

# SHEAF REPRESENTATION OF MONOIDAL CATEGORIES

RUI SOARES BARBOSA AND CHRIS HEUNEN

**ABSTRACT.** Every small monoidal category with universal (finite) joins of central idempotents is monoidally equivalent to the category of global sections of a sheaf of (sub)local monoidal categories on a topological space. Every small stiff monoidal category monoidally embeds into such a category of global sections. These representation results are functorial and subsume the Lambek–Moerdijk–Awodey sheaf representation for toposes, the Stone representation of Boolean algebras, and the Takahashi representation of Hilbert modules as continuous fields of Hilbert spaces. Many properties of a monoidal category carry over to the stalks of its sheaf, including having a trace, having exponential objects, having dual objects, having limits of some shape, and the central idempotents forming a Boolean algebra.

## 1. INTRODUCTION

Representation theorems make abstract structures easier to work with by showing that they always have a more concrete form. For example, the category  $\mathbf{Vect}$  of vector spaces is easy in the sense that it is not a product of other categories, whereas the category  $\mathbf{Vect} \times \mathbf{Vect}$  is also monoidal, but in a more complicated way. The goal of this article is to prove that any monoidal category embeds into a product of easier ones. In fact we prove something stronger: nice enough monoidal categories are always (equivalent to) a dependent product of easy ones.

To explain what we mean by “easy” categories, “dependent” products, and “nice enough” monoidal categories, consider again  $\prod_{i \in \{0,1\}} \mathbf{Vect}$ . Its decomposable nature can be detected by its *central idempotents*, which correspond to the open sets of the discrete topological space  $\{0,1\}$ . We consider a monoidal category “nice” when its central idempotents are respected by tensor products, and have the structure of a (distributive lattice or even a) frame that is respected by tensor products; in the former case we call the category *stiff* and in the latter we say that the category has *universal (finite) joins* of central idempotents. We consider a category “easy” when its central idempotents are *(sub)local* in that any (finite) cover already contains the covered element; topologically this means there is a single focal point that every net converges to; logically this is the (disjunction or) existence property that if a (finite) disjunction holds then one of the disjoints holds. Finally, by a “dependent” product we mean that the fibre  $\mathbf{Vect}$  does not have to be constant but can vary continuously with the index  $i$ , that is, the category consists of global sections of a sheaf. With this terminology, made more precise below, here are our main results.

**Theorem.** Any small monoidal category with universal (finite) joins of central idempotents is monoidally equivalent to the category of global sections of a sheaf of (sub)local categories.

**Corollary.** Any small stiff monoidal category monoidally embeds into a category of global sections of a sheaf of (sub)local categories.

This subsumes sheaf representation theorems for toposes [1, 2, 40, 42, 44], where the lattice of central idempotents corresponds to that of elements of the subobject

classifier. In fact, the simple main insight underlying this work is that in cartesian categories, subterminal objects can be characterised entirely algebraically as central idempotents. This improves on earlier work [23], which focused on the special case of central idempotents called subunits. From a logical point of view, it extends the sheaf representation of (topos) categorical models of higher-order intuitionistic logic to (symmetric monoidal closed) categorical models of multiplicative linear logic. An additional improvement over results from the literature is that our proof is entirely concrete and avoids stacks. Furthermore, by virtue of being monoidal, our results are not just analogous to, but directly capture, sheaf representation theorems for frames and modules [32, 19]. The representation theorem may also be regarded as a structure theorem for higher-dimensional algebra [3, 35], because 2-vector spaces may be seen as sheaves of vector spaces [28, Appendix A]. Finally, our results find applications in physics and concurrent (probabilistic) computing, where a monoidal category models a process theory [29] and its central idempotents may be interpreted as an underlying causal structure [22].

We proceed as follows. Section 2 defines central idempotents and gives examples. Basic properties of central idempotents that make the category “nice” are discussed in Section 3. In Section 4 we construct the topological space on which the structure sheaf will be based. Section 5 details the presheaf structure, and Section 6 establishes the sheaf condition. Next, Section 7 investigates the stalks, and in Section 8 we show that they are (sub)local, finishing the proof of the Theorem above. The sheaf representation is extended in Section 9, by showing that it preserves being Boolean, having limits, being closed, and having a trace. Section 10 works out several examples that were not possible to handle with sheaf representations for toposes, including recovering the Stone representation of Boolean algebras, and the Takahashi representation of Hilbert modules. Section 11 settles functoriality of the main construction, after which Section 12 proves the Corollary above. Finally, in Section 13 we discuss several open questions that may be attacked using the representation theorem. Appendix A compares central idempotents with the special case of subunits [23].

## 2. CENTRAL IDEMPOTENTS

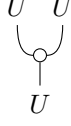
This section introduces central idempotents, gives examples, and discusses basic properties. If  $\mathbf{C}$  is a monoidal category, and  $I$  its tensor unit, then the slice category  $\mathbf{C}/I$  is again monoidal: the (terminal) tensor unit is the identity  $I \rightarrow I$ , the tensor product of objects  $u: U \rightarrow I$  and  $v: V \rightarrow I$  is  $\rho_I \circ (u \otimes v): U \otimes V \rightarrow I$ , and the tensor product of morphisms is as in  $\mathbf{C}$ . An object  $u: U \rightarrow I$  in  $\mathbf{C}/I$  may be idempotent in the sense that  $u \simeq u \otimes u$ . We are interested in objects in  $\mathbf{C}/I$  that are idempotent in a canonical way.

**Definition 2.1.** A morphism  $u: U \rightarrow I$  in a monoidal category is a *left idempotent* when  $\lambda_U \circ (u \otimes U): U \otimes U \rightarrow U$  is invertible, and a *right idempotent* when  $\rho_U \circ (U \otimes u): U \otimes U \rightarrow U$  is invertible.

We will freely use the graphical calculus for monoidal categories [29], and draw a left idempotent  $u: U \rightarrow I$  as:

$$\begin{array}{c} \circ \\ | \\ U \end{array}$$

For a left idempotent we draw the inverse of  $\lambda_U \circ (u \otimes U): U \otimes U \rightarrow U$  as:



Recall that the *centre* of a monoidal category  $\mathbf{C}$  has as objects  $U \in \mathbf{C}$  equipped with a *half-braiding* (see [36, XIII.4] and [33]): a natural transformation  $\sigma_A: U \otimes A \rightarrow A \otimes U$  such that  $\sigma_{A \otimes B} = (A \otimes \sigma_B) \circ (\sigma_A \otimes B)$ . A morphism  $(U, \sigma) \rightarrow (V, \tau)$  in the centre is a morphism  $f: U \rightarrow V$  in  $\mathbf{C}$  satisfying  $(A \otimes f) \circ \sigma_A = \tau_A \circ (f \otimes A)$ . The tensor unit  $I$  of a monoidal category always carries a half-braiding  $\rho_A^{-1} \circ \lambda_A: I \otimes A \rightarrow A \otimes I$ . We draw  $\sigma_A$  as:



**Definition 2.2.** A *central idempotent* in a monoidal category is a left idempotent  $u: U \rightarrow I$  in its centre such that:

$$(2.1) \quad \lambda_U \circ (u \otimes U) = \rho_U \circ (U \otimes u): U \otimes U \rightarrow U$$

Explicitly, it is a morphism  $u: U \rightarrow I$  such that (2.1) holds and is invertible, equipped with a half-braiding satisfying:

$$(2.2) \quad \begin{array}{c} A \\ \circ \\ \swarrow \quad \searrow \\ U \quad A \end{array} = \begin{array}{c} A \\ \circ \\ | \\ U \quad A \end{array}$$

We will identify two central idempotents  $u: U \rightarrow I$  and  $v: V \rightarrow I$  when  $u = v \circ m$  for an isomorphism  $m: U \rightarrow V$  that respects the half-braidings, and write  $\text{ZI}(\mathbf{C})$  for the set<sup>1</sup> of central idempotents of  $\mathbf{C}$ .

The central equation (2.1) graphically becomes:

$$\begin{array}{c} \circ \\ | \\ UU \end{array} = \begin{array}{c} \circ \\ | \\ UU \end{array}$$

If  $u: U \rightarrow I$  and  $v: V \rightarrow I$  are central idempotents, we draw a morphism  $m: U \rightarrow V$  satisfying  $u = v \circ m$  as:



All in all, the following graphical identities will be useful:

$$\begin{array}{c} U \\ \circ \\ \swarrow \quad \searrow \\ U \quad U \end{array} = \begin{array}{c} U \\ | \\ U \end{array} = \begin{array}{c} U \\ \circ \\ \swarrow \quad \searrow \\ U \quad U \end{array} \quad \begin{array}{c} V \\ \circ \\ | \\ U \end{array} = \begin{array}{c} \circ \\ | \\ U \end{array} \quad \begin{array}{c} V \quad V \\ \circ \\ \swarrow \quad \searrow \\ U \quad U \end{array} = \begin{array}{c} V \quad V \\ \circ \\ | \quad | \\ U \quad U \end{array}$$

<sup>1</sup>Throughout this article we consider only *small* categories  $\mathbf{C}$  to prevent size subtleties.

The following proposition ensures that these graphical notations are well-defined.

**Proposition 2.3.** *For central idempotents  $u: U \rightarrow I$  and  $v: V \rightarrow I$  in a monoidal category  $\mathbf{C}$ , the following are equivalent:*

- $u = v \circ m$  for a necessarily unique  $m: U \rightarrow V$  that respects half-braidings;
- $U \otimes v: U \otimes V \rightarrow U \otimes I$  is invertible;
- $v \otimes U: V \otimes U \rightarrow I \otimes U$  is invertible.

We say  $u \leq v$  when these conditions hold. This partially orders  $\text{ZI}(\mathbf{C})$ .

*Proof.* First we prove that the morphism  $m$  is unique. If  $u = v \circ m = v \circ n$ , then:

$$\begin{array}{c} V \\ | \\ \boxed{m} \\ | \\ U \end{array} = \begin{array}{c} V \\ | \\ \boxed{m} \\ | \circ \\ \cup \\ | \\ U \end{array} = \begin{array}{c} V \quad V \\ | \quad | \\ \boxed{m} \quad \boxed{n} \\ | \quad | \\ \cup \\ | \\ U \end{array} = \begin{array}{c} V \quad V \\ | \quad | \\ \boxed{m} \quad \boxed{n} \\ | \quad | \\ \cup \\ | \\ U \end{array} = \begin{array}{c} V \\ | \\ \boxed{n} \\ | \circ \\ \cup \\ | \\ U \end{array} = \begin{array}{c} V \\ | \\ \boxed{n} \\ | \\ U \end{array}$$

To see that the first point implies the second:

$$\begin{array}{c} U \\ | \\ \boxed{f} \\ | \circ \\ \cup \\ | \\ U \end{array} = \begin{array}{c} U \\ | \\ \boxed{f} \\ | \\ U \end{array} = \begin{array}{c} U \\ | \\ U \end{array} \quad \begin{array}{c} U \quad V \\ | \quad | \\ \boxed{f} \\ | \quad | \\ \cup \\ | \\ U \quad V \end{array} = \begin{array}{c} U \quad V \\ | \quad | \\ \boxed{f} \\ | \quad | \\ \cup \\ | \\ U \quad V \end{array} = \begin{array}{c} U \quad V \\ | \quad | \\ \boxed{f} \\ | \quad | \\ \cup \\ | \\ U \quad V \end{array} = \begin{array}{c} U \quad V \\ | \quad | \\ \boxed{f} \\ | \quad | \\ \cup \\ | \\ U \quad V \end{array} = \begin{array}{c} UV \\ | \\ UV \end{array}$$

Hence the dashed morphism inverts  $U \otimes v$ .

To see that the second point implies the first, suppose  $f$  inverts  $U \otimes v$ . Then:

$$\begin{array}{c} \circ \\ | \\ U \end{array} = \begin{array}{c} U \quad V \\ | \quad | \\ \boxed{f} \\ | \\ U \end{array} = \begin{array}{c} U \quad V \\ | \quad | \\ \boxed{f} \\ | \\ U \end{array}$$

Hence the dashed morphism  $m$  satisfies  $u = v \circ m$ .

The first and third points are similarly equivalent. These conditions clearly satisfy transitivity, reflexivity, and anti-symmetry.  $\square$

The previous proposition justifies drawing the mediating morphism  $m: U \rightarrow V$  as an unlabeled dot. It also follows from the previous proposition that a central idempotent is completely determined by its domain: if  $u, u': U \rightarrow I$  both represent central idempotents, then  $u = u' \circ m$  for a unique isomorphism  $m: U \rightarrow U$ . This justifies drawing  $u$  as an unlabeled dot on a wire labelled  $U$ .

**Lemma 2.4.** *If  $\mathbf{C}$  is any monoidal category,  $\text{ZI}(\mathbf{C})$  is a (meet-)semilattice, with*

$$u \wedge v = \begin{array}{c} \circ \quad \circ \\ | \quad | \\ UV \end{array}$$

and largest element  $1: I \rightarrow I$ .

*Proof.* Clearly  $u \geq u \wedge v \leq v$ . If  $u \geq w \leq v$ , then:

$$\begin{array}{c} U \quad V \\ | \quad | \\ \circ \quad \circ \\ | \quad | \\ \cup \\ | \\ W \end{array} = \begin{array}{c} \circ \quad \circ \\ | \quad | \\ \cup \\ | \\ W \end{array} = \begin{array}{c} \circ \\ | \\ W \end{array}$$

So  $w \leq u \wedge v$ , and  $u \wedge v$  is the greatest lower bound of  $u$  and  $v$ . That 1 is the greatest element is clear.  $\square$

We have taken some pains to define central idempotents for monoidal categories. The next two lemmas show that for braided monoidal categories, the half-braiding is superfluous, and hence the definition of central idempotents simplifies.

**Lemma 2.5.** *A half-braiding  $\sigma$  makes a left idempotent  $u$  in a monoidal category into a central idempotent if and only if  $\sigma_{U,U} = U \otimes U$ :*

$$\begin{array}{c} U \\ \circ \\ | \\ U \end{array} \begin{array}{c} U \\ | \\ U \end{array} = \begin{array}{c} U \\ | \\ U \end{array} \begin{array}{c} U \\ \circ \\ | \\ U \end{array} \iff \begin{array}{c} U \quad U \\ \diagdown \quad \diagup \\ U \quad U \end{array} = \begin{array}{c} U \quad U \\ | \quad | \\ U \quad U \end{array}$$

*Proof.* If  $u$  is a left idempotent satisfying (2.1), then:

$$\begin{array}{c} U \quad U \\ \diagdown \quad \diagup \\ U \quad U \end{array} = \begin{array}{c} U \quad U \\ \circ \quad \circ \\ \diagdown \quad \diagup \\ U \quad U \end{array} = \begin{array}{c} U \quad U \\ \diagup \quad \diagdown \\ U \quad U \end{array} \begin{array}{c} U \\ \circ \\ | \\ U \end{array} = \begin{array}{c} U \quad U \\ \diagup \quad \circ \\ \diagdown \quad | \\ U \quad U \end{array} = \begin{array}{c} U \quad U \\ | \quad | \\ U \quad U \end{array}$$

Conversely, if  $\sigma_{U,U} = U \otimes U$ , then  $\lambda_U \circ (u \otimes U) = \lambda_U \circ (u \otimes U) \circ \sigma_{U,U} = \rho_U \circ (U \otimes u)$ .  $\square$

**Lemma 2.6.** *If  $u$  is a central idempotent in a braided monoidal category, then the half-braiding  $\sigma_u$  equals the braiding  $U \otimes - \Rightarrow - \otimes U$ .*

*Proof.* For this proof only, to distinguish them we will draw the half-braiding of  $U$  as  $\curvearrowright$ , the braiding as  $\curvearrowleft$ , and the inverse braiding as  $\curvearrowright$ . Then, graphically:

$$\begin{array}{c} B \quad U \\ \curvearrowleft \\ | \quad | \\ U \quad B \end{array} = \begin{array}{c} B \quad U \\ \circ \\ \curvearrowleft \\ | \quad | \\ U \quad B \end{array} = \begin{array}{c} B \quad U \\ \curvearrowleft \\ \curvearrowright \\ | \quad | \\ U \quad B \end{array} = \begin{array}{c} B \quad U \\ \curvearrowright \\ \curvearrowleft \\ | \quad | \\ U \quad B \end{array} \\ \\ = \begin{array}{c} B \quad U \\ \curvearrowright \\ \curvearrowleft \\ | \quad | \\ U \quad B \end{array} = \begin{array}{c} B \quad U \\ \curvearrowright \\ \curvearrowright \\ | \quad | \\ U \quad B \end{array} = \begin{array}{c} B \quad U \\ \curvearrowright \\ \curvearrowright \\ | \quad | \\ U \quad B \end{array} = \begin{array}{c} B \quad U \\ \circ \\ \curvearrowright \\ | \quad | \\ U \quad B \end{array} = \begin{array}{c} B \quad U \\ \curvearrowright \\ | \quad | \\ U \quad B \end{array}$$

These equalities used, respectively: invertibility of  $u \otimes U$ , invertibility of the braiding, Lemma 2.5, naturality of the braiding in the first argument, Lemma 2.5, naturality of the braiding in the second argument, equation (2.2), and finally invertibility of  $u \otimes U$ . Notice that (the proof of) Lemma 2.5 indeed holds for the braiding rather than the half-braiding, and really only depends on invertibility of  $u \otimes U$  and the central equation 2.1.  $\square$

The rest of this section elaborates some examples.

**Lemma 2.7.** *In a cartesian category, an object is (the domain of) a central idempotent if and only if it is subterminal.*

*Proof.* Suppose the unique morphism  $u: U \rightarrow 1$  is a central idempotent. Let  $f, g: A \rightarrow U$ . Write  $a$  for the unique morphism  $A \rightarrow 1$ . Then  $A \times a: A \otimes A \rightarrow A$  has an inverse  $\Delta: A \rightarrow A \otimes A$ , and:

$$\begin{array}{c} U \\ | \\ \boxed{f} \\ | \\ A \end{array} = \begin{array}{c} U \\ | \\ \boxed{f} \quad \boxed{a} \\ | \quad | \\ A \quad A \\ \downarrow \Delta \\ A \end{array} = \begin{array}{c} U \\ | \\ \boxed{f} \quad \boxed{g} \\ | \quad | \\ A \quad A \\ \downarrow \Delta \\ A \end{array} = \begin{array}{c} U \\ | \\ \boxed{f} \quad \boxed{g} \\ | \quad | \\ A \quad A \\ \downarrow \Delta \\ A \end{array} = \begin{array}{c} U \\ | \\ \boxed{a} \quad \boxed{g} \\ | \quad | \\ A \quad A \\ \downarrow \Delta \\ A \end{array} = \begin{array}{c} U \\ | \\ \boxed{g} \\ | \\ A \end{array}$$

So  $u$  is monic. □

In particular, if  $X$  is a topological space, then central idempotents in its category of sheaves  $\text{Sh}(X)$  correspond to open subsets of  $X$  [10, Corollary 2.2.16]. Also, if  $M$  is a monoid, then central idempotents in the category  $[M^{\text{op}}, \mathbf{Set}]$  of  $M$ -actions correspond to right ideals of  $M$ .

