal pairs for x in ${\downarrow} z$ are (x, z) and (z, x). Suppose that $d_1(y) = d_2(y) = (x, y)$. As $d_1(z) \neq d_2(z)$, one of them is (x, z) and the other is (z, x). We apply condition (3) of Definition 4.10 and find an upper bound $u = y \vee z$ of y, z. According to Proposition 2.18, d_1, d_2 are uniquely determined on ${\downarrow} u$ by $d_1(y) = d_2(y)$, a contradiction with $d_1(z) \neq d_2(z)$, $z \in {\downarrow} u$. The case $d_1(y) = d_2(y) = (y, x)$ is excluded analogously with the role of y, z interchanged in (3) of Definition 4.10. This contradiction shows x has at most two directions. □

Now we begin putting structure on the set of directions of an orthodomain.

Proposition 4.13. *Let* d *be a direction for a basic element* x *of an orthodomain* X. *There is a direction* d' *for* x *given by* $d'(w) = (z, y)$ *if* $d(w) = (y, z)$. *Further, there are directions* 0 *and* 1 *for the basic element* $\bot \in X$, *given by*

$$0(w) = (\bot, w) \qquad \textit{and} \qquad 1(w) = (w, \bot).$$

Proposition 4.14. *For an orthodomain with no basic maximal elements, the following are equivalent:*

(1) *each basic element has a direction;*
(2) *each basic element has exactly two directions.*

Proof. The direction (2) ⇒ (1) is trivial. For the converse, let d be a direction for x. Then so is d' given by Proposition 4.13. If $d = d'$, then for a maximal element w above x we have that $d(w) = d'(w)$, so w is basic, contrary to our assumptions. Thus each basic element has at least two directions, so by Proposition 4.12 has exactly two directions. □

Definition 4.15. Call an orthodomain *proper* if it has no maximal elements that are basic. Say it *has enough directions* if it is proper and each basic element has a direction.

Note that the situation is somewhat analogous to that of spatial frames, which are defined through the existence of a sufficient supply of points.

Definition 4.16. For X an orthodomain with enough directions, let $\oplus$ be a partial binary operation on $\mathrm{Dir}(X)$ defined by the following three cases. For each direction d set

(1) $d \oplus 0 = d = 0 \oplus d$
(2) $d \oplus d' = 1$

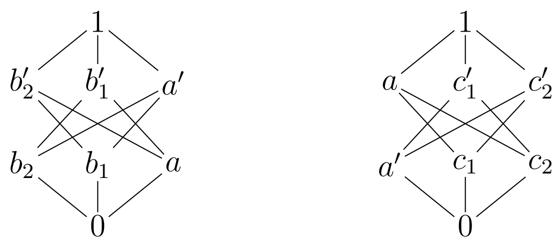
For d a direction for x and e a direction for y with x, y near and z the third atom beneath $x \vee y = w$, then $d \oplus e$ is defined if $d(w) = (x, w)$ and $e(w) = (y, w)$, and in this case

(3) $d \oplus e$ is the direction for z with $(d \oplus e)(w) = (w, z)$

We will write $d \perp e$ and say d *is orthogonal to* e if $d \oplus e$ is defined.

Theorem 4.17. *Let A be an orthoalgebra without small blocks. Then the orthodomain* $\mathrm{BSub}(A)$ *has enough directions and A is isomorphic to* $\mathrm{Dir}(\mathrm{BSub}(A))$.

Proof. By Proposition 4.8, $X = \mathrm{BSub}(A)$ is an orthodomain. Let $a \in A$ and $x = \{0, a, a', 1\}$. Let us verify the three conditions of Definition 4.10 for the direction d_a given by Definition 4.9. Condition (1) follows because $d_a(y)$ is a principal pair in $\downarrow y$ for x. Condition (2) follows by construction of d_a. For condition (3), consider first the case $x = \perp$, where a is either 0 or 1. If $a = 0$ then $d_a(y) = (\perp, y)$ for all y, and if $a = 1$ then $d_a(y) = (y, \perp)$ for all y, so (3) holds vacuously. Suppose x is an atom of X with $x \lessdot y, z$ and that $d_a(y) = (x, y)$ and $d_a(z) = (z, x)$. This means that y, z are 8-element Boolean algebras, that a is an atom in y, and a is a coatom in z. Let b_1, b_2 be the atoms of y distinct from a and c_1, c_2 be the atoms of z distinct from a'. We depict y below on the left, and z on the right.



Then $c_1 \oplus c_2 = a$ and $b_1 \oplus b_2 = a'$ in A. Therefore b_1, b_2, c_2, c_2 are the atoms of a 16-element Boolean subalgebra u of A. Clearly $u = y \vee z$ and $y, z \lessdot u$, establishing condition (3). Thus d_a is a direction. Since this holds for each $a \in A$, the orthodomain X has enough directions.

Define $\varphi\colon A \to \mathrm{Dir}(X)$ by $\varphi(a) = d_a$. Every basic element of X is of the form $\{0, a, a', 1\}$ and has 2 directions. Since d_a and $d_{a'}$ are directions for $\{0, a, a', 1\}$, the map φ is surjective. If $\varphi(a) = \varphi(b)$, then since d_a is a direction given by a and d_b is a direction given by b, we must have that $b = a$ or $b = a'$. But $d_a \neq d_{a'}$, so φ is injective.

To show that φ is an isomorphism, it is easily seen that φ maps $0, 1$ of A to the directions $0, 1$ of X, and that $\varphi(a') = \varphi(a)'$. It remains to show that $a \perp b$ if and only if $\varphi(a) \perp \varphi(b)$ and that then $\varphi(a \oplus b) = \varphi(a) \oplus \varphi(b)$. Consider the possibilities to have $a \perp b$. For any a we have $a \perp 0$, $\varphi(a) \perp \varphi(0)$, and $\varphi(a \oplus 0) = \varphi(a) = \varphi(a) \oplus \varphi(0)$. For any a we have $a \perp a'$ and $a \oplus a' = 1$. Since $\varphi(a') = \varphi(a)'$, then $\varphi(a) \perp \varphi(a')$ and $\varphi(a \oplus a') = 1 = \varphi(a) \oplus \varphi(a')$.

The remaining possibility to have $a \perp b$ is when a, b are distinct atoms of an 8-element Boolean subalgebra w of A. In this case, $x = \{0, a, a', 1\}$ and $y = \{0, b, b', 1\}$ are basic elements that are near, $x \vee y = w$, $z = \{0, a \oplus b, (a \oplus b)', 1\}$ is the third atom beneath w, and $d_a(w) = (x, w)$, $d_b(w) = (y, w)$. Thus $\varphi(a) \perp \varphi(b)$, and as $d_a \oplus d_b$ is the direction for z with $(d_a \oplus d_b)(w) =$

(w, z), we have $\varphi(a) \oplus \varphi(b) = \varphi(a \oplus b)$. Conversely, suppose $\varphi(a) \perp \varphi(b)$ via condition (3) of Definition 4.16. Since d_a is a direction for $x = \{0, a, a', 1\}$ and d_b is a direction for $y = \{0, b, b', 1\}$, this condition assumes x, y are near and generate an 8-element Boolean subalgebra of A. Further, since $d_a(w) = (x, w)$ and $d_b(w) = (y, w)$, we have that a, b are atoms of w, hence $a \perp b$ in A. Finally, $d_a \oplus d_b$ is the direction for the third atom $z = \{0, a \oplus b, (a \oplus b)', 1\}$ beneath w with $(d_a \oplus d_b)(w) = (w, z)$, so $d_a \oplus d_b = d_{a \oplus b}$. □

Remark 4.18. The previous theorem achieves one of our primary aims: a means to reconstruct an orthoalgebra A from its poset of Boolean subalgebras. It only applies if A has no small blocks, but, with one exception, we can still recover A from $\mathrm{BSub}(A)$ without this restriction. The exception is when $\mathrm{BSub}(A)$ has a single element, which occurs when A is a 1-element orthoalgebra and also when A is a 2-element orthoalgebra. In these cases it is clearly impossible to recover A from $\mathrm{BSub}(A)$.

