

AXIOMS FOR THE CATEGORY OF HILBERT SPACES

CHRIS HEUNEN AND ANDRE KORNEILL

ABSTRACT. We provide axioms that guarantee a category is equivalent to that of continuous linear functions between Hilbert spaces. The axioms are purely categorical and do not presuppose any analytical structure. This addresses a question about the mathematical foundations of quantum theory raised in reconstruction programmes such as those of von Neumann, Mackey, Jauch, Piron, Abramsky, and Coecke.

Quantum mechanics has mathematically been firmly founded on Hilbert spaces and operators between them for nearly a century [21]. There has been continuous inquiry into the special status of this foundation since [18, 5, 12]. How are the mathematical axioms to be interpreted physically? Can the theory be reconstructed from a different framework whose axioms can be interpreted physically? Such reconstruction programmes involve a mathematical reformulation of (a generalisation of) the theory of Hilbert spaces and their operators, such as operator algebras [16], orthomodular lattices [10, 17], and, most recently, categorical quantum mechanics [1, 4]. The latter uses the framework of category theory [15], and emphasises operators more than their underlying Hilbert spaces. It postulates a category with structure that models physical features of quantum theory [9]. The question of how “to justify the use of Hilbert space” [17] then becomes: which axioms guarantee that a category is equivalent to that of continuous linear functions between Hilbert spaces? This article answers that mathematical question. The axioms are purely categorical in nature, and do not presuppose any analytical structure such as continuity, complex numbers, or probabilities. The approach is similar to Lawvere’s categorical characterisation of the theory of sets [14].

1. AXIOMS

We consider six axioms on a (locally small) category \mathbf{C} . The first two impose additional structure: an involution called a dagger and a tensor product. All axioms have to respect the dagger. In particular, the right notion of inclusion is a dagger subobject, which permeates the last four axioms. Axioms three and four demand finite (co)completeness; roughly, direct sums and equalisers. The last two axioms ask that dagger subobjects behave well: intuitively, one should be able to divide by them and take directed suprema of them. More precisely:

- (D) The category is equipped with a *dagger* \dagger . This is an operation that maps each morphism $f: H \rightarrow K$ to a morphism $f^\dagger: K \rightarrow H$ such that $f^{\dagger\dagger} = f$, and $(g \circ f)^\dagger = f^\dagger \circ g^\dagger$, and $\text{id}_H^\dagger = \text{id}_A$. If $f^\dagger \circ f = \text{id}_H$ we call f a *dagger monomorphism*, and if additionally $f \circ f^\dagger = \text{id}_K$ we call it a *dagger isomorphism*.
- (T) The category is equipped with a *dagger symmetric monoidal* structure \otimes whose unit I is a *simple monoidal separator*. The former means that all the isomorphisms $(H \otimes K) \otimes L \simeq H \otimes (K \otimes L)$, $I \otimes H \simeq H \simeq H \otimes I$, and $H \otimes K \simeq K \otimes H$ are

Chris Heunen was supported by EPSRC Fellowship EP/R044759/1. Andre Kornell was supported by AFOSR MURI grant FA9550-16-1-0082.

dagger isomorphisms. The latter means that I has exactly two subobjects, and that $f = g : H \otimes K \rightarrow L$ as soon as $f \circ (h \otimes k) = g \circ (h \otimes k)$ for all $h : I \rightarrow H$ and $k : I \rightarrow K$.

- (B) Any two objects H and K have a *dagger biproduct* $H \oplus K$, and there is a zero object 0 . The latter means that for every object H there exist unique morphisms $0 \rightarrow H$ and $H \rightarrow 0$. Hence any two objects H and K have a unique morphism $0_{H,K} : H \rightarrow K$ that factors through 0 . The former means that there are dagger monomorphisms $i : H \rightarrow H \oplus K$ and $j : K \rightarrow H \oplus K$ that make $H \oplus K$ a coproduct of H and K and satisfy $j^\dagger \circ i = 0_{H,K}$ [13].
- (E) Any two morphisms $H \rightarrow K$ have a *dagger equaliser* $f : E \rightarrow H$, that is, an equaliser f that is a dagger monomorphism.
- (K) Any *dagger monomorphism* $N \rightarrow H$ is a dagger equaliser of some morphism $H \rightarrow K$ and $0_{H,K} : H \rightarrow K$.
- (C) The wide subcategory of dagger monomorphisms has *directed colimits*. This means that if (N, \leq) is a directed partially ordered set, H_n are objects, and $H_{m \leq n} : H_m \rightarrow H_n$ are dagger monomorphisms, then there is a universal object H with dagger monomorphisms $H_n \rightarrow H$ commuting with $H_{m \leq n}$.