**Example 2.8.** Any (meet-)semilattice may be regarded as a (strict) monoidal category: objects are elements of the semilattice, there is a unique morphism  $u \rightarrow v$  if and only if  $u \leq v$ , tensor products are given by greatest lower bounds, and the tensor unit is the greatest element. In this special case of Lemma 2.7, any object is a central idempotent. Write  $\mathbf{SLat}_{\leq}$  for the category of semilattices and monotone functions that preserve the greatest element. Write  $\mathbf{MonCat}$  for the category of monoidal categories and functors  $F$  that are (lax) monoidal and whose coherence morphism  $I \rightarrow F(I)$  is invertible (see also Definition 3.5 below). The former is a coreflective subcategory of the latter:

$$\mathbf{SLat}_{\leq} \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\perp} \\ \xleftarrow{\text{ZI}} \end{array} \mathbf{MonCat}$$

**Example 2.9.** A quantale [47] is a complete lattice  $Q$  with an element  $e \in Q$  and a multiplication  $Q \times Q \rightarrow Q$  such that:

$$u(\bigvee v_i) = \bigvee uv_i \quad (\bigvee u_i)v = \bigvee u_iv \quad eu = u = ue$$

Any quantale may be regarded as a monoidal category: objects are elements of the quantale, (composition of) morphisms is induced by the partial order, and the tensor product is induced by the multiplication. The central idempotents of this category are the elements  $q^2 = q \leq e$ . Taking as morphisms between quantales functions that preserve  $\bigvee$ ,  $\cdot$ , and  $e$ , this gives a functor  $\mathbf{Quantale} \rightarrow \mathbf{MonCat}$ .

For an important special case, recall that a *frame* is a complete lattice in which finite joins distribute over suprema [32]: a frame is a commutative quantale in which the multiplication is idempotent and whose unit is the largest element. Frames form a coreflective subcategory of commutative quantales [23, Proposition 3.5], where a morphism of frames is a function that preserves  $\bigvee$ ,  $\wedge$ , and 1:

$$\mathbf{Frame} \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\perp} \\ \xleftarrow{\text{ZI}} \end{array} \mathbf{cQuantale}$$

**Example 2.10.** Consider the monoidal category  $\mathbf{Mod}_R$  of *modules* over a commutative ring  $R$ . Centrality of  $a: A \rightarrow R$  means that:

$$(2.3) \quad a(x) \cdot y = a(y) \cdot x$$

for every  $x, y \in A$ . Idempotency means that:

$$(2.4) \quad \forall x \exists x_i, y_i: x = \sum_{i=1}^n a(x_i) \cdot y_i$$

$$(2.5) \quad a(x) \cdot y = 0 \iff x \otimes y = 0$$

If  $e \in R$  is a (central) idempotent element of  $R$ , then the (inclusion  $a$  into  $R$  of the) ideal  $A = eR$  is a central idempotent in  $\mathbf{Mod}_R$ . Conversely, (2.4) implies that  $a(A) \subseteq R$  is an idempotent ideal. In general we cannot say much more, but if  $A$  is faithfully flat [39, Section 4], then  $A \otimes a$  being injective implies that  $a: A \rightarrow R$  is injective, and hence central idempotents correspond to idempotent ideals. This includes free modules  $A$ , and hence vector spaces  $A$ .

**Example 2.11.** Let  $A$  be a *bialgebra* over a field  $k$ . Then the category  $\mathbf{Mod}_A$  of left  $A$ -modules is monoidal [45, Example 9.1.3]. Note that  $A$  need not be commutative, and  $\mathbf{Mod}_A$  is only braided when  $A$  is quasitriangular. If  $u: U \rightarrow A$  is a central idempotent in  $\mathbf{Mod}_A$ , then composition with the counit  $\varepsilon: A \rightarrow I$  makes  $\varepsilon \circ u: U \rightarrow I$  a central idempotent in the category of vector spaces over  $k$ . By Example 2.10 and [23, Proposition 3.11], this means that up to isomorphism  $U = 0$ , or  $U = I$  and  $\varepsilon \circ U = I$ . That  $u$  is  $A$ -linear means  $\varepsilon(a) \cdot u = a \cdot u$ . Thus central idempotents are in bijection with elements  $u \in U$  satisfying  $u \cdot u = u$  and  $a \cdot u = u \cdot a$  for all  $a \in A$ ; the former case is  $u = 0$ , the latter case has  $\varepsilon(u) = 1$ . Thus central idempotents in  $\mathbf{Mod}_A$  correspond to central idempotent elements of  $A$ , that is,  $\text{ZI}(\mathbf{Mod}_A) \simeq \{u \cdot u = u \in Z(A)\}$ .

**Example 2.12.** Consider the category  $\mathbf{Hilb}_{C_0(X)}$  of *Hilbert  $C^*$ -modules* [43] for a locally compact Hausdorff space  $X$ . This is not an abelian category [26, Appendix A], nor a topos. It is monoidally equivalent to the category of fields of Hilbert spaces over  $X$  [28, Corollary 4.9]. Under this equivalence the monoidal structure is fibrewise, so a central idempotent  $a: A \rightarrow C_0(X)$  in  $\mathbf{Hilb}_{C_0(X)}$  becomes a continuous functions that is fibrewise a central idempotent  $a_x: A_x \rightarrow \mathbb{C}$  in the category of Hilbert spaces and bounded linear maps. It follows from Example 2.10 that each  $a_x$  is injective. Finally, because composition of maps between fields of Hilbert spaces is fibrewise too, we conclude that the central idempotent  $a$  in  $\mathbf{Hilb}_{C_0(X)}$  is monic. It follows [23, Proposition 3.16] that  $\text{ZI}(\mathbf{Hilb}_{C_0(X)}) \simeq \{U \subseteq X \text{ open}\}$ .

**Example 2.13.** If  $\mathbf{C}$  is any category, the functor category  $[\mathbf{C}, \mathbf{C}]$  is monoidal with composition as tensor product. If  $\varepsilon: T \Rightarrow \mathbf{C}$  is a left idempotent satisfying (2.1), then the inverse  $\varepsilon T = T\varepsilon: T^2 \Rightarrow T$  is a comultiplication that makes  $T$  into an *idempotent comonad* on  $\mathbf{C}$ . Conversely, if  $T$  is an idempotent comonad on  $\mathbf{C}$ , then its counit is a left idempotent in  $[\mathbf{C}, \mathbf{C}]$  satisfying (2.1) by [9, Proposition 4.2.3]. (In fact, the counit is central if and only if the comonad is idempotent [17, Proposition 2.11, e $\Leftrightarrow$ f].) Thus idempotents in  $[\mathbf{C}, \mathbf{C}]$  satisfying (2.1) correspond (up to natural isomorphism) to idempotent comonads. Notice that the counit of an idempotent comonad need not be monic [41, Lemma 3.4]. See also [37].

An idempotent comonad  $\varepsilon: T \Rightarrow \mathbf{C}$  is in the centre of the monoidal category  $[\mathbf{C}, \mathbf{C}]$  when  $FT \simeq TF$  for all endofunctors  $F: \mathbf{C} \rightarrow \mathbf{C}$ , and  $\varepsilon$  respects these

natural isomorphisms. As in Lemma 2.5, it follows that:

$$\begin{array}{c} \cup \\ \diagdown \quad \diagup \\ T \quad F \quad T \\ \text{---} \end{array} \circ \begin{array}{c} \circ \\ | \\ T \end{array} = \begin{array}{c} \circ \\ | \\ T \end{array} \begin{array}{c} | \\ | \\ F \quad T \end{array}$$

Hence  $\varepsilon_{FT(A)} = FT(\varepsilon_A) \circ \sigma_{T(A)}^F$  is a composition of invertible maps and so an isomorphism for every  $A \in \mathbf{C}$  and  $F: \mathbf{C} \rightarrow \mathbf{C}$ . Taking  $F$  to be the functor that maps every object to  $A$  and every morphism to the identity on  $A$  shows that  $\varepsilon$  is a natural isomorphism. Hence  $\text{ZI}([\mathbf{C}, \mathbf{C}]) = 1$ .

For more examples, see [23, 3.7–3.8, 3.12–3.14].

### 3. UNIVERSAL JOINS

This section defines properties of central idempotents that make them more well-behaved, starting with the weakest one.

**Definition 3.1.** A monoidal category is *stiff* if

$$\begin{array}{ccc} A \otimes U \otimes V & \longrightarrow & A \otimes V \\ \downarrow \lrcorner & & \downarrow \\ A \otimes U & \longrightarrow & A \end{array} \quad \begin{array}{c} A \\ | \\ \circ \\ | \\ A \quad U \quad V \end{array} = \begin{array}{c} A \\ | \\ \circ \\ | \\ A \quad U \quad V \end{array}$$

is a pullback for all objects  $A$  and central idempotents  $u$  and  $v$ .

A monoidal category  $\mathbf{C}$  has *finite universal joins* of central idempotents when it has an initial object  $0$  satisfying  $A \otimes 0 \simeq 0$  for all objects  $A$ , and  $\text{ZI}(\mathbf{C})$  has binary joins such that

$$(3.1) \quad \begin{array}{ccc} A \otimes U \otimes V & \twoheadrightarrow & A \otimes V \\ \downarrow \lrcorner & & \downarrow \\ A \otimes U & \twoheadrightarrow & A \otimes (U \vee V) \end{array} \quad \begin{array}{c} A \quad U \vee V \\ | \quad \circ \quad \circ \\ | \quad | \quad | \\ A \quad U \quad V \end{array} = \begin{array}{c} A \quad U \vee V \\ | \quad \circ \quad \circ \\ | \quad | \quad | \\ A \quad U \quad V \end{array}$$

is both a pullback and pushout for all objects  $A$  and central idempotents  $u$  and  $v$ .

A monoidal category has *universal joins* of central idempotents when  $\text{ZI}(\mathbf{C})$  has all joins such that the cocone

$$\begin{array}{c} A \vee U_i \\ | \\ A \quad U_i \end{array}$$

is a colimit of the diagram with morphisms

$$\begin{array}{c} A \quad U_j \\ | \\ A \quad U_i \end{array}$$

if  $u_i \leq u_j$  for any set  $\{u_i\}$  of central idempotents satisfying  $\{u_i\} = \{u_i \wedge u_j\}$ .



If  $\mathbf{C}$  has universal finite joins of central idempotents, then  $\text{ZI}(\mathbf{C})$  is a distributive lattice with least element 0. If  $\mathbf{C}$  has universal joins of central idempotents, then  $\text{ZI}(\mathbf{C})$  is even a frame.

Any coherent category has universal finite joins of central idempotents [23, Proposition 7.4]. Any cocomplete Heyting category has universal joins of central idempotents [23, Proposition 7.12]; this includes all cocomplete toposes, such as Grothendieck toposes. Any quantale, regarded as a monoidal category as in Example 2.9, has universal joins of central idempotents. The category of Hilbert  $\mathbf{C}^*$ -modules of Example 2.12 has universal joins of central idempotents.

Next, we show that the properties of being equal, being monic, being epic, and being invertible, may be verified ‘central idempotent-wise’.

**Lemma 3.2.** *If (finitely many) central idempotents  $u_i$  in a monoidal category with universal (finite) joins of central idempotents satisfy  $\bigvee u_i = 1$ , then:*

- *morphisms  $f, g: A \rightarrow B$  are equal if and only if  $f \otimes u_i = g \otimes u_i$  for each  $i$ ;*
- *a morphism  $f$  is epic if  $f \otimes u_i$  or  $f \otimes U_i$  is epic for each  $i$ ;*
- *a morphism  $f$  is monic if  $f \otimes u_i$  is monic for each  $i$ ;*
- *a morphism  $f$  is invertible if and only if  $f \otimes U_i$  is invertible for each  $i$ .*

*Proof.* The point about equality of morphisms follows directly from Definition 3.1.

Next, let  $f: A \rightarrow B$  and suppose that  $g \circ f = h \circ f$  for  $g, h: B \rightarrow C$ . Then also  $(g \otimes u_i) \circ (f \otimes U_i) = (h \otimes u_i) \circ (f \otimes U_i)$ . Therefore  $g \otimes u_i = h \otimes u_i$  because  $f \otimes U_i$  is epic. But then  $g = h$  as before because  $\bigvee u_i = 1$ . If instead each  $f \otimes u_i$  is epic, then immediately  $g = h$ . In either case  $f$  is epic.

We turn to monomorphisms. Let  $f: B \rightarrow C$ . Suppose that  $f \circ g = f \circ h$  for  $g, h: A \rightarrow B$ . Because  $f \otimes u_i$  is epic, then:

$$\begin{array}{c} C \\ \boxed{f} \\ B \\ \boxed{g} \\ A \end{array} \circ \begin{array}{c} \\ \\ \\ U_i \end{array} = \begin{array}{c} C \\ \boxed{f} \\ B \\ \boxed{h} \\ A \end{array} \circ \begin{array}{c} \\ \\ \\ U_i \end{array} \implies \begin{array}{c} B \\ \boxed{g} \\ A \end{array} \begin{array}{c} U_i \\ | \\ U_i \end{array} = \begin{array}{c} B \\ \boxed{h} \\ A \end{array} \begin{array}{c} U_i \\ | \\ U_i \end{array} \implies \begin{array}{c} B \\ \boxed{g} \\ A \end{array} \circ \begin{array}{c} \\ \\ \\ U_i \end{array} = \begin{array}{c} B \\ \boxed{h} \\ A \end{array} \circ \begin{array}{c} \\ \\ \\ U_i \end{array}$$

So  $g = h$  because  $\bigvee u_i = 1$ . Thus  $f$  is monic.

Finally, suppose  $f: A \rightarrow B$  makes each  $f \otimes U_i: A \otimes U_i \rightarrow B \otimes U_i$  invertible. Write  $g_i: B \otimes U_i \rightarrow A \otimes U_i$  for the inverses. Now the morphisms  $g_i \otimes u_i: B \otimes U_i \rightarrow A$  form a cocone for the diagram  $B \otimes u_i: B \otimes U_i \rightarrow B$ , giving a mediating morphism  $g: B \rightarrow A$ . It follows from uniqueness that  $g \circ f = B$  and  $f \circ g = A$ , making  $f$  invertible. Conversely, if  $f$  has inverse  $g$ , then  $f \otimes U_i$  has inverse  $g \otimes U_i$ .  $\square$

In Section 12 below we will see that any stiff monoidal category can be freely completed with universal (finite) joins of central idempotents. We end this section by considering which functors between monoidal categories preserve central idempotents.

**Definition 3.3.** A lax monoidal functor  $F: \mathbf{C} \rightarrow \mathbf{D}$  with coherence morphisms  $\theta_{A,B}: F(A) \otimes F(B) \rightarrow F(A \otimes B)$  and  $\theta_I: I \rightarrow F(I)$  preserves a half-braiding  $\sigma_A^U: U \otimes A \rightarrow A \otimes U$  in  $\mathbf{C}$  when there is a chosen half-braiding  $F(\sigma^U)_D: F(U) \otimes D \rightarrow D \otimes F(U)$  in  $\mathbf{D}$  satisfying  $\theta_{A,U} \circ F(\sigma^U)_{F(A)} = F(\sigma_A^U) \circ \theta_{U,A}$ .

Any braided lax monoidal functor preserves all half-braidings by Lemma 2.6. Conversely, if a lax monoidal functor  $F: \mathbf{C} \rightarrow \mathbf{D}$  preserves all half-braidings, then it induces a functor from the centre of  $\mathbf{C}$  to the centre of  $\mathbf{D}$  (but this functor need not be lax monoidal without further assumptions).

**Lemma 3.4.** *Let  $F: \mathbf{C} \rightarrow \mathbf{D}$  be an lax monoidal functor with coherence morphisms  $\theta_{A,B}: F(A) \otimes F(B) \rightarrow F(A \otimes B)$  and  $\theta_I: I \rightarrow F(I)$ . Suppose that  $F$  preserves half-braidings of central idempotents, and that  $\theta_I$  and  $\theta_{A,U}$  are invertible for all objects  $A$  and  $u \in \text{ZI}(\mathbf{C})$ . If  $u: U \rightarrow I$  is a central idempotent in  $\mathbf{C}$ , then*

$$F(U) \xrightarrow{F(u)} F(I) \xrightarrow{\theta_I^{-1}} I$$

*is a central idempotent in  $\mathbf{D}$ . This induces a semilattice morphism  $\text{ZI}(\mathbf{C}) \rightarrow \text{ZI}(\mathbf{D})$  if additionally:*

$$\begin{aligned} F(\sigma^I)_D &= (D \otimes \theta_I) \circ \rho_D^{-1} \circ \lambda_D \circ (\theta_I^{-1} \otimes D) \\ F(\sigma^{U \otimes V})_D &= (D \otimes \theta_{U,V}) \circ (F(\sigma_U)_D \otimes F(V)) \circ (F(U) \otimes F(\sigma^V)_D) \circ (\theta_{U,V}^{-1} \otimes D) \end{aligned}$$

*Proof.* The central equation (2.1) holds:

$$\begin{array}{ccccc} F(U) \otimes F(U) & \xrightarrow{F(u) \otimes F(U)} & F(I) \otimes F(U) & \xrightarrow{\theta_I^{-1} \otimes F(U)} & I \otimes F(U) \\ \theta_{U,U} \downarrow & & \downarrow \theta_{I,U} & & \downarrow \lambda_{F(U)} \\ F(U \otimes U) & \xrightarrow{F(u \otimes U)} & F(I \otimes U) & \xrightarrow{F(\lambda)} & F(U) \\ F(\sigma_U) \downarrow & & \downarrow F(\sigma_I) & & \parallel \\ F(U \otimes U) & \xrightarrow{F(U \otimes u)} & F(U \otimes I) & \xrightarrow{F(\rho)} & F(U) \\ \theta_{U,U} \uparrow & & \uparrow \theta_{U,I} & & \uparrow \rho_{F(U)} \\ F(U) \otimes F(U) & \xrightarrow{F(U) \otimes F(u)} & F(U) \otimes F(I) & \xrightarrow{F(U) \otimes \theta_I^{-1}} & F(U) \otimes I \end{array}$$

To see that (2.1) is invertible:

$$\begin{array}{ccccc} & & F(U \otimes U) & \xrightarrow{\theta_{U,U}^{-1}} & F(U) \otimes F(U) \\ & \nearrow^{F((U \otimes u)^{-1})} & \downarrow F(U \otimes u) & & \downarrow F(U) \otimes F(u) \\ F(U) & & F(U \otimes I) & \xrightarrow{\theta_{U,I}^{-1}} & F(U) \otimes F(I) \\ & \searrow & \downarrow F(\rho_U) & & \downarrow F(U) \otimes \theta_I^{-1} \\ & & F(U) & \xleftarrow{\rho_{F(U)}} & F(U) \otimes I \end{array}$$

By assumption  $F(\sigma)$  extends to a half-braiding.

That  $F(u \wedge v) = F(u) \wedge F(v)$  follows because  $\theta_{U,V}$  is invertible and from the condition on  $F(\sigma^{U \otimes V})$ . Finally, top element of the semilattice is preserved by the condition on  $F(\sigma^I)$ .  $\square$

We say that a lax monoidal functor *preserves central idempotents* when it satisfies the conditions of the previous lemma. Braided strong monoidal functors between braided monoidal categories do so automatically.

**Definition 3.5.** Write  $\mathbf{MonCat}$  for the category of (small) monoidal categories and lax monoidal functors  $F$  whose coherence morphism  $I \rightarrow F(I)$  is invertible. Write  $\mathbf{MonCat}_s$  for the subcategory of stiff monoidal categories and lax monoidal functors  $F$  that preserve central idempotents. Write  $\mathbf{MonCat}_{\mathfrak{f}}$  for the further subcategory of monoidal categories with universal finite joins of central idempotents and functors that preserve finite joins of central idempotents, and  $\mathbf{MonCat}_j$  for the subcategory of monoidal categories with universal joins of central idempotents and functors that preserve joins of central idempotents.

Example 2.8 above also gives a coreflection between  $\mathbf{MonCat}_s$  and  $\mathbf{SLat}$ , the category of semilattices and semilattice morphisms, that is, functions that preserve meets and the greatest element.