Suppose then that A has more than two elements. If it does have small blocks, these appear in $\mathrm{BSub}(A)$ as maximal atoms. Provided A has a block that is not small, removing these blocks from A yields an orthoalgebra A^*, and $\mathrm{BSub}(A^*)$ is obtained from $\mathrm{BSub}(A)$ from removing maximal atoms. Since we can reconstruct A^* as $\mathrm{Dir}(\mathrm{BSub}(A^*))$, we can then reconstruct A by adding a number of horizontal summands equal to the number of maximal atoms of $\mathrm{BSub}(A)$. If A consists of only small blocks, it is determined by the cardinality of the set of its maximal atoms.

5. Characterizing orthodomains of the form BSub(A)

In this section, we show, for any orthodomain X with enough directions, that $\mathrm{Dir}(X)$ is an orthoalgebra, and characterize those orthodomains that are of the form $\mathrm{BSub}(A)$ for some orthoalgebra A.

Definition 5.1. For an orthodomain X, let X^* be the set of elements of X of height 3 or less. A *shadow* of X is a nonempty subset $S \subseteq X^*$ satisfying:

(1) S is a downset of X^*;
(2) S is closed under existing joins in X^*.

Note, the second condition means that if $T \subseteq S$ and there is $w \in X^*$ that is the least upper bound of T in X, which will imply that w is also the least upper bound of T in X^*, then $w \in S$.

Proposition 5.2. *Let X be an orthodomain, S be a shadow of X, x be a basic element of X with $x \in S$, and d be a direction of X for x. Then:*

(1) *S is an orthodomain;*
(2) *the restriction $d\,|\,S$ of d to $\uparrow x \cap S$ is a direction of S.*

Hence if X has enough directions and S has no maximal elements which are basic, then S has enough directions.

Proof. Since X^*, and hence S, has finite height, every directed set has a maximal element and hence a join, and each element is compact. Since X is atomistic and S is a downset, it is atomistic. Since S is a downset of X, for each $s \in S$ the ideal $\downarrow s$ is a Boolean domain. Finally, if x, y are atoms of S and $x, y \lessdot w$, then $x \vee y = w$ in X, hence $x \vee y = w$ in S as well. Thus S is an orthodomain, establishing part (1).

To see part (2) we verify the three conditions of Definition 4.10. The first two are trivial consequences of restricting. For the third, suppose there are $x \lessdot y, z$ with $y, z \in S$ and $d(y) = (x, y)$, $d(z) = (z, x)$. Since d is a direction of X, then $y \vee z = w$ exists in X and $y, z \lessdot w$. Since

x is basic and $x \lessdot y, z \lessdot w$ then w has height at most 3. So $w \in X^*$ and w is the join of y, z in X^*. Since S is a shadow, it is closed under joins in X^*, so $w \in S$ and $w = y \vee z$ in S. □

Definition 5.3. Let X be an orthodomain with enough directions and S be a shadow of X without maximal elements which are basic. Write $\mathrm{Dir}_S(X)$ for the set of directions of X for basic elements $x \in S$, and let $\mu_S\colon \mathrm{Dir}_S(X) \to \mathrm{Dir}(S)$ be given by $\mu_S(d) = d|_S$.

Proposition 5.4. *Let X be an orthodomain with enough directions and S be a shadow of X without maximal elements which are basic. Then:*

(1) $\mathrm{Dir}_S(X)$ *contains* $0, 1$ *and is closed under* $'$ *and* $\oplus$*;*
(2) μ_S *is a bijection from* $\mathrm{Dir}_S(X)$ *to* $\mathrm{Dir}(S)$*;*
(3) μ_S *preserves* $0, 1$ *and* $'$*;*
(4) $d \perp e$ *if and only if* $\mu_S(d) \perp \mu_S(e)$*, and in this case* $\mu_S(d \oplus e) = \mu_S(d) \oplus \mu_S(e)$*.*

Proof. (1) Since $0, 1$ are directions for $\perp$ and $\perp \in S$, we have $0, 1 \in \mathrm{Dir}_S(X)$. If d is a direction for x, then d' is also a direction for x, giving closure under $'$. For closure under $\oplus$, suppose $d, e \in \mathrm{Dir}_S(X)$ with d a direction for $x \in S$, e a direction for $y \in S$, and $d \perp e$. There are several cases for $\perp$. If one of d, e is 0, then $d \oplus e$ equals d or e, and if $e = d'$, then $d \oplus e = 1$, so these cases are trivial. In the remaining case x and y are near. Say $x \vee y = w$ with z the third atom beneath w. Then $w \in S$ since S is closed under joins in X^*, and so $z \in S$ since S is a downset of X. Since $d \oplus e$ is a direction for z, we have $d \oplus e \in \mathrm{Dir}_S(X)$.

(2) For a basic $x \in S$, the two directions for x in X are d and d'. These restrict to directions of S for x and their restrictions are orthocomplements. Then as S has no basic maximal elements, these restrictions are distinct and are the only two directions for x in S. Part (3) is trivial.

For part (4), suppose $d, e \in \mathrm{Dir}_S(X)$ with d a direction for x and e a direction for y. Note that one of d, e is 0 iff one of $\mu_S(d), \mu_S(e)$ is 0, and in this case $\mu_S(d \oplus e) = \mu_S(d) \oplus \mu_S(e)$. Next, $e = d'$ iff $\mu_S(e) = \mu_S(d)'$, and in this case $\mu_S(d \oplus e) = \mu_S(d) \oplus \mu_S(e)$. For the remaining case we have $d \perp e$ if and only if x, y are near and $d(w) = (x, w), e(w) = (y, w)$ where $x \vee y = w$. But this is equivalent to $\mu_S(d) \perp \mu_S(e)$. In this case, $d \oplus e$ is the direction for the third atom z beneath w with $(d \oplus e)(w) = (w, z)$, and thus its restriction is a direction for z taking value (w, z) at w, and hence is $\mu_S(d) \oplus \mu_S(e)$. □

A specific instance of the previous proposition is of particular interest. It is easily seen that X^* is a shadow of X that has no basic maximal elements when X has none. Furthermore, since every basic element of X belongs to X^*, we have $\mathrm{Dir}_{X^*}(X) = \mathrm{Dir}(X)$.

Corollary 5.5. *If X is an orthodomain with enough directions, then so is X^*, and restriction gives an isomorphism* $\mathrm{Dir}(X) \simeq \mathrm{Dir}(X^*)$*.*

We next set out to prove that $\mathrm{Dir}(X)$ is an orthoalgebra for any orthodomain X with enough directions.

