A quantum while loop for amplitude amplification

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Proposal

An alternative algorithm for amplitude amplification such that

- the number of iterations is not specified beforehand and
- it keeps the quantum speed-up: $\mathcal{O}\left(1/\sqrt{\rho}\right)$ oracle queries.

Our approach uses a *while loop*: after every iteration, we test a condition by applying a *weak measurement*. Once the condition is satisfied, the algorithm succeeds.

Background

Amplitude amplification. Let B be a finite set and let $\chi\colon B \to \{0,1\}$ be the *oracle function* that characterises a marked subset of B. Define Hilbert spaces

$$\mathcal{H}_i = \operatorname{span}\{b \in B \mid \chi(b) = i\}$$

for $i \in \{0,1\}$ and $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$. Choose some $|\psi\rangle \in \mathcal{H}$ as the *initial state*. The task, starting from $|\psi\rangle$, is to return a state in $\mathcal{H}_1$ with probability close to 1.

Write P_1 for the orthogonal projection onto $\mathcal{H}_1$ and

$$\rho = \langle\psi|P_1|\psi\rangle$$

for the *initial success probability*. The algorithm is *efficient* if its expected number of queries to χ is $\mathcal{O}\left(1/\sqrt{\rho}\right)$.

A **weak measurement** "gives very little information about the system on average, but also disturbs the state very little" [1]

Let $\mathcal{P} = \operatorname{span}\{\bot, \top\}$ be a Hilbert space known as the *probe*. A weak measurement on $|\phi\rangle \in \mathcal{H}$ is achieved by applying a unitary

$$E_\kappa\colon \mathcal{H} \otimes \mathcal{P} \to \mathcal{H} \otimes \mathcal{P}$$

on state $|\phi\rangle \otimes |\bot\rangle$ and then measuring *only* the probe. Parameter $\kappa \in [0,1]$ determines the strength of the measurement.

$$E_\kappa = P_0 \otimes I_{\mathcal{P}} + P_1 \otimes R_\kappa$$
$$R_\kappa = \begin{pmatrix} \sqrt{1-\kappa} & \sqrt{\kappa} \\ \sqrt{\kappa} & -\sqrt{1-\kappa} \end{pmatrix} \tag{1}$$

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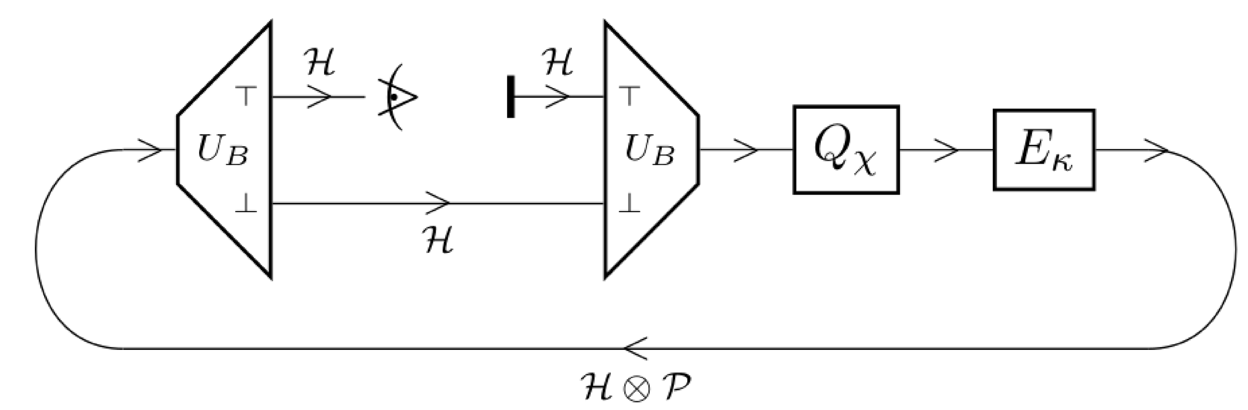
Our Algorithm

Consider the decomposition

$$|\psi\rangle = \cos\alpha|\psi_0\rangle + \sin\alpha|\psi_1\rangle$$

where $|\psi_i\rangle = P_i|\psi\rangle$. For any iteration n, we find (2) the corresponding angle a_n that describes the current state.



Here, U_B is the isomorphism $\mathcal{H} \otimes \mathcal{P} \to \mathcal{H} \oplus \mathcal{H}$ separating $\bot$ from $\top$. Q_χ is the unitary applied on each iteration of the standard algorithm; it increases the angle by 2α. E_κ is given in (1) and it is part of a weak measurement. With probability

$$p_\top = \kappa \sin^2 a_n$$

the outcome is $\top$ and a marked element is found. Otherwise, the angle is reduced by some θ_n and we keep iterating.

$$a_{n+1} = a_n + 2\alpha - \theta_n$$
$$a_0 = \alpha \approx \sqrt{\rho}. \tag{2}$$

The value of θ_n can be calculated using trigonometry. We prove

$$\kappa \leq \sqrt{\rho} \;\Rightarrow\; |\theta_n| \leq \alpha. \tag{3}$$

implying the angle a_n increases at a steady pace throughout the iterations. We argue that, for roughly half of the iterations, $\sin^2 a_n \geq 1/2$ so that $p_\top \geq \kappa/2$.

The algorithm succeeds as soon as $\top$ is measured, hence, the number of iterations our algorithm takes follows a geometric distribution; its expected value is $\mu = 4/\kappa$. Finally, by imposing

$$\kappa = \sqrt{\rho} \tag{4}$$

we achieve a query complexity of $\mathcal{O}\left(1/\sqrt{\rho}\right)$.

References

[1] T. A. Brun.
A simple model of quantum trajectories.
American Journal of Physics, 70(7):719–737, 2002.

[2] A. Mizel.
Critically damped quantum search.
Phys. Rev. Lett., 102:150501, Apr 2009.

Previous Work

In the standard algorithm, the number of iterations is fixed to $K = \frac{\pi}{4\sqrt{\rho}}$. In contrast, the number of iterations is not predetermined in our approach, but instead the strength of the measurement is set to $\kappa = \sqrt{\rho}$. In either case, we must define some parameter according to the value of ρ.

We later found out a paper [2] proposing essentially the same algorithm as ours, although they do not discuss it from the perspective of while loops and weak measurements.

Alternative: test-restart approach

Weak measurements may be replaced by the following procedure:

1. pick a random number $r \in [0,1]$ from a uniform distribution,
2. if and only if $r \leq \kappa$, apply a projective measurement on $\mathcal{H}$.

If the outcome of the measurement is a marked element, the algorithm succeeds. Otherwise, the state is initialised to $|\psi\rangle$ and the algorithm restarts.

The number of iterations between restarts follows a geometric distribution with $\mu = 1/\kappa$. For $\kappa = \sqrt{\rho}$, this is close to the $\frac{\pi}{4\sqrt{\rho}}$ iterations required in standard amplitude amplification.

It follows that the average query complexity matches $\mathcal{O}\left(1/\sqrt{\rho}\right)$.

Further statistical analysis has shown that the variance of the number of queries also roughly matches that of our weak measurement approach.

We conclude that weak measurements *are not providing an algorithmic advantage*. However, there might be other benefits to using them: namely, experimental realisation, as discussed below.

Next Step: continuous measurement

A weak measurement is a natural procedure in a laboratory. However, our weak measurements are applied at discrete points in time, while the simplest experimental realisation would have the probe under *continuous measurement*.

We intend to study whether our algorithm can be adapted to the continuous-time setting while maintaining the quantum speed-up. The main obstacle may be the quantum Zeno effect.

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