#### 4. BASE SPACE

We will build the representation for the case of universal finite joins first, starting with the base space in this section, followed by the presheaf, sheaf, and stalks in following sections. In Section 8 below this will be upgraded to universal joins. Therefore we first consider distributive lattices. As base space, we will take the prime spectrum of the distributive lattice of central idempotents [20, 32]. Recall that a *prime filter* of a lattice  $L$  is a nonempty upward-closed and downward-directed subset  $P$  where  $u \vee v \in P$  implies  $u \in P$  or  $v \in P$ ; equivalently,  $P$  is the inverse image of 1 under a lattice homomorphism  $L \rightarrow \{0, 1\}$ .

**Definition 4.1.** Let  $L$  be a distributive lattice with a least element. Its *prime spectrum* is a topological space  $\text{Spec}(L)$ , whose points are prime filters  $P \subseteq L$ , and whose topology is generated by a basis consisting of the sets

$$B_u = \{P \in \text{Spec}(L) \mid u \in P\}$$

where  $u$  ranges over  $L$ .

**Lemma 4.2.** *The basic opens  $B_u$  of the prime spectrum of a lattice  $L$  are compact.*

*Proof.* The frame  $\mathcal{O}(\text{Spec}(L))$  of opens of  $\text{Spec}(L)$  is in bijection with the frame  $\text{Idl}(L)$  of ideals of  $L$ . The map that sends an element  $v \in L$  to its principal ideal  $\downarrow v = \{u \in L \mid u \leq v\}$  embeds  $L$  in  $\text{Idl}(L)$  as a lattice. On the topological side, it sends  $u \in L$  to  $B_u$ , establishing an isomorphism between  $L$  and the basis  $\{B_u \mid u \in L\} \subseteq \mathcal{O}(\text{Spec}(L))$ . Because the principal ideals are compact in  $\text{Idl}(L)$ , so are  $B_u$  in  $\mathcal{O}(\text{Spec}(L))$ .  $\square$

The following lemma holds more generally for sheaves valued in any category  $\mathbf{V}$ , but we spell it out for  $\mathbf{MonCat}$ -valued sheaves.

**Lemma 4.3.** *To specify a sheaf  $F: \mathcal{O}(\text{Spec}(L))^{\text{op}} \rightarrow \mathbf{MonCat}$  for a distributive lattice  $L$ , it suffices to give a presheaf  $F: \{B_u \mid u \in L\}^{\text{op}} \rightarrow \mathbf{MonCat}$ , such that  $F(B_0)$  is terminal in  $\mathbf{MonCat}$  and the following is an equaliser in  $\mathbf{MonCat}$ :*

$$F(B_{u \vee v}) \xrightarrow{\langle F(B_u \subseteq B_{u \vee v}), F(B_v \subseteq B_{u \vee v}) \rangle} F(B_u) \times F(B_v) \xrightarrow{\begin{array}{c} F(B_{u \wedge v} \subseteq B_u) \circ \pi_1 \\ \phantom{F(B_{u \wedge v} \subseteq B_u) \circ \pi_1} \\ F(B_{u \wedge v} \subseteq B_v) \circ \pi_2 \end{array}} F(B_{u \wedge v})$$

*Proof.* A sheaf on a basis uniquely determines a sheaf on the whole topological space [44, Theorem II.1.3]. The presheaf  $F: \{B_u \mid u \in L\}^{\text{op}} \rightarrow \mathbf{MonCat}$  provides this data. It furthermore needs to satisfy the sheaf condition on the covers of basic opens by basic opens,  $B_v = \bigcup_{u \in J} B_u$  with  $J \subseteq L$ . Since the basic open  $B_v$  is compact by Lemma 4.2, it suffices to consider finitary and hence binary covers,  $B_{u \vee v} = B_u \cup B_v$  for  $u, v \in L$ , and nullary covers,  $B_0 = \bigvee \emptyset$ . The sheaf condition for covers of this form says exactly that the functor in the statement is an equaliser, and  $F(B_0)$  is isomorphic to the terminal monoidal category  $\mathbf{1}$ .  $\square$

#### 5. STRUCTURE PRESHEAF

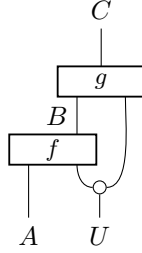
In this section we define the structure presheaf of a monoidal category. For a mere monoidal category this is a presheaf on the semilattice of central idempotents, which in the presence of universal finite joins extends to a presheaf on the topological space of the previous section. So we define a presheaf of monoidal categories on the central

idempotents by describing its behaviour over basic open sets. We will generalise the filter-quotient construction from topos theory to monoidal categories [44, V.9].

Any central idempotent  $U$  in a monoidal category  $\mathbf{C}$  is a (commutative) comonoid with comultiplication  $\smile$  and counit  $\circ$ . Hence  $- \otimes U$  is a comonad on  $\mathbf{C}$ . We are interested in its co-Kleisli category.

**Definition 5.1.** Let  $\mathbf{C}$  be a monoidal category, and  $u: U \rightarrow I$  a chosen representative of a central idempotent. Define a category  $\mathbf{C}\|_u$  as follows:

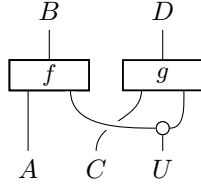
- objects are those of  $\mathbf{C}$ ;
- morphisms  $A \rightarrow B$  in  $\mathbf{C}\|_u$  are morphisms  $A \otimes U \rightarrow B$  in  $\mathbf{C}$ ;
- composition of  $f: A \rightarrow B$  and  $g: B \rightarrow C$  is:



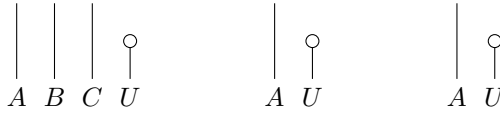
- identity on  $A$  is  $\smile$ .

**Lemma 5.2.** If  $\mathbf{C}$  is a monoidal category, then so is  $\mathbf{C}\|_u$ .

*Proof.* The monoidal structure of  $\mathbf{C}\|_u$  is defined as follows. The tensor product of morphisms  $f: A \rightarrow B$  and  $g: C \rightarrow D$  is:



The associator and left and right unitors are given by:



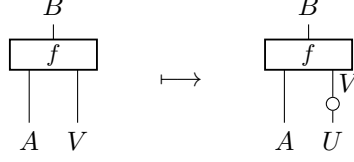
Easy graphical manipulation establishes naturality, the interchange law, as well as the triangle and pentagon equations.  $\square$

**Lemma 5.3.** If  $\mathbf{C}$  is cartesian, there is a monoidal equivalence  $\mathbf{C}\|_u \simeq \mathbf{C}/U$ .

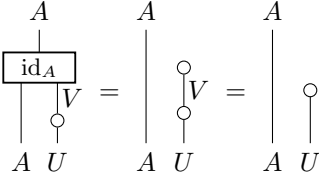
*Proof.* First, notice that  $A \otimes U: A \rightarrow A \otimes U$  is an isomorphism  $A \simeq A \otimes U$  in  $\mathbf{C}\|_u$  with inverse  $A \otimes u \otimes u: A \otimes U \rightarrow A$ . Therefore,  $\mathbf{C}\|_u$  is equivalent to its full subcategory  $\mathbf{C}\|_u$  of objects of the form  $A \otimes U$ . Moreover, this equivalence preserves tensor products on the nose.

There is a functor  $\mathbf{C}\|_u \rightarrow \mathbf{C}/U$  that sends an object  $A \otimes U$  to  $\pi_2: A \otimes U \rightarrow U$ , and that sends a morphism  $f: A \otimes U \otimes U \rightarrow B \otimes U$  to  $f \circ (A \otimes U \otimes u)^{-1}$ . It is monoidal. There is also a functor  $\mathbf{C}/U \rightarrow \mathbf{C}\|_u$  that sends an object  $!: A \rightarrow U$  to  $A \otimes U$ , and that sends a morphism  $f: A \rightarrow B$  to  $f \otimes U \otimes u$ . This functor is monoidal too. These two functors form an equivalence, because  $A \otimes U \otimes U \simeq A \otimes U$  in  $\mathbf{C}\|_u$ .  $\square$

**Lemma 5.4.** *If  $u \leq v$  are central idempotents in a monoidal category  $\mathbf{C}$ , there is a strict monoidal functor  $\mathbf{C}\|_{u \leq v}: \mathbf{C}\|_v \rightarrow \mathbf{C}\|_u$  that acts as identity on objects and on morphisms as:*



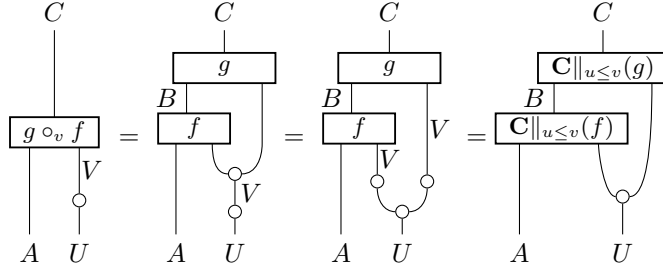
*Proof.* We first show functoriality. To verify that  $\mathbf{C}\|_{u \leq v}$  preserves identity morphisms:



The fact that

$$\mathbf{C}\|_{u \leq v}(g \circ_v f) = \mathbf{C}\|_{u \leq v}(g) \circ_u \mathbf{C}\|_{u \leq v}(f)$$

is proven by:

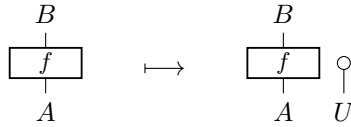


As for monoidality, for  $f: A \otimes V \rightarrow B$  and  $g: C \otimes V \rightarrow D$  in  $\mathbf{C}$ :

$$\mathbf{C}\|_{u \leq v}(f \otimes g) = \begin{array}{c} B \ D \\ \boxed{f \otimes g} \\ \text{---} \\ A \ C \ U \end{array} = \begin{array}{c} B \ D \\ \boxed{f} \quad \boxed{g} \\ \text{---} \\ A \ C \ U \end{array} = \begin{array}{c} B \ D \\ \boxed{f} \quad \boxed{g} \\ \text{---} \\ A \ C \ U \end{array} = \mathbf{C}\|_{u \leq v}(f) \otimes \mathbf{C}\|_{u \leq v}(g)$$

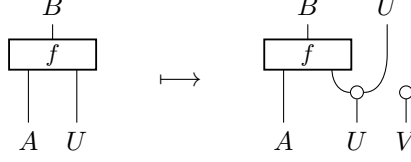
This functor preserves the tensor unit, and hence is strict monoidal.  $\square$

The special case  $v = 1$  gives a functor  $\mathbf{C} \rightarrow \mathbf{C}\|_u$  that acts as the identity on objects and acts on morphisms as:



**Lemma 5.5.** *If  $u \leq v$  are central idempotents in a monoidal category  $\mathbf{C}$ , there is an oplax monoidal functor  $\mathbf{C}\|^{u \leq v}: \mathbf{C}\|_u \rightarrow \mathbf{C}\|_v$ , that acts on objects as  $A \mapsto A \otimes U$ ,*

and on morphisms as:



This functor is left adjoint to  $\mathbf{C}\|_{u \leq v}$ , and the unit of this adjunction is invertible.

*Proof.* It is straightforward to verify that  $\mathbf{C}\|^{u \leq v}$  indeed preserves identities and composition. It is oplax monoidal with  $\theta_I = I \otimes u \otimes v: \mathbf{C}\|_{u \leq v}(I) \rightarrow I$  and:

$$\theta_{A,B} = \left| \begin{array}{c} A \quad U \quad B \quad U \\ \text{---} \quad \text{---} \quad \text{---} \quad \text{---} \\ \text{---} \quad \text{---} \quad \text{---} \quad \text{---} \\ \text{---} \quad \text{---} \quad \text{---} \quad \text{---} \\ A \quad B \quad U \quad V \end{array} \right| : \mathbf{C}\|^{u \leq v}(A \otimes B) \rightarrow \mathbf{C}\|^{u \leq v}(A) \otimes \mathbf{C}\|^{u \leq v}(B)$$

It is routine to verify that  $\theta_{A,B}$  and  $\theta_I$  are natural and respect the coherence isomorphisms of  $\mathbf{C}\|_u$  and  $\mathbf{C}\|_v$ . Notice that in fact  $\theta_{A,B}$  is invertible, with inverse  $\theta_{A,B}^{-1} = A \otimes u \otimes B \otimes U \otimes v: \mathbf{C}\|^{u \leq v}(A) \otimes \mathbf{C}\|^{u \leq v}(B) \rightarrow \mathbf{C}\|^{u \leq v}(A \otimes B)$ , but that  $\theta_I$  is not invertible.

The fact that  $\mathbf{C}\|^{u \leq v}$  is left adjoint to  $\mathbf{C}\|_{u \leq v}$  simply means:

$$\mathbf{C}\|_v(A \otimes U, B) = \mathbf{C}(A \otimes U \otimes V, B) \simeq \mathbf{C}(A \otimes U, B) = \mathbf{C}\|_v(A \otimes U, B)$$

This holds because  $u \wedge v = u$  as  $u \leq v$ , and is easily seen to be natural in  $A$  and  $B$ . The unit of the adjunction is  $\eta_A = A \otimes U: A \rightarrow A \otimes U$  in  $\mathbf{C}\|_u$ . It is inverted by  $\eta_A^{-1} = A \otimes u \otimes u: A \otimes U \rightarrow A$  in  $\mathbf{C}\|_u$ .  $\square$

To finish this section on the structure presheaf, we determine the central idempotents of the categories  $\mathbf{C}\|_u$ .

**Lemma 5.6.** *If  $u$  is a central idempotent in a monoidal category  $\mathbf{C}$ , then there is an isomorphism  $\text{ZI}(\mathbf{C}\|_u) \simeq \text{ZI}(\mathbf{C}) \cap \downarrow u$  of semilattices.*

*Proof.* Let  $q: Q \rightarrow I$  be a morphism in  $\mathbf{C}\|_u$ . That is,  $q$  is a morphism  $Q \otimes U \rightarrow I$  in  $\mathbf{C}$ . By definition  $q$  satisfies (2.1) in  $\mathbf{C}\|_u$  if and only if the following holds in  $\mathbf{C}$ :

$$\left| \begin{array}{c} Q \\ \text{---} \\ \text{---} \\ \text{---} \\ Q \quad Q \quad U \end{array} \right| = \left| \begin{array}{c} Q \\ \text{---} \\ \text{---} \\ \text{---} \\ Q \quad Q \quad U \end{array} \right|$$

But by Lemma 2.5, this means precisely that  $q: Q \otimes U \rightarrow I$  satisfies (2.1) in  $\mathbf{C}$ .

Now,  $q$  is idempotent in  $\mathbf{C}\|_u$  if and only if there is a morphism  $f: Q \otimes U \rightarrow Q \otimes Q$  in  $\mathbf{C}$  satisfying:

$$\left| \begin{array}{c} Q \quad Q \\ \text{---} \\ \text{---} \\ \text{---} \\ Q \quad Q \quad U \end{array} \right| = \left| \begin{array}{c} Q \quad Q \\ \text{---} \\ \text{---} \\ \text{---} \\ Q \quad Q \quad U \end{array} \right| \quad \left| \begin{array}{c} Q \\ \text{---} \\ \text{---} \\ \text{---} \\ Q \quad U \end{array} \right| = \left| \begin{array}{c} Q \\ \text{---} \\ \text{---} \\ \text{---} \\ Q \quad U \end{array} \right|$$

Similarly,  $q$  is idempotent in  $\mathbf{C}$  if and only if there is  $g: Q \otimes U \rightarrow Q \otimes U \otimes Q \otimes U$  in  $\mathbf{C}$  satisfying:

But these two properties are equivalent via:

Finally, if  $\tau_A: Q \otimes A \rightarrow A \otimes Q$  is a half-braiding in  $\mathbf{C}\|_u$ , and  $\sigma_A: U \otimes A \rightarrow A \otimes U$  is the given half-braiding in  $\mathbf{C}$ , then

$$Q \otimes U \otimes A \xrightarrow{Q \otimes \sigma_A} Q \otimes A \otimes U \simeq Q \otimes A \otimes U \otimes U \xrightarrow{\tau_A \otimes U} A \otimes Q \otimes U$$

is a half-braiding in  $\mathbf{C}\|_u$ . Thus any central idempotent  $V$  in  $\mathbf{C}$  induces a central idempotent  $V$  in  $\mathbf{C}\|_u$ , and any central idempotent  $q$  in  $\mathbf{C}\|_u$  is represented by a central idempotent coming from  $\mathbf{C}$  in this way.  $\square$

**Corollary 5.7.** *Let  $\mathbf{C}$  be a monoidal monoidal category and  $u \in \text{ZI}(\mathbf{C})$ .*

- *If  $\mathbf{C}$  is stiff, then so is  $\mathbf{C}\|_u$ .*
- *If  $\mathbf{C}$  has finite joins of central idempotents, then so does  $\mathbf{C}\|_u$ .*
- *If  $\mathbf{C}$  has joins of central idempotents, then so does  $\mathbf{C}\|_u$ .*

*The functor  $\mathbf{C} \rightarrow \mathbf{C}\|_u$  preserves joins of central idempotents.*

*Proof.* Follows from Definition 3.1 and Lemma 5.6.  $\square$

## 6. STRUCTURE SHEAF

This section establishes that the structure presheaf is in fact a sheaf. We start with checking the sheaf condition for binary joins.

**Proposition 6.1.** *If  $\mathbf{C}$  is a monoidal category with universal finite joins of central idempotents, then the following is an equaliser in  $\mathbf{MonCat}$ :*

$$\mathbf{C}\|_{u \vee v} \xrightarrow{\langle \mathbf{C}\|_{u \leq u \vee v}, \mathbf{C}\|_{v \leq u \vee v} \rangle} \mathbf{C}\|_u \times \mathbf{C}\|_v \xrightarrow[\mathbf{C}\|_{u \wedge v \leq v} \circ \pi_2]{\mathbf{C}\|_{u \wedge v \leq u} \circ \pi_1} \mathbf{C}\|_{u \wedge v}$$

*Proof.* We will prove that the following diagram is a pullback in  $\mathbf{MonCat}$ :

$$\begin{array}{ccc} \mathbf{C}\|_{u \vee v} & \xrightarrow{\mathbf{C}\|_{u \leq u \vee v}} & \mathbf{C}\|_u \\ \mathbf{C}\|_{v \leq u \wedge v} \downarrow & & \downarrow \mathbf{C}\|_{u \wedge v \leq u} \\ \mathbf{C}\|_v & \xrightarrow{\mathbf{C}\|_{u \wedge v \leq v}} & \mathbf{C}\|_{u \wedge v} \end{array}$$

Let  $\mathbf{A}$  be a (monoidal) category and  $F: \mathbf{A} \rightarrow \mathbf{C}\|_u$  and  $G: \mathbf{A} \rightarrow \mathbf{C}\|_v$  (lax monoidal) functors satisfying:

$$(6.1) \quad \mathbf{C}\|_{u \wedge v \leq u} \circ F = \mathbf{C}\|_{u \wedge v \leq v} \circ G$$

We will show that there is a unique functor  $H: \mathbf{A} \rightarrow \mathbf{C}\|_{u \vee v}$  satisfying:

$$(6.2) \quad \mathbf{C}\|_{u \leq u \vee v} \circ H = F \quad \mathbf{C}\|_{v \leq u \vee v} \circ H = G$$

by first showing that (6.2) forces a unique choice for how  $H$  must act on objects and morphisms, and then verifying that this indeed defines a (lax monoidal) functor.