Lemma 5.6. *For X an orthodomain with enough directions, the partial binary operation $\oplus$ on* $\mathrm{Dir}(X)$ *is commutative and associative: when one side of an expression $d \oplus e = e \oplus d$ or $(d \oplus e) \oplus f = d \oplus (e \oplus f)$ is defined, so is the other, and the two are equal.*

Proof. Clearly $\oplus$ is commutative. Making use of this and symmetry, it suffices to show that if $(d \oplus e) \oplus f$ is defined, then $d \oplus (e \oplus f)$ is defined, and the two are equal. For this, we consider a number of cases.

If any of d, e, f are 0, then it is easily verified. The only direction orthogonal to 1 is 0, so we may also assume that none of $d, e, d \oplus e, f$ is 1. So there are atoms x, y, z with d a direction for x, e a direction for y, and f a direction for z. Having $d = e'$ gives $d \oplus e = 1$, so x, y are distinct, and therefore to have $d \perp e$ we must have that x, y are near. Let $x \vee y = w$ and let p be the third atom beneath w. Then $d \oplus e$ is the direction for p with $(d \oplus e)(w) = (w, p)$.

Since neither $d \oplus e$ or f equals 0 or 1, there are two possibilities to have $(d \oplus e) \perp f$. We consider first the case that $f = (d \oplus e)'$. Since $d \oplus e$ is the direction for p with $(d \oplus e)(w) = (w, p)$, this means that f is the direction for $z = p$ with $f(w) = (p, w)$. Since x, y, p are the three pairwise near atoms under w, we then have that $e \oplus f$ is defined, and that $e \oplus f$ is the direction for x with $(e \oplus f)(w) = (w, x)$. Thus $e \oplus f = d'$. So $d \oplus (e \oplus f)$ is also defined and both sides of the expression in this case evaluate to 1.

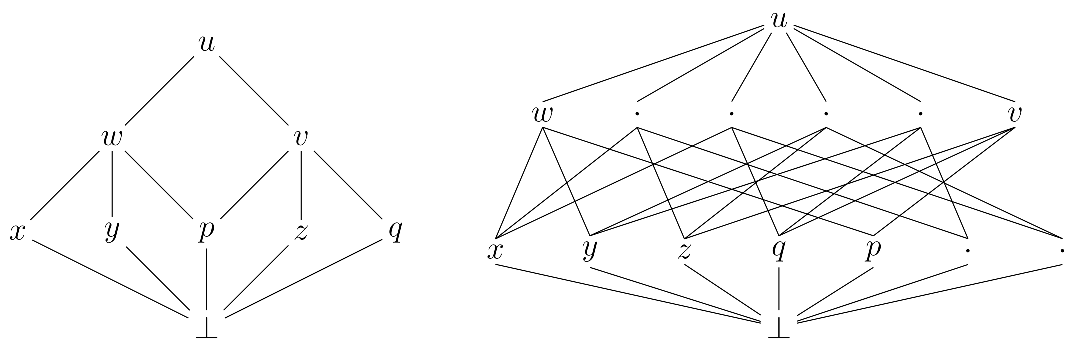


FIGURE 3. A part of the Hasse diagram of the shadow from the proof of Lemma 5.6 (two possible partial diagrams of the same situation)

For the final case (see Figure 3), it must be that item (3) in Definition 4.16 applies to $(d \oplus e) \perp f$. Since $d \oplus e$ is the direction for p with $(d \oplus e)(w) = (w, p)$ and f is a direction for z, the assumptions of (3) give that p, z are near. Say $p \vee z = v$, and let q be the third atom distinct from p, z under v. Then to have $d \oplus e \perp f$ we have $(d \oplus e)(v) = (p, v)$ and $f(v) = (z, v)$, and the sum $(d \oplus e) \oplus f$ is the direction for q with $\big((d \oplus e) \oplus f\big)(v) = (v, q)$.

Since $(d \oplus e)(w) = (w, p)$ and $(d \oplus e)(v) = (p, v)$, we have $w \neq v$. Since the three atoms beneath w are x, y, p, the three atoms beneath v are p, q, z, and w, v cannot have more than one common atom beneath them since they are distinct, we have that x, y, p, q, z, w, v are distinct. Since $d \oplus e$ is a direction for p, by Definition 4.10

$$p \lessdot w, v, \ (d \oplus e)(w) = (w, p) \ \text{ and } \ (d \oplus e)(v) = (p, v) \ \Rightarrow \ w \vee v \text{ exists and } w, v \lessdot w \vee v$$

Let $u = w \vee v$. Since p is an atom and $p \lessdot w, v \lessdot u$ then u has height 3 so belongs to X^*. Let $S = {\downarrow} u$ and note that this is a shadow of X. Since S is isomorphic to $\mathrm{Sub}(B)$ for a 16-element Boolean algebra B, Theorem 4.17 gives that $\mathrm{Dir}(S) \simeq B$. Proposition 5.4 gives $\mathrm{Dir}_S(X) \simeq \mathrm{Dir}(S)$. Since d, e, f all belong to $\mathrm{Dir}_S(X)$, their associativity under $\oplus$ follows. ☐

Theorem 5.7. *If X is an orthodomain with enough directions, then* $\mathrm{Dir}(X)$ *is an orthoalgebra.*

Proof. Lemma 5.6 shows that $\oplus$ is commutative and associative. There are directions $0, 1$. For each direction d also d' is a direction, $d \oplus d'$ is defined, and $d \oplus d' = 1$. Suppose e is another direction with $d \oplus e$ defined and $d \oplus e = 1$. Since 1 is a direction given by the basic element 0, it cannot be that $d \perp e$ is defined because of reason (3) in Definition 4.16. If it is defined because of reason (2), then $e = d'$. If it is defined because of reason (1), then one of d, e is 0, and because we have required $d \oplus e = 1$, the other must be 1, hence again $e = d'$. So d' is the unique direction with $d \oplus d' = 1$. Finally, suppose that d is a direction with $d \oplus d$ defined. This cannot be defined because of reason (3) of Definition 4.16. It cannot be because of reason (2) since $d \neq d'$. So it must be defined because of reason (1), giving $d = 0$. □

Remark 5.8. For an orthodomain X with enough directions, $X \simeq \mathrm{BSub}(\mathrm{Dir}(X))$ does not usually hold. By Corollary 5.5 $\mathrm{Dir}(X) \simeq \mathrm{Dir}(X^*)$, and we clearly do not have $X \simeq X^*$ for each orthodomain X with enough directions. In fact, $X = \mathrm{BSub}(A)$ provides a counterexample for any orthoalgebra A with no small blocks and a block with more than 4 atoms.

Definition 5.9. A shadow $S \subseteq X^*$ of an orthodomain X is a *Boolean shadow* if either:

(1) $S = \downarrow x$ for some basic $x \in X$;
(2) S has enough directions and $\mathrm{Dir}(S)$ is a Boolean orthoalgebra.

Write $\mathrm{BShad}(X)$ for the partially ordered set of Boolean shadows of X under inclusion.

Definition 5.10. Let X be an orthodomain with enough directions, and let B be a Boolean subalgebra of $\mathrm{Dir}(X)$. Define:

$$T_B = \{x \mid x \text{ is basic in } X \text{ and there is some } d \in B \text{ with } d \text{ a direction for } x\},$$
$$S_B = \text{the closure of } T_B \text{ under existing joins in } X^*.$$

Proposition 5.11. *Let X be an orthodomain with enough directions and let B be a Boolean subalgebra of $\mathrm{Dir}(X)$. Then:*

(1) *if B has more than 4 elements, then S_B is proper and $B = \mathrm{Dir}_{S_B}(X)$;*
(2) *S_B is a Boolean shadow of X.*

Proof. We first prove that S_B is a shadow of X.

By definition, $S_B \subseteq X^*$ and is closed under existing joins in X^*. It suffices to show that S_B is a downset of X^*. Clearly if w is a basic element of X that belongs to S_B, then any $x \leq w$ also belongs to S. This covers the case that w is of height 0 or 1. Suppose $w \in S_B$ is of height 2 in X. Then w belonging to S_B means it is the join $w = x \vee y$ of two elements x, y of T_B, both of which are atoms of X. Since $x, y \in T_B$ there are directions $d, e \in \mathrm{Dir}(X)$ with d a direction for x and e a direction for y. Further, these may be chosen so that $d \perp e$. If $z \leq w$, then either z is one of $0, x, y, w$, or z is the third atom beneath $x \vee y$. In the last case $d \oplus e$ is a direction for z and $d \oplus e$ belongs to B, so $z \in T_B \subseteq S_B$.

Our final case is when $w \in S_B$ is of height 3. Since $w \in S_B$, it is the join of atoms of T_B. Since $w \in X$ we have $\downarrow w$ isomorphic to the poset of subalgebras of a 16-element Boolean algebra as shown in Figure 1. Our task is to show that all 7 atoms x in $\downarrow w$ belong to T_B since this then shows that all elements of height 2 in $\downarrow w$ are in S_B. The atoms in $\downarrow w$ can be divided into two groups, the four at left and the three at right. There are two possibilities to consider:

(i) w is the join of two atoms $w = x \vee y$ from the right with $x, y \in T_B$;
(ii) w is the irredundant join of 3 atoms of T_B.

Using the result above that if $z \in S_B$ is of height 2 then $\downarrow z \subseteq S_B$, and that S_B is closed under joins in X^*, the second case can be reduced to the first, so we just consider the first. Let d, d' be the directions for x and e, e' be the directions for y. So $d, d', e, e' \in B$ and neither of d, d' is orthogonal to either of e, e'. Since B is a Boolean subalgebra of $\mathrm{Dir}(X)$, then d, e generate a 16-element subalgebra Y of B. So there are directions $f_1, \ldots, f_4 \in B$ that are the atoms of Y. These must be directions for the four basic atoms at the left of $\downarrow w$. But $f_1, \ldots, f_4$ in B imply that these four atoms at left belong to T_B. Since the remaining atom beneath $\downarrow w$ lies under a join of two of the atoms at left, it also belongs to T_B. This establishes that S_B is a shadow.

For part (1), assume B has more than 4 elements. To see that S_B is proper, first note that it is not the case that 0 is maximal in S_B. Let x be an atom in S_B and hence in T_B. Then there is a direction d in B that is a direction for x. Since B has more than 4 elements, there is a nonzero direction e in B orthogonal and unequal to one of d or d'. If e is a direction for y, then x, y are near, and $w = x \vee y \in S_B$. So no atom is maximal in S_B, and S_B is proper.

It remains to show that $B = \mathrm{Dir}_{S_B}(X)$. Let d be a direction in B for a basic element x. Then by definition, $x \in T_B \subseteq S_B$. Thus by definition $d \in \mathrm{Dir}_{S_B}(X)$. Conversely, let $d \in \mathrm{Dir}_{S_B}(X)$ be a direction for the basic element x of X. By definition, $x \in S_B$. But S_B consists of the elements of X^* that are joins of elements of T_B, and as x is basic, it must be that $x \in T_B$. Thus there is a direction e in B for x. But there are only two directions for x, namely d, d'. So either $e = d$ or $e' = d$, and in either case d is in B since B is closed under orthocomplementation.

For part (2), it remains to show that the shadow S_B is Boolean. If B has 4 or fewer elements, then $T_B = \downarrow x$ for a basic element x, so $S_B = T_B$, and so S_B is Boolean. Suppose B has more than 4 elements. Then by (1) $B = \mathrm{Dir}_{S_B}(X)$. Proposition 5.4 gives $\mathrm{Dir}_{S_B}(X) \simeq \mathrm{Dir}(S_B)$. So $\mathrm{Dir}(S_B)$ is Boolean, giving that S_B is a Boolean shadow. □

Proposition 5.12. *For X an orthodomain with enough directions, there is an isomorphism of posets $\Gamma\colon \mathrm{BSub}(\mathrm{Dir}(X)) \to \mathrm{BShad}(X)$ given by $\Gamma(B) = S_B$.*

Proof. The map is well-defined by Proposition 5.11. If $B_1 \subseteq B_2$, then surely $S_{B_1} \subseteq S_{B_2}$, so Γ preserves order. Suppose $S_{B_1} \subseteq S_{B_2}$. Since elements of S_{B_2} are joins of elements of T_{B_2}, and elements of T_{B_1} are basic and hence join irreducible, this implies that $T_{B_1} \subseteq T_{B_2}$ and this gives that $B_1 \subseteq B_2$. So Γ is an order embedding.

To see that it is surjective, let S be a Boolean shadow of X. If S is either $\{\bot\}$ or $\{\bot, x\}$ for some atom x of X, then $S = \Gamma(B)$ where $B = \{0, 1\}$ or $B = \{0, d, d', 1\}$ where d is a direction for x respectively. Suppose that S has enough directions and $\mathrm{Dir}(S)$ is a Boolean orthoalgebra. Let $B = \mathrm{Dir}_S(X)$. By Proposition 5.4, B is a subalgebra of $\mathrm{Dir}(X)$ and the restriction map from B to $\mathrm{Dir}(S)$ is an isomorphism, so B is a Boolean subalgebra of $\mathrm{Dir}(X)$. Then $\Gamma(B) = S_B$ is the shadow generated by T_B, and the elements of T_B are those basic elements x of X that have a direction $d \in B = \mathrm{Dir}_S(X)$. By definition, the elements of $\mathrm{Dir}_S(X)$ are those directions that are for some basic $x \in S$. Thus T_B consists of the basic elements in S, so $\Gamma(B) = S$. □

Definition 5.13. Let X be an orthodomain. We say X is *short* if $X = X^*$. We say X is *tall* if $m = \bigvee S$ exists and $\downarrow m \cap X^* = S$ for each Boolean shadow S.