For an object  $A$  of  $\mathbf{A}$ , condition (6.1) implies that  $F(A) = G(A)$  since the restriction functors act as the identity on objects, and (6.2) forces  $H(A)$  to be the same object. For a morphism  $m: A \rightarrow B$  in  $\mathbf{A}$ , equation (6.1) says that the maps  $F(m): F(A) \otimes U \rightarrow F(B)$  and  $G(m): G(A) \otimes V \rightarrow G(B)$  in  $\mathbf{C}$  satisfy:

$$\begin{array}{c} H(B) \\ | \\ \boxed{F(m)} \\ | \quad | \\ H(A) \quad U \end{array} \circ \begin{array}{c} \\ \\ \\ \\ V \end{array} = \begin{array}{c} H(B) \\ | \\ \boxed{G(m)} \\ | \quad | \\ H(A) \quad U \end{array} \begin{array}{c} \\ \circ \\ \\ \\ V \end{array}$$

whereas (6.2) says that  $H(m): H(A) \otimes (U \vee V) \rightarrow H(B)$  must satisfy:

$$(6.3) \quad \begin{array}{c} H(B) \\ | \\ \boxed{H(m)} \\ | \quad | \\ H(A) \quad U \vee V \end{array} = \begin{array}{c} H(B) \\ | \\ \boxed{F(m)} \\ | \quad | \\ H(A) \quad U \end{array} \begin{array}{c} \\ \circ \\ \\ \\ V \end{array} \quad \begin{array}{c} H(B) \\ | \\ \boxed{H(m)} \\ | \quad | \\ H(A) \quad U \vee V \end{array} = \begin{array}{c} H(B) \\ | \\ \boxed{G(m)} \\ | \quad | \\ H(A) \quad V \end{array}$$

Both conditions are summarised by commutativity of the following diagram in  $\mathbf{C}$ :

$$\begin{array}{ccc} H(A) \otimes U \otimes V & \longrightarrow & H(A) \otimes U \\ \downarrow & & \downarrow \\ H(A) \otimes V & \longrightarrow & H(A) \otimes (U \vee V) \\ & \searrow & \downarrow \\ & & H(B) \end{array} \begin{array}{c} \\ \\ \\ \text{---} F(m) \\ \text{---} G(m) \end{array}$$

Universal finite joins of central idempotents make this square a pushout. Thus there is only one possible choice for the morphism  $H(m)$ .

It remains to show that this indeed defines a (lax monoidal) functor. Let  $m_1: A \rightarrow B$  and  $m_2: B \rightarrow C$  in  $\mathbf{A}$ . We will show that  $H(m_2) \circ_{u \vee v} H(m_1)$  satisfies the defining conditions of  $H(m_2 \circ m_1)$ . Observe that

$$\begin{aligned} \mathbf{C}\|_{u \leq u \vee v}(H(m_2) \circ_{u \vee v} H(m_1)) &= \mathbf{C}\|_{u \leq u \vee v}(H(m_2)) \circ_u \mathbf{C}\|_{u \leq u \vee v}(H(m_1)) \\ &= F(m_2) \circ_u F(m_1) \\ &= F(m_2 \circ m_1) \end{aligned}$$

by functoriality of restriction, (6.3) instantiated for both  $m_1$  and  $m_2$ , and functoriality of  $F$ . The condition holds analogously for  $G$ . Since  $H(m_2 \circ m_1)$  is the unique map in  $\mathbf{C}$  satisfying these conditions,  $H(m_2) \circ_{u \vee v} H(m_1) = H(m_2 \circ m_1)$ .

Identities are preserved similarly: if  $A$  is an object of  $\mathbf{A}$ , then

$$\mathbf{C}\|_{u \leq u \vee v}(\text{id}_{H(A)}) = \text{id}_{\mathbf{C}\|_{u \leq u \vee v}(H(A))} = \text{id}_{F(A)} = F(\text{id}_A)$$



and analogously  $\mathbf{C}\|_{v \leq u \vee v}(\text{id}_{H(A)}) = G(\text{id}_A)$ , so  $\text{id}_{H(A)} = H(\text{id}_A)$ .

Finally we show that  $H$  is lax monoidal. For objects  $A$  and  $B$  of  $\mathbf{A}$ , write  $\theta_{A,B}^F$  for the structure morphism  $F(A) \otimes F(B) \rightarrow F(A \otimes_{\mathbf{A}} B)$  in  $\mathbf{C}\|_u$  witnessing that  $F$  is lax monoidal. This is a morphism  $\theta_{A,B}^F: F(A) \otimes F(B) \otimes U \rightarrow F(A \otimes B)$  in  $\mathbf{C}$ . Similarly, write  $\theta_{A,B}^G: G(A) \otimes G(B) \otimes V \rightarrow G(A \otimes B)$  in  $\mathbf{C}$ . It follows from (6.1) that:

$$\begin{array}{c} H(A \otimes B) \\ \boxed{\theta_{A,B}^F} \\ | \quad | \quad | \quad | \\ HA \quad HB \quad U \quad V \end{array} \circ = \begin{array}{c} H(A \otimes B) \\ \boxed{\theta_{A,B}^G} \\ | \quad | \quad | \quad | \\ HA \quad HB \quad U \quad V \end{array}$$

That is, the outer square in the following diagram commutes:

$$\begin{array}{ccc} H(A) \otimes H(B) \otimes U \otimes V & \longrightarrow & H(A) \otimes H(B) \otimes U \\ \downarrow & & \downarrow \\ H(A) \otimes H(B) \otimes V & \longrightarrow & H(A) \otimes H(B) \otimes (U \vee V) \\ & \searrow \theta_{A,B}^G & \downarrow \theta_{A,B}^F \\ & & H(A \otimes B) \end{array}$$

(A dashed arrow  $\theta_{A,B}^H$  connects  $H(A) \otimes H(B) \otimes (U \vee V)$  to  $H(A \otimes B)$ .)

This uniquely determines the dashed morphism  $\theta_{A,B}^H$  satisfying:

$$\begin{array}{ccc} \begin{array}{c} H(A \otimes B) \\ \boxed{\theta_{A,B}^F} \\ | \quad | \quad | \\ HA \quad HB \quad U \end{array} = \begin{array}{c} H(A \otimes B) \\ \boxed{\theta_{A,B}^H} \\ | \quad | \quad | \\ HA \quad HB \quad S \end{array} \begin{array}{c} U \vee V \\ \circ \\ \downarrow \end{array} & \begin{array}{c} H(A \otimes B) \\ \boxed{\theta_{A,B}^F} \\ | \quad | \quad | \\ HA \quad HB \quad V \end{array} = \begin{array}{c} H(A \otimes B) \\ \boxed{\theta_{A,B}^H} \\ | \quad | \quad | \\ HA \quad HB \quad V \end{array} \begin{array}{c} U \vee V \\ \circ \\ \downarrow \end{array} \end{array}$$

The structure morphism  $I \rightarrow H(I)$  is defined similarly. Naturality of  $\theta^H$  and the laws of unitality and associativity follow from those for the structure morphisms of  $F$  and  $G$ .  $\square$

We have to pay careful attention to the nullary case of the sheaf condition.

**Lemma 6.2.** *If  $\mathbf{C}$  be a monoidal category with universal finite joins of central idempotents, then  $\mathbf{C}\|_0$  is monoidally equivalent to the terminal category  $\mathbf{1}$ .*

*Proof.* It suffices to show that every object  $A$  is isomorphic to  $0$  in  $\mathbf{C}\|_0$ . Because  $A \otimes 0 \simeq 0$  by universal finite joins of central idempotents, there is a unique morphism  $f: A \otimes 0 \rightarrow 0$  in  $\mathbf{C}$ . Similarly, there is a unique morphism  $g: 0 \otimes 0 \rightarrow A$ . The composition  $g \circ f$  in  $\mathbf{C}\|_0$  is a morphism  $A \otimes 0 \rightarrow A$  in  $\mathbf{C}$ , and therefore unique, so has to equal the identity in  $\mathbf{C}\|_0$ . Similarly, the composition  $f \circ g$  in  $\mathbf{C}\|_0$  is a morphism  $0 \otimes 0 \rightarrow 0$  in  $\mathbf{C}$  and so equals the identity in  $\mathbf{C}\|_0$  by uniqueness.  $\square$

We can now define the desired sheaf of categories. For clarity we have avoided stacks and worked everything out concretely so far, but at this point we have to make a small change because of the previous lemma, as  $\mathbf{C}\|_0$  is not isomorphic to the terminal category but only equivalent to it. In logical terms,  $\mathbf{C}\|_0$  models the theory in which  $0 = 1$ .

**Proposition 6.3.** *Let  $\mathbf{C}$  be a monoidal category with universal finite joins of central idempotents. The functor  $\mathcal{O}(\text{Spec}(\text{ZI}(\mathbf{C})))^{\text{op}} \rightarrow \mathbf{MonCat}$  given by  $s \mapsto \mathbf{C}\|_s$  is naturally monoidally equivalent to a sheaf of monoidal categories defined as follows:*

$$\begin{aligned} F: \mathcal{O}(\text{Spec}(\text{ZI}(\mathbf{C})))^{\text{op}} &\rightarrow \mathbf{MonCat} \\ 0 &\mapsto \mathbf{1} \\ 0 \neq u &\mapsto \mathbf{C}\|_u \end{aligned}$$

*Proof.* Because  $\mathbf{1}$  is terminal,  $F$  is indeed functorial. There is a natural transformation from  $u \mapsto \mathbf{C}\|_u$  to  $F$ , whose component at every  $u \neq 0$  is the identity functor, and whose component at  $0$  is the unique functor to  $\mathbf{1}$ . This natural transformation is a (monoidal) equivalence by Lemma 6.2. Combining Lemma 4.3 with Proposition 6.1 shows that  $F$  defines a sheaf of (monoidal) categories.  $\square$

## 7. STALKS

In this section, we study the stalks of the sheaf of Proposition 6.3. We start by generalising the idea of germs to the monoidal setting.

**Definition 7.1.** Let  $\mathbf{C}$  be a monoidal category, and  $x \subseteq \text{ZI}(\mathbf{C})$  a prime filter. Define a category  $\mathbf{C}\|_x$  as follows:

- objects are those of  $\mathbf{C}$ ;
- morphisms  $A \rightarrow B$  are equivalence classes of pairs of  $v \in x$  and  $f: A \otimes V \rightarrow B$ , where we identify  $(v, f)$  and  $(v', f')$  when  $u \leq v \wedge v'$  for some  $u \in x$ :

$$\begin{array}{c} B \\ | \\ \boxed{f} \\ | \quad | \\ A \quad U \quad V \end{array} = \begin{array}{c} B \\ | \\ \boxed{f'} \\ | \quad | \\ A \quad U \quad V' \end{array}$$

- composition of  $(u, f): A \rightarrow B$  and  $(v, g): B \rightarrow C$  is:

$$\left[ u \wedge v, \begin{array}{c} C \\ | \\ \boxed{g} \\ | \\ \boxed{f} \\ | \quad | \\ A \quad U \quad V \end{array} \right]$$

- identity on  $A$  is  $[1, \lfloor_A]$ .

Notice that  $[u, f] = [u \wedge v, f \otimes v]$  for any central idempotent  $v$ .

**Lemma 7.2.** *If  $\mathbf{C}$  is a monoidal category, and  $x \subseteq \text{ZI}(\mathbf{C})$  a prime filter, then  $\mathbf{C}\|_x$  is monoidal.*

*Proof.* The tensor product of objects is as in  $\mathbf{C}$ , the tensor product of morphisms  $[u, f]: A \rightarrow B$  and  $[v, g]: C \rightarrow D$  is  $[u, f] \otimes [v, g] = [u \wedge v, (f \otimes g) \circ (A \otimes \sigma_{U,V} \otimes D)]: A \otimes C \rightarrow B \otimes D$ . The coherence isomorphisms are  $[1, \alpha]$ ,  $[1, \lambda]$ , and  $[1, \rho]$ .  $\square$

**Lemma 7.3.** *If  $\mathbf{C}$  is a monoidal category, and  $x \subseteq \text{ZI}(\mathbf{C})$  a prime filter, then*

$$\mathbf{C}\|_x = \text{colim}_{u \in x} \mathbf{C}\|_u$$

*in  $\mathbf{MonCat}$ , where the colimit ranges over the diagram induced by the functors  $\mathbf{C}\|_{u \leq v}: \mathbf{C}\|_v \rightarrow \mathbf{C}\|_u$  from Lemma 5.4.*

*Proof.* The functors  $F_u: \mathbf{C}\|_u \rightarrow \mathbf{C}\|_x$  that send  $f: A \otimes U \rightarrow B$  to  $[u, f]$  form a cocone. If  $G_u: \mathbf{C}\|_u \rightarrow \mathbf{D}$  is another cocone, then there is a unique mediating functor  $M: \mathbf{C}\|_x \rightarrow \mathbf{D}$  given by  $M[u, f] = G_u(f)$ . If  $\mathbf{C}$  and  $G_u$  are monoidal, then  $M$  is lax monoidal.  $\square$

The next lemma characterises the central idempotents in  $\mathbf{C}\|_x$ . The equivalence relation of having the same germ specialises to a semilattice congruence of central idempotents as follows for a prime filter  $x$  of central idempotents:

$$v \sim_x w \iff \exists u \in x: u \wedge v = u \wedge w$$

**Lemma 7.4.** *If  $\mathbf{C}$  is a braided monoidal category and  $x \subseteq \text{ZI}(\mathbf{C})$  a prime filter, then there is an isomorphism  $\text{ZI}(\mathbf{C}\|_x) \simeq \text{ZI}(\mathbf{C})/\sim_x$  of semilattices.*

*Proof.* Let  $[v, q]: Q \rightarrow I$  be a morphism in  $\mathbf{C}\|_x$ . That is, choose a representing morphism  $q: Q \otimes V \rightarrow I$  in  $\mathbf{C}$ . Then by definition  $[v, q]$  satisfies (2.1) in  $\mathbf{C}\|_x$  if and only if:

in  $\mathbf{C}$  for some  $u \in x \cap \downarrow v$ . If  $m: U \rightarrow V$  satisfies  $u = v \circ m$ , then  $q \circ (Q \otimes m): Q \otimes U \rightarrow I$  is central in  $\mathbf{C}$  if and only if:

But these two equations are equivalent by Lemma 2.5.

Similarly,  $[v, q]$  is idempotent in  $\mathbf{C}\|_x$  if and only if there exist  $u \in x \cap \downarrow v$ , and a morphism  $p: Q \otimes U \rightarrow Q \otimes Q$  in  $\mathbf{C}$  such that:

If  $u = v \circ m$ , then  $q \circ (Q \otimes m): Q \otimes U \rightarrow I$  is idempotent in  $\mathbf{C}$  if and only if there exists a morphism  $f: Q \otimes U \rightarrow Q \otimes U \otimes Q \otimes U$  in  $\mathbf{C}$  satisfying:

These two properties are equivalent, by choosing:

Just like in Lemma 5.6, a half-braiding on  $Q$  in  $\mathbf{C}\|_x$  induces a half-braiding on  $Q \otimes U$  in  $\mathbf{C}$ . Thus every central idempotent in  $\mathbf{C}\|_x$  is induced by a central idempotent in  $\mathbf{C}$ . But by definition two central idempotents  $u$  and  $v$  in  $\mathbf{C}$  induce the same central idempotent in  $\mathbf{C}\|_x$  exactly when  $u \sim_x v$ .  $\square$

**Lemma 7.5.** *Let  $\mathbf{C}$  be a monoidal monoidal category and  $x \subseteq \text{ZI}(\mathbf{C})$  a prime filter.*

- *If  $\mathbf{C}$  is stiff, then so is  $\mathbf{C}\|_x$ .*
- *If  $\mathbf{C}$  has universal finite joins of central idempotents, then so does  $\mathbf{C}\|_x$ .*
- *If  $\mathbf{C}$  has universal joins of central idempotents, then so does  $\mathbf{C}\|_x$ .*

*The functor  $\mathbf{C} \rightarrow \mathbf{C}\|_x$  preserves joins of central idempotents.*

*Proof.* We start with stiffness. Let  $v, w$  be central idempotents in  $\mathbf{C}$ . It is clear that the inner square below commutes in  $\mathbf{C}\|_x$ .

Suppose the outer square commutes too. This means  $(A \otimes v) \circ (f \otimes u' \otimes q) = (A \otimes w) \circ g \circ (B \otimes u \otimes U' \otimes q)$  for some central idempotent  $q$  in  $\mathbf{C}$ . Because  $\mathbf{C}$  is stiff, there is a morphism  $m: B \otimes U \otimes U' \otimes Q \rightarrow A \otimes V \otimes W$  satisfying:

It follows that the dashed morphism  $[u \wedge u' \wedge q, m]$  makes the two triangles commute. The uniqueness of  $m$  in  $\mathbf{C}$  also guarantees that the dashed morphism is the unique such morphism in  $\mathbf{C}\|_x$ .

Next we turn to joins of central idempotents. The initial object  $0$  satisfying  $A \otimes 0 \simeq 0$  in  $\mathbf{C}$  is still initial and still satisfies  $A \otimes 0 \simeq 0$  in  $\mathbf{C}\|_x$ . The proof that (3.1) is a pullback in  $\mathbf{C}\|_x$  is virtually the same as in the stiff case. We focus on the pushout property. Suppose that  $[u, f] \circ [1, A \otimes V \otimes w] = [u', g] \circ [1, A \otimes v \otimes W]$  in  $\mathbf{C}\|_x$ . This means:

$$f \circ (q \otimes U \otimes u' \otimes A \otimes V \otimes w) = g \circ (q \otimes u \otimes U' \otimes A \otimes v \otimes W)$$

for some central idempotent  $q$  in  $\mathbf{C}$ . If  $\mathbf{C}$  has finite joins of central idempotents, there is a morphism  $m: ((Q \otimes U \otimes V) \vee (Q \otimes U' \otimes W)) \otimes A \rightarrow B$  satisfying:

$$\begin{array}{c}
 \begin{array}{c} B \\ \hline f \\ \hline A \quad Q \otimes U \otimes V \end{array} \\
 = \\
 \begin{array}{c} B \\ \hline m \\ \hline A \quad Q \otimes U \otimes V \\ \circlearrowleft \begin{array}{l} (Q \otimes U \otimes V) \vee \\ (Q \otimes U' \otimes W) \end{array} \end{array} \\
 \begin{array}{c} B \\ \hline g \\ \hline A \quad Q \otimes U' \otimes W \end{array} \\
 = \\
 \begin{array}{c} B \\ \hline m \\ \hline A \quad Q \otimes U' \otimes W \\ \circlearrowleft \begin{array}{l} (Q \otimes U \otimes V) \vee \\ (Q \otimes U' \otimes W) \end{array} \end{array}
 \end{array}$$

Observe that  $(q \wedge u \wedge v) \vee (q \wedge u' \wedge w) = p \wedge (v \vee w)$  for  $p = q \wedge (u \vee u') \wedge (u \vee w) \wedge (v \vee u')$ . Now  $[p, m]$  is the unique mediating morphism in  $\mathbf{C}||_x$  satisfying  $[p, m] \circ [1, A \otimes (v \leq v \vee w)] = [u, f]$  and  $[p, m] \circ [1, A \otimes (w \leq v \vee w)] = [u', g]$ . The above holds equally well for wide pushouts.  $\square$

## 8. SUBLOCAL AND LOCAL

This section proves that the stalks are particularly easy, in the sense that the central idempotents behave well because there are few of them.

**Definition 8.1.** Call a partially ordered set *sublocal* if  $u \vee w = 1$  can only happen when  $u = 1$  or  $w = 1$ . Call a monoidal category  $\mathbf{C}$  *sublocal* when  $\text{ZI}(\mathbf{C})$  is sublocal.