Proposition 5.14. *Let A be an orthoalgebra without small blocks. Then $X = \mathrm{BSub}(A)$ is a tall orthodomain with enough directions.*

Proof. By Theorem 4.17, X is an orthodomain with enough directions, and there is an orthoalgebra isomorphism $\varphi\colon A \to \mathrm{Dir}(X)$ where $\varphi(a) = d_a$ is the direction for $x_a = \{0, a, a', 1\}$ with

$$d_a(w) = (\downarrow_w a \cup \uparrow_w a', \downarrow_w a' \cup \uparrow_w a).$$

Let S be a Boolean shadow of X. If $S = \downarrow x$ for a basic element x it is clear that $x = \bigvee S$ exists and $\downarrow x \cap X^* = S$. Assume that S has enough directions and $\mathrm{Dir}(S)$ is Boolean. By Proposition 5.4 $\mathrm{Dir}(S) \simeq \mathrm{Dir}_S(X)$, hence $\mathrm{Dir}_S(X)$ is Boolean. Let $w = \varphi^{-1}(\mathrm{Dir}_S(X))$. Then w is a Boolean subalgebra of A, and consists of all the $a \in A$ with $\varphi(a) \in \mathrm{Dir}_S(X)$, hence all $a \in A$ with $d_a \in \mathrm{Dir}_S(X)$, and therefore all $a \in A$ with $x_a \in S$. Since each basic element of X is of the form x_a given by some $a \in A$, we have for a basic element $x \in X$, that $x \in S$ exactly when $x \leq w$. Since S is a downset and X is atomistic $w = \bigvee S$, and since S is closed under existing joins in X^* we have $X^* = S$. Thus X is tall. □

Proposition 5.15. *If X is a tall orthodomain with enough directions, then $X \simeq \mathrm{BSub}(\mathrm{Dir}(X))$.*

Proof. By Proposition 5.12, we have $\mathrm{BSub}(\mathrm{Dir}(X)) \simeq \mathrm{BShad}(X)$. Define $\psi\colon \mathrm{BShad}(X) \to X$ by $\psi(S) = \bigvee S$, and $\lambda\colon X \to \mathrm{BShad}(X)$ by $\lambda(m) = \downarrow m \cap X^*$. Since X is tall, $\bigvee S$ exists and $\downarrow(\bigvee S) \cap X^* = S$. For any $w \in X$ we have $\downarrow w \cap X^*$ is a downset of X^* that is closed under existing joins in X^*, hence is a shadow. If w is basic in X, then by definition $\downarrow w$ is a Boolean shadow. Otherwise $\downarrow w$ is a proper Boolean domain, hence has enough directions. Since $\downarrow w \cap X^* = (\downarrow w)^*$, Corollary 5.5 gives that $\downarrow w \cap X^*$ is an orthodomain with enough directions and that $\mathrm{Dir}(\downarrow w \cap X^*)$ is isomorphic to $\mathrm{Dir}(\downarrow w)$, and hence is Boolean. In any case, $\downarrow w \cap X^*$ is a Boolean shadow of X. So ψ and λ are well-defined. Since $\downarrow(\bigvee S) \cap X^* = S$ we have $\lambda \circ \psi = id$. For $w \in X$, by atomisticity $w = \bigvee(\downarrow w \cap X^*)$, so $\psi \circ \lambda = id$. Thus $\mathrm{BShad}(X) \simeq X$, so $\mathrm{BSub}(\mathrm{Dir}(X)) \simeq X$. ☐

Theorem 5.16. *The following are equivalent for an orthodomain X:*

(1) *X is tall and has enough directions;*
(2) *$X \simeq \mathrm{BSub}(A)$ for an orthoalgebra A without small blocks.*

When these conditions hold, $\mathrm{Dir}(X)$ is an orthoalgebra and $X \simeq \mathrm{BSub}(\mathrm{Dir}(X))$.

Proof. The direction (1) ⇒ (2) follows from Theorem 5.7 and Proposition 5.15. The direction (2) ⇒ (1) follows from Theorem 4.17 and Proposition 5.14. ☐

Remark 5.17. The previous theorem only characterizes orthodomains of the form $\mathrm{BSub}(A)$ for an orthoalgebra A without small blocks. This can be extended to orthoalgebras A with small blocks as follows. If A has two or fewer elements, then $\mathrm{BSub}(A)$ is a 1-element orthodomain. If A has more than two elements and all its blocks are small, then $\mathrm{BSub}(A)$ is an orthodomain where all elements are basic, and each orthodomain where all elements are basic arises this way as the horizontal sum of 4-element Boolean algebras, one for each atom of the orthodomain. Otherwise not all blocks of A are small. Let A^* be the orthoalgebra obtained by removing small blocks from A. Then $\mathrm{BSub}(A^*)$ is a tall orthodomain with enough directions, and $\mathrm{BSub}(A)$ is obtained from this by adding a maximal atom to $\mathrm{BSub}(A^*)$ for each small block of A. So the orthodomains isomorphic to $\mathrm{BSub}(A)$ for some orthoalgebra A are exactly those that have one element, have all their elements basic, or are constructed by adding a set of maximal atoms to a tall orthodomain with enough directions.

Theorem 5.18. *The following are equivalent for an orthodomain X:*

(1) *X is short and has enough directions;*
(2) *$X \simeq \mathrm{BSub}(A)^*$ for an orthoalgebra A without small blocks.*

When these conditions hold, $\mathrm{Dir}(X)$ is an orthoalgebra and $X \simeq \mathrm{BSub}(\mathrm{Dir}(X))^*$.

Proof. The direction (2) ⇒ (1) follows from Theorem 4.17 and Corollary 5.5. For the converse, assume (1). By Proposition 5.12 there is an isomorphism $\Gamma\colon \mathrm{BSub}(\mathrm{Dir}(X)) \to \mathrm{BShad}(X)$ given by $\Gamma(B) = S_B$ where S_B is from Definition 5.10. Then Γ restricts to an isomorphism of posets $\Gamma'\colon \mathrm{BSub}(\mathrm{Dir}(X))^* \to \mathrm{BShad}(X)^*$. We will show that $\mathrm{BShad}(X)^*$ is equal to the poset of principle downsets $\downarrow w$ where $w \in X$, hence is isomorphic to X. This will show that $\mathrm{BSub}(\mathrm{Dir}(X))^*$ is isomorphic to X, establishing (2) and the further remark.

Suppose $w \in X$. If w is basic, then by definition $\downarrow w$ is a Boolean shadow that clearly has height at most 1 in $\mathrm{BShad}(X)$. Otherwise $\downarrow w$ is a Boolean domain with enough directions and $\mathrm{Dir}(\downarrow w)$ is a Boolean algebra. Thus $\downarrow w$ is a Boolean shadow. Since X is short, w has height at most 3, so $\downarrow w$ has height at most 4 in $\mathrm{BShad}(X)$, so belongs to $\mathrm{BShad}(X)^*$.

From the isomorphism Γ', the elements of $\mathrm{BShad}(X)^*$ are the S_B where B is a Boolean subalgebra of $\mathrm{Dir}(X)$ with at most 16 elements. We must show that all such S_B are equal to $\downarrow w$ for some $w \in X$. If B has 4 or fewer elements then S_B is equal to $\downarrow w$ for a basic element $w \in X$. Suppose B has 8 elements. Let d_1, d_2, d_3 be the directions that are the atoms of B and assume d_i is a direction for the basic element x_i of X for $i = 1, 2, 3$. Since d_1 is orthogonal to d_2 we have that x_1, x_2 are near, so have a join $w = x_1 \vee x_2$, and this belongs to S_B. By simple counting, S_B must be equal to $\downarrow w$. Finally, suppose that B has 16 elements and $d_1, \ldots, d_4$ are the atoms of B with d_i a direction for x_i for $i = 1, \ldots, 4$. Since d_1, d_2 are orthogonal x_1, x_2 are near, so $z = x_1 \vee x_2$ exists. Suppose x is the third atom beneath z. Then $(d_1 \oplus d_2)(z) = (z, x)$. Let $y = x_3 \vee x_4$. Since $d_3 \oplus d_4 = (d_1 \oplus d_2)'$ we have that the third atom under y is x. Also $(d_3 \oplus d_4)(y) = (y, x)$, so $(d_1 \oplus d_2)(y) = (x, y)$. Then by condition (3) of Definition 4.10 $w = y \vee z$ exists and has height 3. Then simple counting gives that $S_B = \downarrow w$. □

Remark 5.19. The previous theorem extends to small blocks as in Remark 5.17. This provides a bijective correspondence between isomorphism classes of orthoalgebras and isomorphism classes of short orthodomains with enough directions.

Remark 5.20. We have shown that for an orthoalgebra A, the poset $\mathrm{BSub}(A)^*$ of elements of height 3 or less in $\mathrm{BSub}(A)$ is sufficient to reconstruct A. We will show one cannot make due with the order structure of the elements of height 2 or less. Specifically, for an orthodomain X, let $X^\dagger$ be the poset of elements of height 2 or less in X. We will give two non-isomorphic orthoalgebras A and C where $\mathrm{BSub}(A)^\dagger$ and $\mathrm{BSub}(C)^\dagger$ are isomorphic.

Let A be the 16-element Boolean algebra shown in Figure 1. The diagram below shows the Fano plane minus a single line, the circle connecting the middle elements of each side. This figure gives a poset P with bottom $\perp$, seven atoms given by the vertices of this figure, and six elements of height 2 given by the lines of the figure, with the understanding that a vertex lies beneath a line if it lies on the line. Then P is isomorphic to the elements $\mathrm{BSub}(A)^\dagger$ of height 2 or less in $\mathrm{BSub}(A)$.

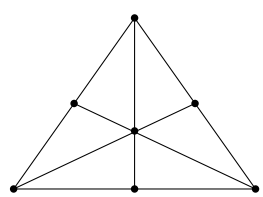


FIGURE 4. The hypergraph view of the subalgebras of a 16-element Boolean algebra

However, it follows from the usual hypergraph representation of orthoalgebras (see [14] with a correction [15]) that this poset also represents the atom structure of an orthoalgebra C. This means that there is an orthoalgebra C whose blocks all have 8 elements where the atoms of C are the vertices of this figure, and the sets of atoms forming a block of C are exactly the vertices lying on a line in the figure. Then $\mathrm{BSub}(C)$ is isomorphic to P, and as every element of it is of height 2 or less, $\mathrm{BSub}(C)^\dagger = \mathrm{BSub}(C)$. Thus $\mathrm{BSub}(A)^\dagger \simeq \mathrm{BSub}(C)^\dagger$, but $A \not\simeq C$.

6. Categorical aspects in the Boolean setting

In Section 2 we gave a correspondence between Boolean algebras and Boolean domains. In this section we include morphisms in our discussion in the Boolean setting. The idea is to take $\mathrm{Sub}(B)$ as the object part of a functor from the category of Boolean algebras to a suitable category of Boolean domains, and to take a Boolean homomorphism $f\colon B_1 \to B_2$ to the direct image map $f[-]\colon \mathrm{Sub}(B_1) \to \mathrm{Sub}(B_2)$. From the outset, there are limitations on what can be achieved. The 2-element and 1-element Boolean algebras both have 1-element Boolean domains as their subalgebra lattices, and the 4-element Boolean algebra has 2 automorphisms, while its subalgebra lattice is a 2-element lattice, and has only the trivial automorphism. So there cannot be an equivalence of categories. Nevertheless, barring these obstacles, we come as close as one could hope to an equivalence.

Definition 6.1. Write **BoolAlg** for the category of Boolean algebras having more than one element and the Boolean algebra homomorphisms between them.

Recall that for a Boolean domain X, we worked initially with principal pairs for a basic element $x \in X$. The approach based on directions for Boolean domains was equivalent to that based on principal pairs, and in some ways more intuitive. But principal pairs are easier to work with, and we will formulate our discussion here in terms of them.

Definition 6.2. Write **BoolDom** for the category whose objects are Boolean domains, and whose morphisms are functions $\alpha\colon X_1 \to X_2$ that preserve arbitrary joins and for each $w \in X_1$:

(1) $\alpha[\downarrow w] = \downarrow \alpha(w)$;
(2) if (y, z) is a principal pair in $\downarrow w$, then $(\alpha(y), \alpha(z))$ is a principal pair in $\downarrow \alpha(w)$;
(3) if $(y_1, z_1) \leq (y_2, z_2)$ in $\mathrm{Pp}(\downarrow w)$, then $(\alpha(y_1), \alpha(z_1)) \leq (\alpha(y_2), \alpha(z_2))$ in $\mathrm{Pp}(\downarrow \alpha(w))$.

Remark 6.3. The third condition in Definition 6.2 is not so elegant, and carries less weight than it may seem. Since α preserves joins, it preserves order. Hence $(y_1, z_1) \leq (y_2, z_2)$ implies $(\alpha(y_1), \alpha(z_1)) \leq (\alpha(y_2), \alpha(z_2))$ except when $(\alpha(y_1), \alpha(z_1))$ is a dual modular atom of $\downarrow \alpha(w)$. The content of condition (3) is to guarantee that in this case $(\alpha(y_2), \alpha(z_2)) \neq (\alpha(y_1), \alpha(z_1))'$. We do not know if (3) is necessary, or whether it is independent of (1) and (2). Perhaps a way around this trouble would be to define $\oplus$ for $\mathrm{Pp}(X)$ instead of $\leq$.

Proposition 6.4. *There is a functor* $\mathrm{Sub}\colon$ **BoolAlg** $\to$ **BoolDom** *that takes a Boolean algebra* B *to its lattice* $\mathrm{Sub}(B)$ *of subalgebras, and that takes a homomorphism* $f\colon B_1 \to B_2$ *to* $f[-]$.

Proof. In general, the image under a homomorphism of the subalgebra generated by the union of a family of sets is equal to the subalgebra generated by the union of the images of the sets. So $f[-]$ preserves arbitrary joins. Let $w \in \mathrm{Sub}(B_1)$ and $q = f[w]$. We verify (1)–(3) of Definition 6.2 for w. If $p \in \mathrm{Sub}(B_2)$ with $p \leq q$, then $y = f^{-1}[p] \cap w$ is a subalgebra of B_1 that is contained in w and $f[y] = p$. So $f[-]$ maps $\downarrow w$ onto $\downarrow f[w]$, giving (1).

Suppose that (y, z) is a principal pair in $\downarrow w$. By Proposition 2.11 there is $a \in w$ with

$$y = \downarrow_w a \cup \uparrow_w a' \qquad \text{and} \qquad z = \downarrow_w a' \cup \uparrow_w a.$$

Since f maps the subalgebra w of B_1 onto the subalgebra q of B_2, we have $f[\downarrow_w a] = \downarrow_q f(a)$ and $f[\uparrow_w a'] = \uparrow_q f(a)'$. So $f[y] = \downarrow_q f(a) \cup \uparrow_q f(a')$ and $f[z] = \downarrow_q f(a)' \cup \uparrow_q f(a)$. Then, by Proposition 2.11 this is a principal pair in $\downarrow q = \downarrow \alpha(w)$, giving (2).