A partially ordered set is sublocal if and only if it has a unique maximal ideal.

**Lemma 8.2.** *If  $\mathbf{C}$  is a monoidal category with universal finite joins of central idempotents, and  $x \subseteq \text{ZI}(\mathbf{C})$  is a prime filter, then  $\mathbf{C}||_x$  is sublocal.*

*Proof.* Consider two elements of  $\text{ZI}(\mathbf{C}||_x)$ . They are represented by  $u, w \in \text{ZI}(\mathbf{C})$  by Lemma 7.4. Now,  $u \vee w \sim_x 1$  if and only if there is  $v$  in the prime filter  $x$  and  $(u \wedge v) \vee (v \wedge w) = v$ . Since  $(u \wedge v) \vee (v \wedge w)$  is in the prime filter  $x$ , either  $u \wedge v \in x$  or  $v \wedge w \in x$ . But this implies  $u \sim_x 1$  or  $w \sim_x 1$ .  $\square$

In fact, everything we've done so far can easily be extended from finite joins to arbitrary joins. Recall that a *completely prime filter* of a complete lattice  $L$  is a nonempty upward-closed and downward-directed subset  $P$  where  $\bigvee s_i \in P$  implies  $s_i \in P$  for some  $i$ ; equivalently,  $P$  is the inverse image of 1 under a complete lattice morphism  $L \rightarrow \{0, 1\}$ .

**Definition 8.3.** Let  $L$  be a frame. Its *completely prime spectrum* is a topological space  $X$ , whose points are completely prime ideals  $P \subseteq L$ , and whose topology is generated by a basis consisting of the sets

$$B_u = \{P \in X \mid u \notin P\}$$

where  $u$  ranges over  $L$ .

**Definition 8.4.** Call a partially ordered set *local* if  $\bigvee u_i = 1$  can only happen when there is an  $i$  with  $u_i = 1$ . Call a monoidal category  $\mathbf{C}$  *local* when  $\text{ZI}(\mathbf{C})$  is local.

**Lemma 8.5.** *If  $\mathbf{C}$  is a monoidal with universal joins of central idempotents, and  $x \subseteq \text{ZI}(\mathbf{C})$  is a completely prime filter, then  $\mathbf{C}||_x$  is local.*

*Proof.* Completely analogous to Lemma 8.2.  $\square$

By a *sheaf of (sub)local categories*, we mean a sheaf  $\mathcal{O}(X)^{\text{op}} \rightarrow \mathbf{MonCat}$  on a topological space  $X$  whose stalks are all (sub)local. Pulling everything together, we now arrive at our main result. A *global section* of a sheaf  $F: \mathcal{O}(X)^{\text{op}} \rightarrow \mathbf{MonCat}$  is an object of  $F(X)$ . For  $\mathbf{Set}$ -valued sheaves, this corresponds to the more usual definition of a global section being a natural transformation  $1 \Rightarrow F$ . Enriching this as usual, we will call  $F(X)$  the category of global sections of  $F$ .

**Theorem 8.6.** *Any small monoidal category with universal (finite) joins of central idempotents is monoidally equivalent to the category of global sections of a sheaf of (sub)local monoidal categories.*

*Proof.* Let  $\mathbf{C}$  be the monoidal category with universal finite joins of central idempotents. Take  $X = \text{Spec}(\text{ZI}(\mathbf{C}))$  as the base space, as in Definition 4.1. Then  $B_u \mapsto \mathbf{C}||_u$  is a presheaf of monoidal categories by Lemma 5.4. It extends to a sheaf as in Proposition 6.3. The stalks  $\mathbf{C}||_x$  for  $x \in X$  are sublocal by Lemma 8.2, so this is a sheaf of sublocal categories. Finally, the global sections of this sheaf form  $\mathbf{C}||_1$ , which is monoidally equivalent to  $\mathbf{C}$  itself.

In case  $\mathbf{C}$  has universal joins of central idempotents, we instead take the completely prime spectrum of  $\text{ZI}(\mathbf{C})$  as the base space. Lemma 5.4 still shows that  $B_u \mapsto \mathbf{C}||_u$  is a presheaf. For the sheaf condition it now no longer suffices to verify binary equaliser of Lemma 4.3, and we have to consider a wide equaliser instead. But the proof of Proposition 6.1 extends easily to this case. The exceptional nullary case is taken care of by Lemma 6.2 as in Proposition 6.3 as before. The stalks are now local by Lemma 8.5.  $\square$

## 9. PRESERVATION

Theorem 8.6 showed that any small monoidal category with universal joins of central idempotents is a category of global sections of a sheaf of local monoidal categories. In this section we extend that main result by showing that it preserves various properties: if the original category has a certain property, then so do the stalks. We consider two kinds of properties: properties associated with monoidal categories and linear logic, such as being compact, and having a trace; and properties associated with toposes and intuitionistic logic, such as having a Boolean algebra of central idempotents, and having limits. We also investigate the property of being closed, which resides in both camps. So for example, we show that any small monoidal category with (finite) universal joins of central idempotents that is closed, is equivalent to the category of global sections of a sheaf of (sub)local closed categories.

We start with compactness. Recall that a symmetric monoidal category is compact when every object has a dual [29, Chapter 3].

**Corollary 9.1.** *If  $\mathbf{C}$  is a compact category,  $u$  is a central idempotent, and  $x \subseteq \text{ZI}(\mathbf{C})$  is a prime filter, then  $\mathbf{C}||_u$  and  $\mathbf{C}||_x$  are compact categories, too.*

*Proof.* Strong monoidal functors preserve dual objects [29, Theorem 3.14], so this follows directly from Lemma 5.4 and Proposition 6.1.  $\square$

Notice, however, that a central idempotent  $U$  has itself as a dual only if it is (represented by) a split monomorphism.

**Lemma 9.2.** *A central idempotent  $u$  is split monic if and only if  $U \dashv U$  with counit  $\rho \circ (u \otimes u)$ .*

*Proof.* If  $e \circ u = U$ , set  $\eta = (U \otimes u)^{-1} \circ (e \otimes I) \circ \lambda^{-1}$  and  $\varepsilon = \rho \circ (u \otimes u)$ , so:

$$\begin{array}{c}
 U \\
 | \\
 \boxed{\varepsilon} \\
 | \\
 \boxed{\eta} \\
 | \\
 U
 \end{array}
 =
 \begin{array}{c}
 U \\
 | \\
 \circ \\
 | \\
 \boxed{e} \\
 | \\
 U
 \end{array}
 =
 \begin{array}{c}
 U \\
 | \\
 \boxed{e} \\
 | \\
 \circ \\
 | \\
 U
 \end{array}$$

Conversely, if  $U \dashv U$  with unit  $\eta$  and counit  $u \otimes u$ , then  $e \circ u = U$  for  $e = (U \otimes u) \circ \eta$ .  $\square$

Another linear property that is preserved is having a trace [34].

**Lemma 9.3.** *If a monoidal category  $\mathbf{C}$  with universal finite joins of central idempotents is traced,  $u$  is a central idempotent, and  $x \subseteq \text{ZI}(\mathbf{C})$  a prime filter, then  $\mathbf{C}\|_u$  and  $\mathbf{C}\|_x$  are traced too, and the functors  $\mathbf{C}\|_{u \leq v}: \mathbf{C}\|_v \rightarrow \mathbf{C}\|_u$  and  $\mathbf{C} \rightarrow \mathbf{C}\|_x$  preserve the trace.*

*Proof.* If  $\mathbf{C}$  is braided, then so are  $\mathbf{C}\|_u$  and  $\mathbf{C}\|_x$ . The trace of  $f \in \mathbf{C}\|_u(A \otimes Z, B \otimes Z) = \mathbf{C}(A \otimes Z \otimes U, B \otimes Z)$  is defined to be the trace of  $f \circ (A \otimes \sigma) \in \mathbf{C}(A \otimes U \otimes Z, B \otimes Z)$ . It is easy to see that this satisfies the axioms for a trace. For example, with the usual graphical calculus [48] of the traced monoidal category  $\mathbf{C}$ , the superposing axiom becomes:

$$g \otimes \text{Tr}(f) = \begin{array}{c} \begin{array}{ccc} B & D & \\ | & | & \\ \boxed{g} & \boxed{f} & \\ | & | & \\ A & C & U \end{array} \\ \text{Tr}(g \otimes f) \end{array} = \begin{array}{c} \begin{array}{ccc} B & D & \\ | & | & \\ \boxed{g} & \boxed{f} & \\ | & | & \\ A & C & U \end{array} \\ \text{Tr}(g \otimes f) \end{array} = \text{Tr}(g \otimes f)$$

Stalks are entirely similar: define the trace of  $[v, f] \in \mathbf{C}\|_x(A \otimes Z, B \otimes Z)$  to be  $[v, g]$  where  $g$  is the trace of  $f$  in  $\mathbf{C}\|_v$ . To verify that this is well-defined, if  $[v, f] = [v', f']$ , say because  $f \otimes u \otimes v' = f' \otimes u \otimes v$  for a central idempotent  $u$  in  $\mathbf{C}$ , then:

$$\begin{array}{c} \begin{array}{ccc} B & & \\ | & & \\ \boxed{f} & & \\ | & & \\ A & V & V'U \end{array} \\ \text{Tr}([v, f]) \end{array} = \begin{array}{c} \begin{array}{ccc} B & & \\ | & & \\ \boxed{f} & & \\ | & & \\ A & V & V'U \end{array} \\ \text{Tr}([v, f]) \end{array}$$

The functors  $\mathbf{C}\|_{u \leq v}: \mathbf{C}\|_v \rightarrow \mathbf{C}\|_u$  and  $\mathbf{C} \rightarrow \mathbf{C}\|_x$  preserve trace by construction.  $\square$

Next we turn to closedness.

**Lemma 9.4.** *If a monoidal category  $\mathbf{C}$  is closed and  $u$  is a central idempotent, then  $\mathbf{C}\|_u$  is closed. If  $u \leq v$ , then the functor  $\mathbf{C}\|_{u \leq v}: \mathbf{C}\|_v \rightarrow \mathbf{C}\|_u$  is closed.*

*Proof.* Suppose that  $(-) \otimes B: \mathbf{C} \rightarrow \mathbf{C}$  has a right adjoint  $B \multimap (-): \mathbf{C} \rightarrow \mathbf{C}$ . Write  $\varepsilon_{B,C}: (B \multimap C) \otimes B \rightarrow C$  for the counit and  $\eta_{A,B}: A \rightarrow (B \multimap (A \otimes B))$  for the unit. Define a functor  $B \multimap^u (-): \mathbf{C}\|_u \rightarrow \mathbf{C}\|_u$  by  $B \multimap^u C = B \multimap C$  on objects, and by sending a morphism  $f \in \mathbf{C}\|_u(C, D) = \mathbf{C}(C \otimes U, D)$  to  $B \multimap^u f$  defined as:

$$\begin{array}{ccc} (B \multimap C) \otimes U & \xrightarrow{B \multimap^u f} & B \multimap D \\ \downarrow \eta_{(B \multimap C) \otimes U, B} & & \uparrow B \multimap f \\ B \multimap ((B \multimap C) \otimes U \otimes B) & \xrightarrow{B \multimap (1 \otimes \sigma_{U,B})} & B \multimap ((B \multimap C) \otimes B \otimes U) & \xrightarrow{B \multimap (\varepsilon_{B,C} \otimes 1)} & B \multimap (C \otimes U) \end{array}$$

It is straightforward to see that  $B \multimap^u (-)$  is functorial. There is a bijection

$$\mathbf{C}\|_u(A, B \multimap^u C) \simeq \mathbf{C}\|_u(A \otimes B, C)$$

that sends  $g \in \mathbf{C}\|_u(A, B \multimap^u C) = \mathbf{C}(A \otimes U, B \multimap C)$  to the morphism

$$\begin{array}{c}
 C \\
 \downarrow \\
 \boxed{\varepsilon_{B,C}} \\
 \downarrow \\
 B \multimap C \\
 \downarrow \\
 \boxed{g} \\
 \downarrow \\
 A \quad B \quad U
 \end{array}$$

in  $\mathbf{C}\|_u(A \otimes B, C)$ . It is easy to see that this bijection is natural in  $A$  and  $C$ . Thus  $B \multimap^u (-)$  is right adjoint to  $(-) \otimes B$ , and  $\mathbf{C}\|_u$  is closed.

It is clear that the functor  $\mathbf{C}\|_{u \leq v}$  preserves  $\multimap$  strictly.  $\square$

**Lemma 9.5.** *If  $\mathbf{C}$  is a closed monoidal category and  $x \subseteq \text{ZI}(\mathbf{C})$  a prime filter, then  $\mathbf{C}\|_x$  is closed, and the functor  $\mathbf{C} \rightarrow \mathbf{C}\|_x$  is closed.*

*Proof.* The proof is very close to that of Lemma 9.4. The functor  $B \multimap^x (-): \mathbf{C}\|_x \rightarrow \mathbf{C}\|_x$  is defined as  $C \mapsto B \multimap C$  on objects and as  $[u, f] \mapsto [u, B \multimap^u f]$  on morphisms. This is well-defined and functorial as before. There is a natural bijection  $\mathbf{C}\|_x(A \otimes B, C) \simeq \mathbf{C}\|_x(A, B \multimap C)$  that sends  $[u, f]$  to  $[u, g]$  where  $g = (B \multimap f) \circ (B \multimap (1 \otimes \sigma)) \circ \eta: A \otimes U \rightarrow B \multimap C$  corresponds to  $f: A \otimes B \otimes U \rightarrow C$  under the bijection  $\mathbf{C}\|_u(A \otimes B, C) \simeq \mathbf{C}\|_u(A, B \multimap C)$ . To see that it is well-defined: if  $[u, f] = [u', f'] \in \mathbf{C}\|_x(A \otimes B, C)$ , then for some central idempotent  $v$  of  $\mathbf{C}$ ,

$$\begin{aligned}
 & (B \multimap f) \circ (B \multimap (1 \otimes \sigma)) \circ \eta \circ (1 \otimes u' \otimes v) \\
 &= (B \multimap (f \otimes u' \otimes v)) \circ (B \multimap (1 \otimes \sigma)) \circ \eta \\
 &= (B \multimap (f' \otimes u \otimes v)) \circ (B \multimap (1 \otimes \sigma)) \circ \eta \\
 &= (B \multimap f') \circ (B \multimap (1 \otimes \sigma)) \circ \eta \circ (1 \otimes u \otimes v \otimes 1)
 \end{aligned}$$

by naturality of  $\eta$ , so  $[u, g] = [u', g']$ . Thus  $\mathbf{C}\|_x$  is closed, and the functor  $\mathbf{C} \rightarrow \mathbf{C}\|_x$  preserves  $\multimap$  strictly.  $\square$

Call a monoidal category *Boolean* when its central idempotent semilattice is a Boolean algebra. In this case, the stalks are also Boolean, and in fact two-valued, as follows.

**Lemma 9.6.** *If  $L$  is a Boolean algebra and  $x$  a prime filter, then  $L/\sim_x$  is again a Boolean algebra.*

*Proof.* It suffices to prove that  $\sim_x$  is a congruence of lattices, because complements are uniquely defined in terms of joins and meets. Suppose  $a \sim_x a'$  and  $b \sim_x b'$ . Say  $a \wedge s = a' \wedge s$  and  $b \wedge t = b' \wedge t$  for  $s, t \in x$ . For  $r = s \wedge t$ , then clearly  $a \wedge b \wedge r = a' \wedge b' \wedge r$  so  $a \wedge b \sim_x a' \wedge b'$ . Similarly,  $(a \vee b) \wedge r = (a \wedge r) \vee (b \wedge r) = (a' \wedge r) \vee (b' \wedge r) = (a' \vee b') \wedge r$ , so  $a \vee b \sim_x a' \vee b'$ .  $\square$

**Corollary 9.7.** *If a monoidal category  $\mathbf{C}$  with universal finite joins of central idempotents is Boolean, and  $x \subseteq \text{ZI}(\mathbf{C})$  is a prime filter, then  $\text{ZI}(\mathbf{C}\|_x) = \{0, 1\}$ .*

*Proof.* By Lemmas 7.4 and 9.6,  $\text{ZI}(\mathbf{C}\|_x)$  is again a Boolean algebra. Hence if  $u \in \text{ZI}(\mathbf{C}\|_x)$ , then  $u \vee \neg u = 1$ . By Lemma 8.2  $\text{ZI}(\mathbf{C}\|_x)$  is also sublocal. Hence  $u = 1$  or  $\neg u = 1$ , that is,  $u \in \{0, 1\}$ .  $\square$

Next, we turn to limits. In the cartesian case, as in Lemma 5.3, it is well known that limits in the slice category are computed as in the base category. In the monoidal setting we need to be more careful.



**Lemma 9.8.** *Let  $\mathbf{J}$  be a small category, and  $\mathbf{C}$  a monoidal category with  $\mathbf{J}$ -shaped limits.*

- *If  $\mathbf{C}$  has universal finite joins of central idempotents, and  $u$  is a central idempotent, then  $\mathbf{C}\|_u$  has  $\mathbf{J}$ -shaped limits, and the functors  $\mathbf{C}\|_{u \leq v}: \mathbf{C}\|_v \rightarrow \mathbf{C}\|_u$  preserve them.*
- *If  $\mathbf{C}$  has universal finite joins of central idempotents,  $x \subseteq \text{ZI}(\mathbf{C})$  is a prime filter, and  $\mathbf{J}$  is finite, then  $\mathbf{C}\|_x$  has  $\mathbf{J}$ -shaped limits, and the functor  $\mathbf{C} \rightarrow \mathbf{C}\|_x$  preserves them.*
- *If  $\mathbf{C}$  has universal joins of central idempotents, and  $x \subseteq \text{ZI}(\mathbf{C})$  is a prime filter, then  $\mathbf{C}\|_x$  has  $\mathbf{J}$ -shaped limits, and the functor  $\mathbf{C} \rightarrow \mathbf{C}\|_x$  preserves them.*

*Proof.* For the sake of clarity we will consider equalisers, but the proof works for arbitrary (finite) shapes  $\mathbf{J}$ . Let  $f, g \in \mathbf{C}\|_u(X, Y)$ . Let  $l: E \rightarrow X \otimes U$  be their equaliser in  $\mathbf{C}$ . Then  $e = (X \otimes u \otimes u) \circ (l \otimes U)$  defines a morphism  $E \rightarrow X$  in  $\mathbf{C}\|_u$ . Now  $f \circ e = f \circ g$  in  $\mathbf{C}\|_u$ , by the central equation 2.1:

Suppose that  $f \circ e' = g \circ e'$  for some  $e': E' \rightarrow X$  in  $\mathbf{C}\|_u$ , so  $e': E' \otimes U \rightarrow X$  in  $\mathbf{C}$ . Setting  $l' = (e' \otimes U) \circ (E' \otimes U \otimes u)^{-1}: E' \otimes U \rightarrow X \otimes U$  in  $\mathbf{C}$ , then  $f \circ l' = g \circ l'$  in  $\mathbf{C}$ . Hence there is a mediating map  $m: E' \otimes U \rightarrow E$  with  $l' = l \circ m$  in  $\mathbf{C}$ .

But then  $e' = e \circ m$  in  $\mathbf{C}\|_u$ :

Conversely, if  $e' = e \circ m$  in  $\mathbf{C}\|_u$ , then  $(e' \otimes U) \circ (E' \otimes U \otimes u)^{-1} = l \circ m$  in  $\mathbf{C}$ , so  $m$  is the unique such morphism. The functors  $\mathbf{C}\|_{u \leq v}: \mathbf{C}\|_v \rightarrow \mathbf{C}\|_u$  preserve these limits by construction.

The same idea works for  $\mathbf{C}\|_x$ , except that we may need the diagram  $\mathbf{J}$  to be finite, depending on the amount of universal joins of central idempotents available in  $\mathbf{C}$ . This is best illustrated for products. Suppose that  $\mathbf{C}$  has universal finite

joins of central idempotents and has finite products, then  $\prod_{i=1}^n X_i$  is also a product in  $\mathbf{C}\|_x$ . Define the projections to be  $[1, \pi_i]: \prod_i X_i \rightarrow X_i$ , and given finitely many  $[u_i, f_i]: A \otimes U_i \rightarrow X_i$ , define their tuple to be  $[u_1 \wedge \dots \wedge u_n, \langle f_i \rangle]: A \rightarrow \prod_i X_i$  in  $\mathbf{C}\|_x$ . It is easily checked that this is well-defined and satisfies the universal property. If  $\mathbf{C}$  had universal joins of central idempotents, we could have defined the tuple  $[\bigwedge u_i, \langle f_i \rangle]: A \rightarrow \prod X_i$  for an arbitrary number of morphisms  $A \rightarrow X_i$ . In either case, the functor  $\mathbf{C} \rightarrow \mathbf{C}\|_x$  preserves these limits by construction.  $\square$

Finally, let us mention two properties that are preserved only in certain cases.

If  $\mathbf{C}$  has colimits of a certain shape  $\mathbf{J}$ , and  $U \otimes (-): \mathbf{C} \rightarrow \mathbf{C}$  preserves these colimits for a central idempotent  $u$ , then  $\mathbf{C}\|_u$  inherits these  $\mathbf{J}$ -shaped colimits, and the functor  $\mathbf{C} \rightarrow \mathbf{C}\|_u$  preserves them. If  $\mathbf{J}$  is finite or  $\mathbf{C}$  has universal joins of central idempotents, then the same holds for  $\mathbf{C}\|_x$ .

If  $\mathbf{C}$  has a dagger [29, Section 2.3] then  $\mathbf{C}\|_u$  inherits it if and only if the central idempotent  $u$  is (represented by) an isometry, and  $\mathbf{C}\|_x$  inherits it if and only if for every central idempotent  $v$  there is a central idempotent  $u \leq v$  that is (represented by) an isometry.

## 10. EXAMPLES

This brief section illustrates the sheaf representation theorem in several example cases that are genuinely monoidal and could not have been handled using sheaf representation theorems for toposes.

**Example 10.1.** A topological space  $X$  has a frame of open sets  $L = \mathcal{O}(X)$  that may be regarded as a stiff monoidal category with universal joins of central idempotents (but that is not a topos unless  $X$  is empty). Because  $\text{ZI}(L) \simeq L$ , the completely prime spectrum  $\text{Spec}(\text{ZI}(L))$  is (homeomorphic to)  $X$  itself. Theorem 8.6 says that  $L$  is isomorphic to the global sections of a sheaf of local monoidal categories on  $X$ . If  $x$  is a point of  $X$ , then the stalk  $L\|_x$  is the quotient of  $L$  by the filter  $x$ , that is, the quotient  $L/\sim$  by the equivalence relation  $\sim$  where  $U \sim V$  when  $U \cap W = V \cap W$  for some  $W \in L$ . This is sometimes called the frame of germs of open subsets of  $X$  with respect to  $x$  [12, I.6.10]. Concretely, the stalk  $L\|_x$  has the same objects as  $L$ , that is, open subsets  $U$  of  $X$ . There is a unique morphism  $U \rightarrow V$  in  $L\|_x$  when there is an open neighbourhood  $W$  of  $x$  with  $U \cap W \subseteq V$ .

- If  $x \in V$ , then there is a morphism  $U \rightarrow V$ : take  $W = V$ ;
- If  $x \in U$  and  $x \notin V$ , then there is no morphism  $U \rightarrow V$ : for any open neighbourhood  $W$  of  $x$ , there is a point  $x$  in  $U \cap W$  that is not in  $V$ ;
- If  $x \notin \bar{U}$ , then there is a morphism  $U \rightarrow V$ : there is an open neighbourhood  $W$  of  $x$  disjoint from  $U$ ;

Hence all open neighbourhoods of  $x$  are isomorphic as objects in  $L\|_x$ , and they are all terminal. Similarly, all open sets whose closure does not contain  $x$  are isomorphic as objects in  $L\|_x$ , and they are all initial. Therefore, if  $x$  is an isolated point,  $L\|_x$  is equivalent to the partially ordered set  $0 \leq 1$  regarded as a 2-object category. But in general more subtle behaviour can occur:

- If  $x \in \bar{U}$  and  $x \notin \bar{V}$ , then there is no morphism  $U \rightarrow V$ : because  $x \notin \bar{V}$  there is an open neighbourhood  $W$  of  $x$  disjoint from  $V$ , but because  $x \in \bar{U}$  this neighbourhood is not disjoint from  $U$ .
- If  $x \in \partial U = \bar{U} \setminus U$  and  $x \in \partial V = \bar{V} \setminus V$ , there may be a morphism  $U \rightarrow V$  or there may not be. For example, take  $X = \mathbb{R}$ ,  $x = 0$ , and  $U = (-1, 0)$ . If  $V = (0, 1)$ , then any open neighbourhood  $W$  of  $x$  contains points of  $U$  that are not in  $V$ , so there is no morphism  $U \rightarrow V$ . But if  $V = (-2, 0)$ , then  $U \subseteq V$ , so  $W = X$  shows that there is a morphism  $U \rightarrow V$ .

Nevertheless, the global sections of the sheaf  $U \mapsto L||_U$  correspond to continuous functions  $X \rightarrow \{0, 1\}$ .

**Example 10.2.** A Boolean algebra  $B$  may be regarded as a stiff monoidal category with finite universal joins of central idempotents. Because  $\text{ZI}(B) \simeq B$ , in this case  $\text{Spec}(\text{ZI}(B))$  is the Stone space of  $B$ . Theorem 8.6 hence says that any Boolean algebra is isomorphic to the global sections of a sheaf of sublocal rings on a Stone space. By Corollary 9.7 the stalks are two-valued, so this becomes Stone's representation theorem [32]: any Boolean algebra is isomorphic to the algebra of clopen subsets of its Stone space. Going from sublocal to local, this extends to complete Boolean algebras and Stonean spaces.

In fact, this holds more generally for a distributive lattice  $L$  and its prime spectrum  $X$ . As the quotient of  $L$  by a prime filter  $x$  is always  $\{0, 1\}$ , elements of  $L$  correspond to continuous functions  $X \rightarrow \{0, 1\}$ .

**Example 10.3.** If  $X$  is a locally compact Hausdorff space, the category  $\mathbf{Hilb}_{C_0(X)}$  of Hilbert  $C_0(X)$ -modules is a stiff monoidal category with universal joins of central idempotents. Here  $\text{ZI}(\mathbf{Hilb}_{C_0(X)}) \simeq \mathcal{O}(X)$  [23, 3.16], so as in Example 10.1, the representation theorem says that  $\mathbf{Hilb}_{C_0(X)}$  is isomorphic to the global sections of a sheaf of local monoidal categories over  $X$ . In this case, every stalk  $\mathbf{Hilb}_{C_0(X)}||_x$  is the category  $\mathbf{Hilb}$  of Hilbert spaces (see also [28, 2.5, 2.8]). Thus we recover Takahashi's representation theorem of Hilbert modules as continuous fields of Hilbert spaces [50].

**Example 10.4.** Any commutative ring  $R$  has a category  $\mathbf{Mod}_R$  of modules that is a stiff monoidal category with finite universal joins of central idempotents. As in Example 2.10, the central idempotents of  $\mathbf{Mod}_R$  are linear maps  $A \rightarrow R$  whose image is an idempotent ideal of  $R$ . If  $R$  is semisimple, any idempotent ideal is generated by an idempotent element, and hence  $\text{ZI}(\mathbf{Mod}_R)$  includes the set  $E(R) = \{e \in R \mid e^2 = e\}$ . Therefore  $\text{Spec}(\text{ZI}(\mathbf{Mod}_R))$  is a quotient of the Pierce spectrum of  $R$  [32, V.2]. Theorem 8.6 now says that an  $R$ -module corresponds to a global section of a sheaf of sublocal modules over a quotient of the Pierce spectrum of  $R$ . Indeed, any semisimple commutative ring is a finite product of fields.

**Example 10.5.** Combining the previous example with Examples 2.11 shows that, if  $A$  is a bialgebra over a field, a left  $A$ -module corresponds to a global section of a sheaf of sublocal modules over the Pierce spectrum of  $A$ .

**Example 10.6.** Consider a linear-non-linear model of linear logic [7], that is, a symmetric monoidal adjunction between a symmetric monoidal closed category  $\mathbf{L}$  and a cartesian closed category  $\mathbf{C}$ .

$$\mathbf{C} \begin{array}{c} \xrightarrow{F} \\ \perp \\ \xleftarrow{G} \end{array} \mathbf{L}$$

Because  $F$  is automatically strong monoidal [7, Proposition 1], there is a semilattice morphism  $\text{ZI}(\mathbf{C}) \rightarrow \text{ZI}(\mathbf{L})$ , and a functor  $\mathbf{C}||_s \rightarrow \mathbf{L}||_{F(s)}$  for each  $s \in \text{ZI}(\mathbf{C})$ . If  $\mathbf{C}$  and  $\mathbf{L}$  both have universal joins of (finite) central idempotents, then  $\text{ZI}(\mathbf{C}) \rightarrow \text{ZI}(\mathbf{L})$  preserves (finite) joins, because  $F$  is a left adjoint and so preserves colimits. If  $G$  preserves central idempotents, we obtain a Galois connection between  $\text{ZI}(\mathbf{C})$  and  $\text{ZI}(\mathbf{L})$ , and a functor  $\mathbf{L}||_u \rightarrow \mathbf{C}||_{G(u)}$  for each  $u \in \text{ZI}(\mathbf{L})$ .

**Example 10.7.** For a monoid  $M$ , the category  $\mathbf{C} = [M^{\text{op}}, \mathbf{Set}]$  is a topos, and its central idempotents correspond to right ideals of  $M$ . The category  $\mathbf{C}$  is local if and only if  $M$  has a right-absorbing element [25]. Hence Theorem 8.6 shows that any

$M$ -action corresponds to a global section of a sheaf of categories of monoid actions whose stalks are categories of actions of monoids with right-absorbing elements.

**Example 10.8.** A quantale  $Q$  may be regarded as a monoidal category with universal joins of central idempotents. Theorem 8.6 shows that any commutative quantale is equivalent to the category of global sections of a sheaf of monoidal categories. This gives a different view on works combining sheaves and quantales [11, 46, 30]. The central idempotents in  $Q$  are the elements  $u$  satisfying  $u \cdot u = u \leq e$ , which form the largest frame contained in the quantale [23, 3.5]. The category  $Q||_u$ :

- has objects  $q \in Q$ ;
- there is a unique morphism  $p \rightarrow q$  when  $p \cdot u \leq q$ ;
- composition is uniquely determined:  $p \cdot u \leq q$  and  $q \cdot u \leq v$  imply  $p \cdot u \leq v$ ;
- the identity morphism is the equality  $q \cdot e = q$ .

If  $x \subseteq \text{ZI}(Q)$  is a completely prime filter,  $Q||_x$  has:

- objects  $q \in Q$ ;
- a unique morphism  $p \rightarrow q$  if there exists  $u \in x$  satisfying  $p \cdot u \leq q$ .

The partially ordered sets  $Q||_x$  still have finite meets and joins, bottom and top elements, and a multiplication that distributes over joins. To see that  $Q||_x$  inherits meets:  $p_i \cdot 1 = p_i \leq p_1 \wedge p_2$  and  $1 \in x$  so  $p_i \leq_x p_1 \wedge p_2$ , and if  $q \leq_x p_i$ , say because  $q \cdot u_i \leq p_i$ , then  $q \cdot (u_1 \wedge u_2) \leq q \cdot u_1 \wedge q \cdot u_2 \leq p_1 \wedge p_2$  with  $u_1 \wedge u_2 \in x$ , and so  $q \leq_x p_1 \wedge p_2$ ; a similar reasoning holds for joins.

## 11. FUNCTORIALITY

This section shows that the main construction of Theorem 8.6 is functorial. We start by defining the appropriate category of sheaves of monoidal categories, and prove that it is dual to the category of monoidal categories. Lemma 5.5 shows that the sheaves of monoidal categories that arise from the representation theorem are (a categorification of) flabby sheaves [24].

**Definition 11.1.** A sheaf  $P: \mathcal{O}(X)^{\text{op}} \rightarrow \mathbf{MonCat}$  of monoidal categories is *flabby* when the functors  $P_{u \leq v}: P_v \rightarrow P_u$  all act the same on objects, and each has a left adjoint  $P^{u \leq v}: P_v \rightarrow P_u$  and the unit of the adjunction is invertible.

Recall that a topological space is *spectral* when it is  $T_0$ , compact, sober, and its compact open subsets form a base and are closed under finite intersections. A continuous function between spectral spaces is a *morphism of spectral spaces* when the preimage of a compact open subset is again compact. The category of spectral spaces is dually equivalent to the category of distributive lattices [31, 32].

**Definition 11.2.** Write  $\mathbf{MonScheme}$  for the following category. Objects are pairs of a topological space  $X$  and a flabby sheaf  $P: \mathcal{O}(X)^{\text{op}} \rightarrow \mathbf{MonCat}$  of monoidal categories where all  $P_u$  have the same objects and  $X = \text{ZI}(P_1)$ . A morphism  $(X, P) \rightarrow (Y, Q)$  consists of a continuous function  $\varphi: X \rightarrow Y$  and a natural transformation  $F_\varphi: Q(\varphi^{-1}(v)) \rightarrow P(\varphi^{-1}(v))$  such that

$$(11.1) \quad F_1(u) = \theta_{I,U}$$

or in other words  $\varphi^{-1}(u) = \theta_I \circ F_1(u)$ . Write  $\mathbf{MonScheme}_1$  for the full subcategory of sheaves of local categories. Write  $\mathbf{MonScheme}_{\text{sl}}$  for the subcategory of sheaves of sublocal categories where  $X$  is a spectral space, and morphisms where  $\varphi$  is a morphism of spectral spaces.

**Theorem 11.3.** *The representation of Proposition 6.3 extends to equivalences of categories  $\mathbf{MonCat}_{\text{rj}} \rightarrow \mathbf{MonScheme}_{\text{sl}}^{\text{op}}$  and  $\mathbf{MonCat}_{\text{j}} \rightarrow \mathbf{MonScheme}_1^{\text{op}}$ .*

*Proof.* If  $F: \mathbf{C} \rightarrow \mathbf{D}$  is a morphism in  $\mathbf{MonCat}_{\mathfrak{f}}$  then it induces a semilattice morphism  $\mathrm{ZI}(\mathbf{C}) \rightarrow \mathrm{ZI}(\mathbf{D})$ , and hence a continuous function  $\varphi: \mathrm{Spec}(\mathrm{ZI}(\mathbf{D})) \rightarrow \mathrm{Spec}(\mathrm{ZI}(\mathbf{C}))$ , by Lemma 3.4. Furthermore, if  $u \in \mathrm{ZI}(\mathbf{C})$ , then  $F$  induces a morphism  $\mathbf{C}\|_u \rightarrow \mathbf{D}\|_{F(u)}$  in  $\mathbf{MonCat}$  that acts as  $F$  on objects and that sends a morphism  $f: A \otimes U \rightarrow B$  to  $F(f) \circ \theta_{A,U}: F(A) \otimes F(U) \rightarrow F(B)$ . This is natural with respect  $\mathbf{C}\|_{u \leq v}$ , and satisfies the condition  $F_1(u) = \theta_{I,U}$ , and so gives a well-defined morphism  $\mathbf{D}\|_{(-)} \rightarrow \mathbf{C}\|_{(-)}$  in  $\mathbf{MonScheme}_{\mathrm{sl}}$ . This assignment and its counterpart  $\mathbf{MonCat}_{\mathfrak{f}} \rightarrow \mathbf{MonScheme}_1^{\mathrm{op}}$  are clearly functorial, faithful, and injective on objects.

To see that these functors are full, suppose  $(\varphi, \{F_u\}_u)$  is a map in  $\mathbf{MonScheme}$  from  $P_- = \mathbf{D}\|_-$  to  $Q_- = \mathbf{C}\|_-$ . Then  $F = F_1$  is a morphism  $\mathbf{C} \rightarrow \mathbf{D}$  in  $\mathbf{MonCat}$ . It follows from naturality of that all  $F_u$  act the same on objects. Regarding a morphism  $f: A \otimes U \rightarrow B$  in  $\mathbf{C}$  as a morphism in  $\mathbf{C}\|_u$ , we find:

$$\begin{aligned} & F_u(A \otimes U \xrightarrow{f} B) \\ &= F_u(A \otimes U \otimes U \xrightarrow{f \otimes u} B) \circ_{F_u} F_u(A \otimes U \xrightarrow{(A \otimes U \otimes u)^{-1}} A \otimes U \otimes U) \\ &= F_u(\mathbf{C}\|_{u \leq 1}(A \otimes U \xrightarrow{f} B)) \circ_{F_u} F_u(A \otimes U \xrightarrow{(A \otimes U \otimes u)^{-1}} A \otimes U \otimes U) \\ &= \mathbf{D}\|_{F(u) \leq 1}(F_1(A \otimes U \xrightarrow{f} B)) \circ_{F_u} F_u(A \otimes U \xrightarrow{(A \otimes U \otimes u)^{-1}} A \otimes U \otimes U) \\ &= F_1(f) \circ F_u(A \otimes U \xrightarrow{A \otimes U} A \otimes U) \end{aligned}$$

Now, observe that  $A \otimes U: A \otimes U \rightarrow A \otimes U$ , regarded as a morphism  $A \rightarrow A \otimes U$  in  $\mathbf{C}\|_u$ , is an isomorphism in  $\mathbf{C}\|_u$ , with inverse  $A \otimes u \otimes u: A \otimes U \otimes U \rightarrow A$ , regarded as a morphism  $A \rightarrow_u A \otimes U$  in  $\mathbf{C}\|_u$ . Hence  $F_u(A \otimes U)$  is inverse to  $F_u(A \otimes u \otimes u)$  in  $\mathbf{D}\|_{F(u)}$ . But now it follows from  $F_1(A \otimes u) \circ \theta_{A,U} = F(A) \otimes F(U)$  that  $F_u(A \otimes u \otimes u) = F_1(A \otimes u) \otimes F(U)$  is inverted by  $\theta_{A,U}: F(A) \otimes F(U) \rightarrow F(A \otimes U)$ , regarded as a morphism  $F(A) \rightarrow F(A \otimes U)$  in  $\mathbf{D}\|_{F(u)}$ . Thus  $F_u(A \otimes U) = \theta_{A,U}$ , and so  $F_u(f) = F_1(f: A \otimes U \rightarrow B) \circ \theta_{A,U}$ . Therefore the morphism  $(\varphi, \{F_u\})$  in  $\mathbf{MonScheme}$  is indeed induced by the morphism  $F$  in  $\mathbf{MonCat}$ .