For part (3), if $(y_1, z_1) \leq (y_2, z_2)$ are principal pairs in $\downarrow w$, then there are $a \leq b$ in w with $y_1 = \downarrow_w a \cup \uparrow_w a'$, $z_1 = \downarrow_w a' \cup \uparrow_w a$, $y_2 = \downarrow_w b \cup \uparrow_w b'$, and $z_2 = \downarrow_w b' \cup \uparrow_w b$. By the argument used in part (2) we have $(f[y_1], f[z_1])$ is the principal pair in $\downarrow f[w]$ given by $f(a)$ and $(f[y_2], f[z_2])$ is the principal pair in $\downarrow f[w]$ given by $f(b)$. Since $f(a) \leq f(b)$ part (3) follows. □

Remark 6.5. The opening of this section mentioned several obstacles to an equivalence between the categories **BoolAlg** and **BoolDom**. There are more. Suppose that B_1 and B_2 are Boolean algebras with B_2 having more than 1 element. Then every prime ideal of B_1 gives a distinct homomorphism $f\colon B_1 \to B_2$ whose image has 2 elements. For each of these homomorphisms, $f[-]$ is the map sending each element of $\mathrm{Sub}(B_1)$ to $\bot \in \mathrm{Sub}(B_2)$. For another pathology, suppose $f\colon B_1 \to B_2$ is a Boolean algebra homomorphism whose image has 4 elements. Then composing with the nontrivial automorphism of this image gives a homomorphism $g\colon B_1 \to B_2$ distinct from f with $f[-]$ and $g[-]$ equal. Thus the functor Sub is not faithful with respect to homomorphisms whose image has four or fewer elements.

A functor $F\colon \mathbf{C} \to \mathbf{D}$ gives an equivalence of categories if it is full, faithful, and essentially surjective on objects, i.e. each object of $\mathbf{D}$ is isomorphic to some $F(C)$ for an object C of $\mathbf{C}$. The functor $\mathrm{Sub}\colon \mathbf{BoolAlg} \to \mathbf{BoolDom}$ does not have these properties, but it comes close.

Lemma 6.6. *Let B be a Boolean algebra with more than 4 elements, and let $(y,z),(y_1,z_1),(y_2,z_2)$ be principal pairs for the basic elements x,x_1,x_2 in $\mathrm{Sub}(B)$.*

1. *If x is dual modular, one of y,z is basic and the other equals $\top$.*
2. *If x is not dual modular, neither of y,z is basic or equal to $\top$.*
3. *If $y_1 = y_2$ and either $y_1 \neq \top$ or $x_1 = x_2$, then $z_1 = z_2$.*
4. *The join $(y_1,z_1) \vee (y_2,z_2)$ in $\mathrm{Pp}(B)$ has first component $y_1 \vee y_2$.*

Proof. Proposition 2.11 gives $a,a_1,a_2 \in B$ with $(y,z) = ({\downarrow} a \cup {\uparrow} a', {\downarrow} a' \cup {\uparrow} a)$ and with $(y_i,z_i) = ({\downarrow} a_i \cup {\uparrow} a_i', {\downarrow} a_i' \cup {\uparrow} a_i)$ for $i = 1,2$. Part (1) now follows from Definition 2.9. Part (2) follows from Lemma 2.7 because a,a' are not $0,1$, atoms, or coatoms. For part (3), since ${\downarrow} a_1 \cup {\uparrow} a_1' = {\downarrow} a_2 \cup {\uparrow} a_2'$ either $a_1 = a_2$, which implies $z_1 = z_2$, or $y_1 = y_2 = \top$. If $y_1 = y_2 = \top$, then (1) gives that $(y_i,z_i) = (\top, x_i)$ for $i = 1,2$ and the result follows. Finally, part (4). Proposition 2.14 shows the first component of $(y_1,z_1) \vee (y_2,z_2)$ is $S = {\downarrow}(a_1 \vee a_2) \cup {\uparrow}(a_1 \vee a_2)' = \varphi(a_1 \vee a_2)$. Clearly S contains y_1,y_2 hence S contains the subalgebra $y_1 \vee y_2$ they generate. If $b \in {\downarrow}(a_1 \vee a_2)$, then $b = (b \wedge a_1) \vee (b \wedge a_2)$. It follows that $S \subseteq y_1 \vee y_2$. ☐

Lemma 6.7. *Let B_1 and B_2 be Boolean algebras with B_2 having more than 4 elements, and let $f\colon B_1 \to B_2$ be a function that preserves order and complementation. If $f(a \vee b) = f(a) \vee f(b)$ whenever $f(a) \vee f(b)$ is not a coatom or 1, then f is a homomorphism.*

Proof. Since f preserves order and complementation, it preserves orthogonality. We will show that f preserves orthogonal joins. From this it follows that f preserves binary joins, so is a homomorphism. Indeed, for any a,b there are pairwise orthogonal c,d,e with $a = c \vee d$ and $b = d \vee e$. Then if f preserves orthogonal joins, $f(a \vee b) = f(c \vee d \vee e) = f(c) \vee f(d) \vee f(e) = (f(c) \vee f(d)) \vee (f(d) \vee f(e)) = f(a) \vee f(b)$.

We must show that f preserves joins of orthogonal a and b. By assumption f preserves their join if $f(a) \vee f(b)$ is not a coatom or 1, and since f is order preserving it clearly preserves their join if $f(a) \vee f(b) = 1$. The case remains when $f(a) \vee f(b)$ is a coatom c of B_2. Since $f(a)$ and $f(b)$ are orthogonal, they cannot both be coatoms of B_2, and we assume that $f(b)$ is not a coatom. Suppose for a contradiction that $f(a \vee b) \neq f(a) \vee f(b)$, hence $f(a \vee b) = 1$. Since f preserves complementation, $f(a' \wedge b') = 0$. So $f(a' \wedge b') \vee f(b) = f(b)$ is not a coatom or 1. This implies that $f(b) = f(a' \wedge b') \vee f(b) = f((a' \wedge b') \vee b) = f(a' \vee b)$. Then $f(a') \leq f(b) \leq c$ and $f(a) \leq f(a) \vee f(b) = c$. This contradicts that f preserves complementation. ☐

Lemma 6.8. *Suppose that B_1 and B_2 are Boolean algebras and $\alpha\colon \mathrm{Sub}(B_1) \to \mathrm{Sub}(B_2)$ is a Boolean domain morphism with $\alpha(B_1) = T$ having more than 4 elements. Then there is a unique homomorphism $f\colon B_1 \to B_2$ with $\alpha = f[-]$.*

Proof. Note first that $\alpha(B_1) = T$ implies by the definition of a morphism of Boolean domains that α maps $\mathrm{Sub}(B_1)$ onto $\mathrm{Sub}(T)$. Since T has more than 4 elements, it follows that B_1 has more than 4 elements. Then by Theorem 2.15 there are isomorphisms

$$\begin{aligned} B_1 &\simeq \mathrm{Pp}(\mathrm{Sub}(B_1)) & \qquad a &\mapsto (\downarrow a \cup \uparrow a', \downarrow a' \cup \uparrow a) = \varphi_{B_1}(a), \\ T &\simeq \mathrm{Pp}(\mathrm{Sub}(T)) & \qquad b &\mapsto (\downarrow_T b \cup \uparrow_T b', \downarrow_T b' \cup \uparrow_T b) = \varphi_T(b). \end{aligned}$$