Similarly, any object  $P$  in  $\mathbf{MonScheme}$  is (monoidally naturally isomorphic to)  $\mathbf{C}\|_-$  for  $\mathbf{C} = P_1$ , because there is a bijection between morphism  $f: A \rightarrow B$  in  $P_u$  and morphisms  $g: A \otimes U \rightarrow B$  in  $P(1)$  by flabbiness. Thus the functors  $\mathbf{MonCat}_{\mathfrak{f}} \rightarrow \mathbf{MonScheme}_{\mathrm{sl}}^{\mathrm{op}}$  and  $\mathbf{MonCat}_{\mathfrak{f}} \rightarrow \mathbf{MonScheme}_1^{\mathrm{op}}$  are essentially surjective, and so equivalences.  $\square$

The previous theorem appears not to have been worked out in the special case of toposes before [2]. Next we adapt the equivalence between étale bundles and sheaves (see [44, II.6.5] and [16]) to the current setting.

**Lemma 11.4.** *Morphisms  $(X, P) \rightarrow (Y, Q)$  in  $\mathbf{MonScheme}$  that do not necessarily satisfy (11.1) are equivalently given by continuous functions  $\varphi: X \rightarrow Y$  together with a family  $F_x: Q_{\varphi(x)} \rightarrow P_x$  of morphisms in  $\mathbf{MonCat}$  between stalks indexed by  $x \in X$  satisfying:*

- all  $F_x$  act the same on objects;
- for all  $f: A \rightarrow B$  in  $Q(v)$ , the functions  $X \supseteq \varphi^{-1}(v) \rightarrow \Lambda P$  given by

$$x \mapsto F_x([f]_{\varphi(x)})$$

are continuous, where  $[-]_x: P_u \rightarrow P_x = \mathrm{colim}_{x \in u} P_u$  are the stalk maps, and  $\Lambda P = \coprod_{A, B \in P_x, x \in X} P_x(A, B)$  has a base of open sets  $\{[g]_x \mid x \in u\}$  for  $g: A \rightarrow B$  in  $P(u)$ ;

- the coherence morphisms  $\theta_I^x: I \rightarrow F_x(I)$  in  $Q_x$  form a continuous function  $X \rightarrow \prod_{x \in X} Q_x(I, F_x(I))$ , where the latter has a base of open sets  $\{[\theta_I^u]_x \mid x \in u\}$  for  $\theta_I^u: I \rightarrow F_x(I)$  in  $Q_u$ ;
- the coherence morphisms  $\theta_{A,B}^x: F_x(A) \otimes F_x(B) \rightarrow F_x(A \otimes B)$  in  $Q_x$  form a continuous function  $X \rightarrow \prod_{A,B \in P(x), x \in X} Q_x(F_x(A) \otimes F_x(B), F_x(A \otimes B))$ , where the latter has a base of open sets  $\{[\theta_{A,B}^u]_x \mid x \in u\}$  for  $\theta_{A,B}^u: F_x(A) \otimes F_x(B) \rightarrow F_x(A \otimes B)$  in  $Q_u$ .

*Proof.* Fix a continuous function  $\varphi: X \rightarrow Y$ . Given  $\{F_v\}_{v \in \mathcal{O}(Y)}$ , take  $F_x$  to be the morphism  $Q_{\varphi(x)} \rightarrow P_x$  in **MonCat** induced by the fact that the stalk  $P_x$  is the colimit of  $P_{\varphi^{-1}(v)}$  over  $x \in \varphi^{-1}(v)$ , and the stalk  $Q_{\varphi(x)}$  is the colimit of  $Q_v$  over  $\varphi(x) \in v$ . This satisfies the conditions by construction.

Conversely, suppose given  $\{F_x\}_{x \in X}$ , and fix  $v \in \mathcal{O}(Y)$ . On objects, set  $F_v(A) = F_x(A)$  for any  $x \in X$ . Given  $f: A \rightarrow B$  in  $Q_v$ , the continuity condition lets us pick opens  $v_x$  around each point  $x \in u = \varphi^{-1}(v)$  and  $g_x: A \rightarrow B$  in  $P(\varphi^{-1}(v_x))$  with  $F_x([f]_{\varphi(x)}) = [g_x]_x$ . By universality of joins in  $P_u$ , we can paste these into  $F_v(f) := \bigvee_{x \in u} g_x: A \rightarrow B$  in  $P_u$ . In the sublocal case,  $X$  is a spectral space and hence compact, so finitely many  $v_x$  already cover  $X$ , and universal finite joins of central idempotents in  $P_u$  suffice to define  $F(f)$ . Lemma 3.2 makes  $F$  functorial.

Similarly, there are opens  $v_x$  around each point  $x \in u$  and  $\theta_I^{v_x}: I \rightarrow F(I)$  and  $\theta_{A,B}^{v_x}: F(A) \otimes F(B) \rightarrow F(A \otimes B)$  in  $P_{\varphi^{-1}(v_x)}$  with  $[\theta_I^{v_x}]_x = \theta_I^x$  and  $[\theta_{A,B}^{v_x}]_x = \theta_{A,B}^x$ . These paste into  $\theta_I := \bigvee_x \theta_I^{v_x}: I \rightarrow F(I)$  and  $\theta_{A,B} := \bigvee_x \theta_{A,B}^{v_x}: F(A) \otimes F(B) \rightarrow F(A \otimes B)$  in **D**. These make the functor  $F_v$  into a morphism in **MonCat** by Lemma 3.2 because  $\theta_I^x$  and  $\theta_{A,B}^x$  make  $F_x$  into a morphism in **MonCat** for each point  $x$ . Thus we have defined a natural transformation  $F_v: Q_v \rightarrow P_{\varphi^{-1}(v)}$ .

It is straightforward to verify that these two constructions are inverses.  $\square$

We can now recognise (11.1) as a condition on the morphisms  $F_x$  between (sub)local categories, to make the previous lemma into an equivalence.

**Definition 11.5.** A morphism of (sub)local monoidal categories  $F: \mathbf{C} \rightarrow \mathbf{D}$  is a morphism in  $\mathbf{MonCat}_{(\mathbf{f}, \mathbf{j})}$  that is conservative on central idempotents: if  $F(u) = 1$ , then the central idempotent  $u$  itself must be 1.

If  $\mathbf{C}$  and  $\mathbf{D}$  are toposes, the previous definition means that  $F$  reflects all isomorphisms [15, Definition 3.3.2]. If  $\mathbf{C}$  and  $\mathbf{D}$  are the frames of opens of topological spaces, it is equivalent to the continuous function preserving focal points.

**Proposition 11.6.** Let  $P: \mathcal{O}(X)^{\text{op}} \rightarrow \mathbf{MonCat}$  and  $Q: \mathcal{O}(Y)^{\text{op}} \rightarrow \mathbf{MonCat}$  be sheaves of (sub)local categories, let  $\varphi: X \rightarrow Y$  be a continuous function, and let  $F: Q \Rightarrow \varphi_* P$  be a natural transformation. Condition (11.1) holds if and only if the induced functors  $F_x$  on stalks are morphisms of (sub)local monoidal categories.

*Proof.* It suffices to see that if  $F: \mathbf{C} \rightarrow \mathbf{D}$  is a morphism in  $\mathbf{MonCat}_{(\mathbf{f}, \mathbf{j})}$  such that each  $F_x$  is a morphism of (sub)local monoidal categories,  $F_1(u) \in x \iff u \in \varphi(x)$ .

For a central idempotent  $u$  in  $\mathbf{C}$ , observe that  $F_1(u) \in x$  if and only if  $F_1(u) \sim_x 1$ , if and only if  $F_x([u]_{\varphi(x)}) = 1$ . Definition 11.5 makes this equivalent to  $[u]_{\varphi(x)} = 1$ . But that holds exactly when  $u \in \varphi(x)$ .  $\square$

## 12. EMBEDDING

The goal of this section is to prove that any small stiff monoidal category can be freely completed with universal (finite) joins of central idempotents. The guiding idea will be that objects of the completion of  $\mathbf{C}$  with universal (finite) joins can be thought of as formal colimits of the appropriate kind.

**Definition 12.1.** A morphism  $f: A \rightarrow B$  in a monoidal category *restricts* to a central idempotent  $u$  if it factors through  $B \otimes u$  via some  $g: A \rightarrow U$ .

$$\begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 & \searrow g & \uparrow \rho_B \circ (B \otimes u) \\
 & & B \otimes U
 \end{array}
 \quad = \quad
 \begin{array}{ccc}
 B & & B \\
 | & & | \circ U \\
 \boxed{f} & = & \boxed{g} \\
 | & & | \\
 A & & A
 \end{array}$$

It is clear that if  $u \leq v$  are central idempotents, then any morphisms  $f: A \otimes U \rightarrow B$  restricts to  $v$ .

**Proposition 12.2.** For a small stiff monoidal category  $\mathbf{C}$ , there is a category  $D[\mathbf{C}]$ :

- objects are pairs  $\langle D, A \rangle$  of a down-closed  $D \subseteq \text{ZI}(\mathbf{C})$  and an object  $A \in \mathbf{C}$ ;
- morphisms  $\langle D, A \rangle \rightarrow \langle E, B \rangle$  in  $D[\mathbf{C}]$  are families  $\{\eta_u: A \otimes U \rightarrow B\}_{u \in D}$  of morphisms in  $\mathbf{C}$  such that each  $\eta_u$  restricts to some  $v \in E$ , and if  $u \leq u'$ :<sup>2</sup>

$$\begin{array}{ccc}
 B & & B \\
 | & & | \\
 \boxed{\eta_u} & = & \boxed{\eta_{u'}} \\
 | \quad | & & | \quad | \\
 A \quad U & & A \quad U'
 \end{array}$$

- the identity on  $\langle D, A \rangle$  has components  $A \otimes u: A \otimes U \rightarrow D$ ;
- Composition of  $\eta: \langle D, A \rangle \rightarrow \langle E, B \rangle$  and  $\zeta: \langle E, B \rangle \rightarrow \langle F, C \rangle$  is given by  $(\zeta \hat{\circ} \eta)_u = \zeta_v \circ \tilde{\eta}_{u,v}$  for (any)  $v \in E$  and  $\tilde{\eta}_{u,v}: U \otimes A \rightarrow V \otimes B$  satisfying  $\eta_u = (B \otimes v) \circ \tilde{\eta}_{u,v}$ .

$$\begin{array}{ccc}
 A \otimes U & \xrightarrow{\eta_u} & B \\
 (\zeta \hat{\circ} \eta)_u \downarrow & \searrow \tilde{\eta}_{u,v} & \uparrow B \otimes v \\
 C & \xleftarrow{\zeta_v} & B \otimes V
 \end{array}$$

*Proof.* We prove that the composition is independent of the choice of  $v$  and  $\tilde{\eta}_{u,v}$ . Suppose that  $v, v' \in E$  satisfy  $v \leq v'$ , and that  $\eta_u$  restricts to  $v$  via  $\tilde{\eta}: A \otimes U \rightarrow B \otimes V$ , and hence also to  $v'$  via a morphism  $\tilde{\eta}' = (B \otimes m_{v,v'}) \circ \tilde{\eta}$ . Then:

$$(*) \quad
 \begin{array}{ccccc}
 & A \otimes U & \xrightarrow{\tilde{\eta}'} & B \otimes V' & \\
 & \eta_u \swarrow & \downarrow \tilde{\eta} & \searrow & \downarrow \zeta_{v'} \\
 B & \xleftarrow{B \otimes v'} & B \otimes V & \xrightarrow{\zeta_v} & C
 \end{array}$$

Now suppose  $\eta_u$  restricts to  $v_1$  and  $v_2$  in  $E$  via  $\tilde{\eta}_i: A \otimes U \rightarrow B \otimes V_i$ . By stiffness, it then restricts to  $v_1 \wedge v_2$  via  $\tilde{\eta}: A \otimes U \rightarrow B \otimes V_1 \otimes V_2$ :

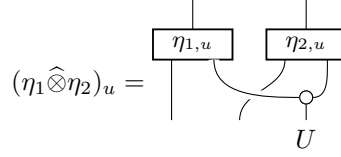
$$\begin{array}{ccccc}
 B \otimes V_1 \otimes V_2 & \xrightarrow{B \otimes V_1 \otimes v_2} & B \otimes V_1 & & \\
 \downarrow B \otimes v_1 \otimes V_2 & \swarrow \tilde{\eta} & \uparrow \tilde{\eta}_1 & & \downarrow B \otimes v_1 \\
 & A \otimes U & & & \\
 & \swarrow \tilde{\eta}_2 & & & \\
 B \otimes V_2 & \xrightarrow{B \otimes v_2} & Y & & 
 \end{array}$$

<sup>2</sup>That is,  $\eta$  is a compatible family for the presheaf  $\mathbf{C} \parallel_{(-)}: D^{\text{op}} \rightarrow \mathbf{MonCat}$ .

Applying  $(*)$  twice, we conclude that  $\zeta_{v_1} \circ \tilde{\eta}_1 = \zeta_{v_1 \wedge v_2} \circ \tilde{\eta} = \zeta_{v_2} \circ \tilde{\eta}_1$ , so  $\zeta \hat{\circ} \eta$  is well defined. It is routine to verify that the composition is associative and satisfies the identity laws.  $\square$

**Proposition 12.3.** *If  $\mathbf{C}$  is a small stiff monoidal category, then  $D[\mathbf{C}]$  is monoidal:*

- the tensor product of objects is  $\langle D_1, A_1 \rangle \hat{\otimes} \langle D_2, A_2 \rangle = \langle D_1 \cap D_2, A_1 \otimes A_2 \rangle$ ;
- the tensor unit is  $\hat{I} = \langle \text{ZI}(\mathbf{C}), I \rangle$ ;
- the tensor product of morphisms is:



*Proof.* To see that the tensor product of morphisms is well-defined, two conditions must be verified. The first holds because  $\mathbf{C}|_{u \leq u'}$  is strict monoidal by Lemma 5.4. For the second, if  $\eta_i$  restricts to  $v_i \in E_i$ , then  $(\eta_1 \hat{\otimes} \eta_2)_u$  restricts to  $v_1 \wedge v_2$ , which is in  $E_1 \cap E_2$  because these sets are down-closed. The associator and unitors, and the coherence conditions these satisfy, now follow routinely from those in  $\mathbf{C}$ .  $\square$

Notice that if  $A \simeq B$  in  $\mathbf{C}$ , then  $\langle D, A \rangle \simeq \langle D, B \rangle$  in  $D[\mathbf{C}]$  for any  $D$ .

**Lemma 12.4.** *There is a strictly monoidal full embedding  $\mathbf{C} \rightarrow D[\mathbf{C}]$  that sends an object  $A$  to  $\hat{A} = \langle \text{ZI}(\mathbf{C}), A \rangle$  and a morphism  $f$  to  $\{f \otimes u\}_{u \in \text{ZI}(\mathbf{C})}$ .*

*Proof.* This is clearly a functor that is injective on objects. If  $E \subseteq \text{ZI}(\mathbf{C})$  is down-closed and  $u \in E$ , then  $D[\mathbf{C}](\langle \downarrow u, A \rangle, \langle (\downarrow) E, B \rangle) \simeq \mathbf{C}(A \otimes U, Y) \simeq \mathbf{C}|_u(A, B)$ . It follows that  $D[\mathbf{C}](\hat{A}, \hat{B}) \simeq \mathbf{C}(A, B)$ , so the functor is full. Finally, observe that  $\hat{I}$  is the monoidal unit in  $D[\mathbf{C}]$ , and that  $\widehat{A \otimes B} = \hat{A} \otimes \hat{B}$ .  $\square$

**Proposition 12.5.** *If  $\mathbf{C}$  is a small stiff monoidal category, there is an isomorphism of partially ordered sets  $\text{ZI}(D[\mathbf{C}]) \simeq \{D \subseteq \text{ZI}(\mathbf{C}) \mid D = \downarrow D\}$  that sends a down-closed  $D \subseteq \text{ZI}(\mathbf{C})$  to the morphism  $\langle D, I \rangle \rightarrow \hat{I}$  with component  $u \circ \lambda_I$  at  $u \in D$ .*

*Proof.* For down-closed  $D \subseteq \text{ZI}(\mathbf{C})$ , define  $\hat{D} = \langle D, I \rangle$ , and write  $d: \hat{D} \rightarrow \hat{I}$  for the morphism in  $D[\mathbf{C}]$  given by  $d_u = u \circ \lambda_I$  for each  $u \in D$ .

We first prove that  $d$  is a central idempotent in  $D[\mathbf{C}]$ . Note that  $\hat{D} \hat{\otimes} \hat{D} = \langle D, I \otimes I \rangle$ . Now  $d_u = u \circ \lambda_I = \lambda_I \circ (I \otimes u) = \rho_I \circ (I \otimes u) = \hat{D}_u$ , so the components of  $d$  and the identity on  $\hat{D}$  are the same morphisms in  $\mathbf{C}$ . Hence  $d \hat{\otimes} \hat{D} = \hat{D} \hat{\otimes} d$ , and this has an inverse with components  $I \otimes u: I \otimes U \rightarrow I \otimes I$ .

The assignment  $D \mapsto d$  is a semilattice morphism, because  $\widehat{D \cap E} \simeq \hat{D} \hat{\otimes} \hat{E}$  via an isomorphism with components  $\rho_U \circ (u \otimes \lambda_I) = \rho_u \circ (u \otimes \rho_I): U \otimes I \otimes I \rightarrow I$ .

Moreover, if  $d \leq e$  in  $D[\mathbf{C}]$  then  $d$  factors through  $e$  via a map  $\hat{D} \rightarrow \hat{E}$ . Its component at  $u \in D$  must be  $u: U \rightarrow I$  factoring via  $v: V \rightarrow I$  for some  $v \in E$ . So any  $u \in D$  is below some  $v \in E$ , so  $D \subseteq E$ . Thus the map  $D \mapsto d$  is injective. It remains to show that it is surjective.

Let  $\eta: \langle D, A \rangle \rightarrow \hat{I}$  be a central idempotent in  $D[\mathbf{C}]$ . It suffices to prove that each  $\eta_u: A \otimes U \rightarrow I$  is a central idempotent in  $\mathbf{C}$ , for then the maps  $\eta_u$  themselves evidently restrict to  $\eta_u$  and form an isomorphism  $\langle D, A \rangle \rightarrow \widehat{\{\eta_u\}_{u \in D}}$ .



From  $(\langle D, A \rangle \widehat{\otimes} \eta)_u = (\eta \widehat{\otimes} \langle D, A \rangle)_u$ , it follows that

$$\begin{array}{c} | \\ A \end{array} = \begin{array}{c} \text{---} \eta_u \text{---} \\ \text{---} \text{---} \text{---} \\ A \quad A \quad U \end{array} = \begin{array}{c} \text{---} \eta_u \text{---} \\ \text{---} \text{---} \text{---} \\ A \quad A \quad U \end{array}$$

and upon simplification and tensoring with  $U$ , we find  $\eta \otimes A \otimes U = A \otimes U \otimes \eta$ .

Invertibility of  $\langle D, A \rangle \widehat{\otimes} \eta$  gives a map  $\zeta: \langle D, A \rangle \rightarrow \langle D, A \otimes A \rangle$  such that:

$$\begin{array}{c} A \quad A \\ \text{---} \zeta_u \text{---} \quad \text{---} \eta_u \text{---} \\ \text{---} \text{---} \text{---} \\ A \quad A \quad U \end{array} = \begin{array}{c} A \quad A \\ | \quad | \\ A \quad A \quad U \end{array} \quad \begin{array}{c} A \\ \text{---} \eta_u \text{---} \\ \text{---} \zeta_u \text{---} \\ A \quad U \end{array} = \begin{array}{c} A \\ | \\ A \quad U \end{array}$$

Precomposing with  $(U \otimes u)^{-1}$  (twice on the left, once on the right) and simplifying shows that

$$\begin{array}{c} A \quad A \quad U \quad U \\ \text{---} \zeta_u \text{---} \quad \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ A \quad U \end{array}$$

inverts  $A \otimes U \otimes \eta_u$ , and so  $\eta_u$  is a central idempotent in  $\mathbf{C}$ .  $\square$

**Lemma 12.6.** *If  $\mathbf{C}$  is a small stiff monoidal category, then  $D[\mathbf{C}]$  has universal joins of central idempotents.*

*Proof.* It follows from Proposition 12.5 that  $\text{ZI}(D[\mathbf{C}])$  is the free frame on the semilattice  $\text{ZI}(\mathbf{C})$ , so it certainly has arbitrary suprema.

Let  $\{D_i\}$  be a family of down-closed subsets of  $\text{ZI}(\mathbf{C})$  closed under intersection, so  $\{\widehat{D}_i\}$  is a family of central idempotents of  $D[\mathbf{C}]$  closed under meets. Let  $\langle E, B \rangle$  be an object in  $D[\mathbf{C}]$ . Consider the diagram in  $D[\mathbf{C}]$  consisting of the objects  $\langle E, B \rangle \widehat{\otimes} \widehat{D}_i \simeq \langle D_i \cap E, B \rangle$  and morphisms  $\langle E, B \rangle \widehat{\otimes} \mu_{D_i, D_j}: \langle E, B \rangle \widehat{\otimes} \widehat{D}_i \rightarrow \langle E, B \rangle \widehat{\otimes} \widehat{D}_j$  whenever  $D_i \subset D_j$ . The components of these maps are just  $B \otimes u: B \otimes U \rightarrow B$  for each  $u \in D_i \cap E$ , and evidently restrict to  $u \in D_i \cap E \subset D_j \cap E$  itself.

We show that  $\langle E, B \rangle \widehat{\otimes} \bigvee \widehat{D}_i = \langle E, B \rangle \widehat{\otimes} \widehat{\bigcup D_i} \simeq \langle D_i \cap E, B \rangle$  with cocone maps

$$\langle E, B \rangle \widehat{\otimes} \mu_{D_i, \bigcup D_i}: \langle E, B \rangle \widehat{\otimes} \widehat{D}_i \rightarrow \langle E, B \rangle \widehat{\otimes} \widehat{\bigcup D_i}$$

is the colimit of this diagram. Suppose that there were another cocone with vertex  $\langle F, C \rangle$  and morphisms  $\zeta_i: \langle E \cap D_i, B \rangle \rightarrow \langle F, C \rangle$ . A mediating morphism  $\eta: \langle E \cap \bigcup D_i, B \rangle \rightarrow \langle F, C \rangle$  must satisfy  $\zeta_i = \eta \widehat{\otimes} (\langle E, B \rangle \widehat{\otimes} \mu_{D_i, \bigcup D_i})$ . Component-wise, by the definition of composition in  $D[\mathbf{C}]$ , this yields for any  $u \in E \cap D_i$ :

$$\begin{array}{ccc} Y \otimes U & \xrightarrow{\mu_{D_i, \bigcup D_i}} & U \\ \zeta_{i,u} \downarrow & \text{---} & \uparrow Y \otimes u \\ Z & \xleftarrow{\eta_u} & Y \otimes U \end{array}$$

Hence,  $u \in E \cap \bigcup D_i$  forces  $\eta_u = \zeta_{i,u}$  for any  $i$  with  $u \in E \cap D_i$ . Hence mediating morphisms is unique. It remains to show that this mediating morphism is well defined.

First, we show that the definition of  $\nu_u$  does not depend on  $i$ . Suppose that  $u \in E \cap D_i$  and  $u \in E \cap D_j$ . Then  $u \in E \cap (D_i \cap D_j)$ , and we write  $i \wedge j$  for the index such that  $D_{i \wedge j} = D_i \cap D_j$ . By the fact that  $\{\zeta_i\}$  is a cocone,  $\zeta_{i \wedge j} = \zeta_i \widehat{\circ} (\langle E, B \rangle \widehat{\otimes} \mu_{i \wedge j, i})$ . Since the component of  $\mu_{i \wedge j, i}$  at  $u$  is  $Y \otimes u = (B \otimes u) \circ (B \otimes U)$ , we get  $\zeta_{i \wedge j, u} = \zeta_{i, u} \circ (B \otimes U) = \zeta_{i, u}$ . Consequently,  $\zeta_{i, u} = \zeta_{i \wedge j, u} = \zeta_{j, u}$  as required.

Next, we check that  $\eta$  is indeed a morphism in  $D[\mathbf{C}]$ . Given  $u \leq u' \in E \cap \bigcup D_i$ , the restriction condition  $\eta_u = \eta_{u'} \circ m_{u, u'}$  follows from the corresponding condition for  $\zeta_i$  for any  $i$  such that  $u' \in E \cap D_i$  (and thus also  $u \in E \cap D_i$ ). Finally, each  $\eta_u$  evidently restricts to a central idempotent in  $F$  since all  $\zeta_{i, u}$  do.  $\square$

An analogous construction can be carried out for finite joins of central idempotents, by considering only finitely generated downsets, which form free distributive lattice on a meet-semilattice. This proves the Corollary from the introduction.

**Corollary 12.7.** *Any small stiff monoidal category allows a central idempotent-preserving monoidal embedding into a category of global sections of a sheaf of (sub)local monoidal categories.*

*Proof.* Combine Theorem 8.6 with Lemmas 12.4 and 12.6.  $\square$

Finally, we show that  $D[\mathbf{C}]$  is in fact the free such completion of  $\mathbf{C}$ .

**Theorem 12.8.** *If  $\mathbf{C}$  is a small stiff monoidal category,  $\mathbf{D}$  is a monoidal category with universal joins of central idempotents, and  $F: \mathbf{C} \rightarrow \mathbf{D}$  a morphism in  $\mathbf{MonCat}_s$ , then there is a unique morphism  $\widehat{F}: D[\mathbf{C}] \rightarrow \mathbf{D}$  in  $\mathbf{MonCat}_j$  with:*

$$\begin{array}{ccc} \mathbf{C} & \xrightarrow{\widehat{(-)}} & D[\mathbf{C}] \\ & \searrow F & \downarrow \widehat{F} \\ & & \mathbf{D} \end{array}$$

*Proof.* Let  $\langle D, A \rangle$  be an object in  $D[\mathbf{C}]$ . Consider the diagram in  $\mathbf{D}$  with objects  $F(U) \otimes F(A)$  for  $u \in D$  and morphisms whenever  $u \leq u' \in D$ . Note that  $F(U) \rightarrow F(U')$  is the morphism in  $\mathbf{D}$  witnessing  $F(u) \leq F(u')$  as central idempotents in  $\mathbf{D}$ . Because  $D$  is closed under meets, and  $F$  induces a semilattice homomorphism  $\text{ZI}(\mathbf{C}) \rightarrow \text{ZI}(\mathbf{D})$ , also  $\{F(u) \mid u \in D\}$  is closed under meets. Because  $\mathbf{D}$  has universal joins of central idempotents, the diagram under consideration has a colimit, given by the image of  $\langle D, A \rangle$  under  $\widehat{F}$ :

$$\widehat{F}(\langle D, A \rangle) = \text{colim}_{u \in D} F(A) \otimes F(U)$$

Now, let  $\eta: \langle D, A \rangle \rightarrow \langle E, B \rangle$  be a morphism in  $D[\mathbf{C}]$ . Writing  $\theta$  for the coherence morphism witnessing that  $F$  is monoidal, the morphisms

$$F(A) \otimes F(U) \xrightarrow{\theta_{A, U}} F(A \otimes U) \xrightarrow{F(\eta_u)} Y$$

form a cocone for the diagram in  $\mathbf{D}$  with objects  $\{F(A) \otimes F(U)\}_{u \in D}$  and morphisms induced by  $u \leq u'$ , since the following diagram commutes:

$$\begin{array}{ccccc} F(A) \otimes F(U') & \xrightarrow{\theta_{A, U'}} & F(A \otimes U') & \xrightarrow{F(\eta_{u'})} & F(B) \\ \uparrow & & \uparrow & \nearrow F(\eta_u) & \\ F(A) \otimes F(U) & \xrightarrow{\theta_{A, U}} & F(A \otimes U) & & \end{array}$$

This gives a mediating morphism  $h: \widehat{F}(\langle D, A \rangle) \rightarrow F(B)$  in  $\mathbf{D}$ .

Finally, each  $\eta_u$  restricts to some  $v \in E$ :

$$\begin{array}{ccc} A \otimes U & \xrightarrow{\eta_u} & B \\ & \searrow \tilde{\eta} & \nearrow B \otimes v \\ & & B \otimes V \end{array}$$

Hence the morphisms  $F(\eta_u) \circ \theta_{A,U}$  in the cocone restrict to  $F(v)$ :

$$\begin{array}{ccccc} & & & & F(\eta_u) \\ & & & & \curvearrowright \\ F(A) \otimes F(U) & \xrightarrow{\theta_{A,U}} & F(A \otimes U) & \xrightarrow{F(\tilde{\eta})} & F(B \otimes V) & \xrightarrow{F(B \otimes v)} & F(Y) \\ & & \searrow \tilde{\eta} & & \uparrow \theta_{B,V} & & \uparrow F(B) \otimes \theta_I^{-1} \simeq \\ & & & & F(B) \otimes F(V) & \xrightarrow{F(B) \otimes F(v)} & F(B) \otimes F(I) \end{array}$$

Consequently, the mediating morphism  $h$  factors through the colimit of the diagram in  $\mathbf{D}$  of objects  $\{F(B) \otimes F(V)\}_{v \in E}$  and morphisms induced by  $v \leq v' \in E$ , that is, through  $\widehat{F}(\langle E, B \rangle)$ . We define  $\widehat{F}(\eta)$  to be this factor  $\widehat{F}(\langle D, A \rangle) \rightarrow \widehat{F}(\langle E, B \rangle)$ .

It is routine to verify that this is functorial. Because

$$\widehat{F}\left(\bigvee_i \overline{D_i}\right) = \widehat{F}\left(\overline{\bigcup_i D_i}\right) = \text{colim}_{u \in \bigcup_i D_i} F(U) = \bigvee_i \text{colim}_{u \in D_i} F(U) = \bigvee_i \widehat{F}(\overline{D_i}).$$

it preserves joins of central idempotents.  $\square$

In other words,  $\mathbf{C} \mapsto D[\mathbf{C}]$  is functorial and left adjoint to the forgetful functor  $\mathbf{MonCat}_j \rightarrow \mathbf{MonCat}_s$ . Similarly, the forgetful functor  $\mathbf{MonCat}_{\mathfrak{f}} \rightarrow \mathbf{MonCat}_s$  has a left adjoint.

This generalises and improves [23, Theorem 10.4] in two ways. First, that result is based on subunits (see Appendix A). Second, it encodes objects of  $D[\mathbf{C}]$  as presheaves on  $\mathbf{C}$ . Both lead to somewhat ad-hoc steps in the proof that do not generalise to central idempotents easily (namely Lemma 8.3 and Proposition 9.6 of [23]). The results of this section are more general and conceptually cleaner.

### 13. FURTHER WORK

Theorem 8.6 and Lemma 9.5 do for multiplicative linear logic what earlier results [1] do for higher-order intuitionistic logic. This raises several questions.

- Does our construction preserve additive connectives too, so that our main result can be extended from the multiplicative fragment to full linear logic? We expect that weakly distributive categories [18] are global sections of sheaves of (sub)local weakly distributive categories.
- The representation result for toposes is often used to obtain logical completeness theorems [42, 1]. Can we derive a similar completeness result for (multiplicative) linear logic? Concrete questions towards this aim include: if a morphism is stalkwise epi/mono/iso, must it be epi/mono/iso? Similarly, one could study converses to the results in Section 9, for example: if all stalks  $\mathbf{C}|_x$  are closed, must  $\mathbf{C}$  itself be closed?
- Completeness theorems for linear logic give rise to coherence proofs for symmetric monoidal closed categories [49]. Does a coherence theorem follow from our construction?

- What does a central idempotent signify in a model of linear logic, and when is the model local? Example 10.6 gives one first step. For another, consider the model of linear logic given by a *Petri net* [21]. Every Petri net generates a monoidal category [4] whose objects are markings and whose morphisms are firing sequences. Central idempotents then correspond to full subnets with one place and two transitions, where both transitions have no other incoming or outgoing edges, and the place is marked by either none or a nonzero number of tokens.



Additionally, the structure of the sheaf representation raises several questions.

- The stalks  $\mathbf{C}|_x$  are closely related to certain restriction categories called tensor-restriction categories [27]. Is there a general representation theorem for restriction categories as categories of global sections?
- Corollary 12.7 first embedded a semilattice of central idempotents into a distributive lattice (or frame) to obtain a sheaf representation. But a spectrum can be defined directly for a semilattice [8]. Does the representation Theorem 8.6 extend to a sheaf on the spectrum of the semilattice of central idempotents of a stiff monoidal category without universal joins directly?
- The prime ideal (Zariski) spectrum of the central idempotents of a monoidal category is analogous to the Pierce spectrum, which is the prime ideal spectrum of the central idempotents, of a ring [32]. Is there an analogon for monoidal categories of the prime spectrum of a ring?

Finally, there are many applications of localisation of rings in algebra, algebraic geometry, and tensor-triangular geometry [5, 6, 14, 13]. We have not yet explored such applications of the representation theorem. For example, Theorem 8.6 may carry generalisations of simplicity assumptions in linear categories [38], and Corollary 12.7 may let us drop simplicity assumptions from embedding theorems [26].

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## APPENDIX A. SUBUNITS

In this appendix we compare central idempotents with *subunits*, which are central idempotents  $u: U \rightarrow I$  that are monic. There is an inclusion  $\text{ISub}(\mathbf{C}) \subseteq \text{ZI}(\mathbf{C})$ ; see [27, Lemma 2.2]. In a so-called *firm* braided monoidal category, the subunits form a semilattice, and this inclusion is an embedding of semilattices. However,  $\text{ZI}(\mathbf{C})$  does not need  $\mathbf{C}$  to be braided, allowing more examples such as Example 2.9 and Example 2.13 above. Additionally, several conditions simplify; for example, central idempotents do not require firmness. The next definition exhibits a condition that holds very often, under which the subunits and central idempotents coincide.

**Definition A.1.** Call a monoidal category *bilinear* when morphisms  $f, g: A \rightarrow B \otimes U$  for a central idempotent  $u$  are equal if (and only if)  $f \otimes U = g \otimes U$ .

**Lemma A.2.** *If a braided monoidal category  $\mathbf{C}$  is bilinear, then  $\text{ISub}(\mathbf{C}) = \text{ZI}(\mathbf{C})$ .*

*Proof.* Let  $u: U \rightarrow I$  be a central idempotent. Suppose that  $u \circ f = u \circ g$  for  $f, g: A \rightarrow U$ . Then  $(u \otimes U) \circ (f \otimes U) = (u \otimes U) \circ (g \otimes U)$ , and hence  $f \otimes U = g \otimes U$ . But bilinearity now implies  $f = g$ .  $\square$

Any posetal category is bilinear, including semilattices, frames, and quantales. Any sheaf category is bilinear, because there is only one function into the empty set (namely the empty function, which has the empty set as domain). More generally, the following lemma shows that any cartesian category is bilinear.

**Lemma A.3.** *Any cartesian category is bilinear.*

*Proof.* Observe that a cartesian category satisfies the following property:

$$\begin{array}{ccc}
 & \overset{A \otimes U}{\curvearrowright} & \\
 & A \otimes U & \\
 A \otimes u \nearrow & & \searrow f \otimes u \\
 A & \xrightarrow{f} & B \otimes U \\
 & \dashrightarrow e & \\
 & & 
 \end{array}$$

Namely, given  $f$ , take  $e = \langle A, f \circ \pi_2 \rangle$ . In fact, notice that  $e$  is independent of  $f$ , because  $f \circ \pi_2$  is the unique morphism from  $A$  into the subterminal object  $U$  by Lemma 5.6. Now if  $f, g: A \rightarrow B \otimes U$  satisfy  $f \otimes U = g \otimes U$ , then  $f = (f \otimes u) \circ e = (g \otimes u) \circ e = g$ .  $\square$

The next two lemmas show that categories of (Hilbert) modules (over nonunital rings) are bilinear too.

**Lemma A.4.** *If  $R$  is a nondegenerate firm commutative ring, then the category  $\mathbf{Mod}_R$  of nondegenerate firm  $R$ -modules and linear maps is bilinear.*

*Proof.* By Example 2.10, a central idempotent in  $\mathbf{Mod}_R$  is a morphism  $u: U \rightarrow S$  for an ideal  $S \subseteq R$  that is nondegenerate, firm, and idempotent [23, Proposition 3.11]. If  $B$  is an  $R$ -module, then  $B \otimes U$  is the submodule  $\{\varphi \cdot b \mid \varphi \in S, b \in B\}$ . Suppose that  $f, g: A \rightarrow B \otimes U$  are linear maps, such that  $f \otimes U = g \otimes U$ . The latter

means that  $f(u(\varphi) \cdot a) = g(u(\varphi) \cdot a)$  for any  $a \in A$  and  $\varphi \in U$ . Let  $a \in A$ . Because  $U \otimes U \simeq U$ , then  $f(a) \in B \otimes U$  corresponds to  $u(\varphi) \cdot f(a) \in B \otimes U \otimes U$  for some  $\varphi \in U$ . But  $u(\varphi) \cdot f(a) = f(u(\varphi) \cdot a) = g(u(\varphi) \cdot a) = u(\varphi) \cdot g(a)$ , so  $f(a) = g(a)$ . Thus  $f = g$ .  $\square$

**Lemma A.5.** *If  $X$  is a locally compact Hausdorff space,  $\mathbf{Hilb}_{C_0(X)}$  is bilinear.*

*Proof.* As in Example 2.12, a central idempotent  $U$  in  $\mathbf{C} = \mathbf{Hilb}_{C_0(X)}$  is of the form  $U = \{\varphi \in C_0(X) \mid \varphi(X \setminus U) = 0\}$  for an open set  $U \subseteq X$  [23, Proposition 3.16]. If  $B$  is a Hilbert  $C_0(X)$ -module, then  $B \otimes U$  is the submodule  $\{b \in B \mid \langle b \mid b \rangle(X \setminus U) = 0\}$ . Suppose that  $f, g: A \rightarrow B \otimes U$  are bounded  $C_0(X)$ -linear maps, and that  $f \otimes U = g \otimes U$ . The latter means that  $f(a) = g(a)$  for  $a \in A$  with  $\langle a \mid a \rangle(X \setminus U) = 0$ . Now pick a net  $\varphi_n \in C(X)$  such that  $\varphi_n(X \setminus U) = 0$  but  $\lim_n \varphi_n(x) = 1$  for every  $x \in U$ . Then  $\varphi_n \cdot f(a) = f(\varphi_n \cdot a) = g(\varphi_n \cdot a) = \varphi_n \cdot g(a)$  for every  $a \in A$ . Moreover,  $\lim_n \langle (\varphi_n - 1)f(a) \mid (\varphi_n - 1)g(a) \rangle(x)$  vanishes for every  $x \in X$ : for if  $x \notin U$  then the limit equals  $\lim \langle f(a) \mid f(a) \rangle(x) = 0$ ; but if  $x \in U$  then  $\varphi_n - 1$  tends to zero and so the limit vanishes too. Thus  $f(a) = \lim_n \varphi_n \cdot f(a) = \lim_n \varphi_n \cdot g(a) = g(a)$  for every  $a$ , that is,  $f = g$ .  $\square$

INL – INTERNATIONAL IBERIAN NANOTECHNOLOGY LABORATORY, BRAGA, PORTUGAL  
*Email address:* `rui.soaesbarbosa@inl.int`

UNIVERSITY OF EDINBURGH, UNITED KINGDOM  
*Email address:* `chris.heunen@ed.ac.uk`