For (y, z) a principal pair of $\mathrm{Sub}(B_1)$, Definition 6.2 gives that $(\alpha(y), \alpha(z))$ is a principal pair in $\downarrow \alpha(B_1) = \downarrow T = \mathrm{Sub}(T)$. So there is a function $(\alpha, \alpha)\colon \mathrm{Pp}(\mathrm{Sub}(B_1)) \to \mathrm{Pp}(\mathrm{Sub}(T))$ taking (y, z) to $(\alpha(y), \alpha(z))$. Let $f\colon B_1 \to T$ be the unique function making the square below commute.

$$\begin{array}{ccc} \mathrm{Pp}(\mathrm{Sub}(B_1)) & \xrightarrow{(\alpha,\alpha)} & \mathrm{Pp}(\mathrm{Sub}(T)) \\ \varphi_{B_1} \uparrow & & \uparrow \varphi_T \\ B_1 & \xrightarrow[f]{} & T \end{array}$$

Explicitly, f is given for each $a \in B_1$ by $f(a) = b$ where $b \in T$ is the unique element with

$$\begin{aligned} \alpha(\downarrow a \cup \uparrow a') &= \downarrow_T b \cup \uparrow_T b', \\ \alpha(\downarrow a' \cup \uparrow a) &= \downarrow_T b' \cup \uparrow_T b. \end{aligned}$$

Suppose $g\colon B_1 \to B_2$ is a homomorphism with $\alpha = g[-]$. Then $g[B_1] = \alpha(B_1) = T$, so the image of g is T. Thus $g[\downarrow a \cup \uparrow a'] = \downarrow_T g(a) \cup \uparrow_T g(a)'$, and $g[\downarrow a' \cup \uparrow a] = \downarrow_T g(a)' \cup \uparrow_T g(a)$, and this gives $g(a) = f(a)$. So there is at most one homomorphism from B_1 to B_2 whose image mapping is α, and the candidate for this homomorphism is the function f.

It remains to show that f is a homomorphism and the image map $f[-]$ equals α. To show that f is a homomorphism, it suffices to show that (α, α) is a homomorphism, which will follow from Lemma 6.7, whose conditions we verify. Condition (3) of Definition 6.2 shows that (α, α) is order preserving. For $(y, z) \in \mathrm{Pp}(\mathrm{Sub}(B_1))$ with $(\alpha(y), \alpha(z)) = (v, w)$, we see that (α, α) preserves complementation since

$$(\alpha, \alpha)((y, z)') = (\alpha(z), \alpha(y)) = (\alpha(y), \alpha(z))' = ((\alpha, \alpha)(y, z))'.$$

To show the final condition of Lemma 6.7, suppose that (y_1, z_1) and (y_2, z_2) are such that $(\alpha(y_1), \alpha(z_1)) \vee (\alpha(y_2), \alpha(z_2))$ is not 1 or a coatom of $\mathrm{Pp}(\mathrm{Sub}(T))$. This means that the first coordinate of this join is not T. Using Lemma 6.6 and the fact that α preserves joins, we have the following, where $\star$ indicates an irrelevant second component:

$$\begin{aligned} (\alpha, \alpha)((y_1, z_1) \vee (y_2, z_2)) &= (\alpha, \alpha)(y_1 \vee y_2, \star) = (\alpha(y_1) \vee \alpha(y_2), \star) \\ (\alpha, \alpha)(y_1, z_1) \vee (\alpha, \alpha)(y_2, z_2) &= (\alpha(y_1), \alpha(z_1)) \vee (\alpha(y_2), \alpha(z_2)) = (\alpha(y_1) \vee \alpha(y_2), \star) \end{aligned}$$

These are principal pairs in $\mathrm{Sub}(T)$ with the same first component that is different from T, so by Lemma 6.6 they are equal. Thus (α, α) satisfies the hypotheses of Lemma 6.7, so it, and hence also f, is a homomorphism.

Finally, it remains to show that $f[-] = \alpha$. Let $a \in B_1$. Then $\big(\downarrow a \cup \uparrow a', \downarrow a' \cup \uparrow a\big)$ is a principal pair for $x = \{0, a, a', 1\}$. By Definition 6.2, $\big(\alpha(\downarrow a \cup \uparrow a'), \alpha(\downarrow a' \cup \uparrow a)\big)$ is a principal pair for $\alpha(x)$. By construction this is the principal pair $\big(\downarrow f(a) \cup \uparrow f(a)', \downarrow f(a)' \cup \uparrow f(a)\big)$ for $\{0, f(a), f(a)', 1\}$. Thus $\alpha(x) = f[x]$. So α and $f[-]$ agree on the basic elements of $\mathrm{Sub}(B_1)$. Preservation of joins then shows $\alpha = f[-]$. $\square$

Theorem 6.9. *The functor* Sub: **BoolAlg** $\to$ **BoolDom** *is essentially surjective on objects, faithful except with regards to Boolean algebra homomorphisms whose image has* 4 *or fewer elements, and full except possibly with regards to Boolean domain morphisms whose image has* 2 *elements.*

Proof. Essential surjectivity follows from Sachs' result [17], which determines a Boolean algebra B up to isomorphism except when $\mathrm{Sub}(B)$ has a single element. The statements about fullness and faithfulness follow from Lemma 6.8. □

7. Concluding remarks

Preceding descriptions of quantum structures were mostly based on *maximal* Boolean subalgebras (blocks) and their intersections, which are again *maximal* common Boolean subalgebras. This approach led to serious combinatorial problems in applications. In contrast to it, we propose here two other approaches. First, we use the structure of *all* Boolean subalgebras. This description is not optimized, but it is well understood thanks to the previous work of Sachs on the Boolean case. Besides, we have shown that the lattice of Boolean subalgebras itself determines an orthoalgebra, without refering to their elements at all. The second step is that we prove that even the structure of *"small"* Boolean subalgebras is sufficient to reconstruct an orthoalgebra. (Here "small" means with up to 16-elements.) There were attempts to describe quantum structures by the orthogonality relation only [11], thus refering only to Boolean subalgebras with up to 8-elements, but there was no progress for decades. Although our descriptions are not very concise, they at least offer two alternatives to the standard approach which did not bring substantial progress during the last 30 years.

We have reconstructed up to isomorphism a nontrivial Boolean algebra B from its Boolean domain $\mathrm{Sub}(B)$ first with principal pairs, and then with directions. The posets of the form $\mathrm{Sub}(B)$ for a Boolean algebra B had been earlier characterized [6]. These results provide a bijection between isomorphism classes of nontrivial Boolean algebras and isomorphism classes of Boolean domains. This object level correspondence is lifted to a functorial correspondence between the category of nontrivial Boolean algebras and Boolean domains. This functor is shown to be nearly an equivalence.

We extended our method to reconstruct up to isomorphism a nontrivial orthoalgebra A from its orthodomain $\mathrm{BSub}(A)$ using directions. We further characterized the posets that arise as $\mathrm{BSub}(A)$ for some orthoalgebra A as the tall orthodomains with enough directions. This provides a bijection between isomorphism classes of nontrivial orthoalgebras and isomorphism classes of tall orthodomains with enough directions. We also provide an isomorphism between isomorphism classes of nontrivial orthoalgebras and isomorphism classes of short orthodomains with enough directions.

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