The category **Hilb** of (real or complex) Hilbert spaces and continuous linear functions satisfies these axioms: (D) is given by adjoints, (T) by tensor product, (B) by direct sum, (E) and (K) by closed subspaces, and (C) by the completion of the directed union.

From now on, we assume a category **C** satisfying the above axioms. We will progressively investigate the structure of **C**, and eventually prove that it is in fact equivalent to **Hilb**.

2. SCALARS

We begin by looking at the morphisms $z : I \rightarrow I$. These are called *scalars*, because in any category they form a monoid under composition, with $\text{id}_I : I \rightarrow I$ as multiplicative unit. In a monoidal category as in axiom (T), this monoid is commutative, because the multiplication of two scalars $z, w : I \rightarrow I$ equals

$$(1) \quad I \simeq I \otimes I \xrightarrow{w \otimes z} I \otimes I \simeq I$$

for which an Eckmann-Hilton argument holds [9, Lemma 2.3].

The dagger of axiom (D) provides an involution on scalars. The biproducts of axiom (B) affords a matrix calculus on morphisms [9, Section 2.2.4]. In particular, there are a diagonal map $(i \ j) : I \oplus I \rightarrow I$ and a codiagonal map $(i \ j)^\dagger : I \rightarrow I \oplus I$ that let us add scalars $w, z : I \rightarrow I$:

$$(2) \quad I \xrightarrow{(i \ j)^\dagger} I \oplus I \xrightarrow{w \oplus z} I \oplus I \xrightarrow{(i \ j)} I.$$

Together with the zero morphism $0_{I,I} : I \rightarrow I$, the scalars hence form an involutive commutative semiring [9, Lemma 2.15].

Lemma 1. $\mathcal{I} = \mathbf{C}(I, I)$ is a field with involution $z \mapsto z^\dagger$.

Proof (sketch). To see that there are multiplicative inverses, let $z \in \mathcal{I}$. By axiom (E), it factors as an epimorphism $I \rightarrow H$ followed by a dagger monomorphism $f : H \rightarrow I$. It then follows from the simplicity of I in axiom (T) that f is either zero or invertible. If $f = 0$ then

$z = 0$. If f is invertible then z is an epimorphism. But then z^\dagger is monic and hence zero or invertible. In either case z is zero or invertible.

Thus \mathcal{I} must either have negative inverses, or be zerosumfree. The latter means that $w + z = 0$ implies $w = z = 0$. But this would contradict axiom (K), because the equaliser $\ker(f)$ of $f: H \rightarrow K$ and $0_{H,K}$ is zero if and only if f is a monomorphism, but according to (2) $\ker(i \ j) = 0$ if $w + z = 0$ whereas $(i \ j) \neq 0$. For details, see [7, Lemma 4.5]. \square

We will see in Proposition 5 below that in fact \mathcal{I} must be \mathbb{R} (with the trivial involution) or \mathbb{C} . Furthermore, any morphism $f: H \rightarrow K$ can be multiplied with a scalar $z: I \rightarrow I$ to get:

$$(3) \quad H \simeq I \otimes H \xrightarrow{z \otimes f} I \otimes K \simeq K.$$

Similarly, the addition of (2) works just as well for parallel morphisms $f, g: H \rightarrow K$ of arbitrary type. It follows from Lemma 1 that the morphisms $H \rightarrow K$ always form a vector space over \mathcal{I} . In particular, we may consider morphisms $I \rightarrow H$ to be *vectors*.

3. PROJECTIONS

Now consider *projections* on an object H , that is, morphisms $p: H \rightarrow H$ satisfying $p^\dagger \circ p = p$. They are partially ordered by $p \leq q$ if and only if $q \circ p = p$. We next show that arbitrary subsets of projections have least upper bounds, and there is an *orthocomplement* p^\perp satisfying $p^{\perp\perp} = p$, and $p \leq q$ if and only if $q^\perp \leq p^\perp$.

Lemma 2. *The projections on an object H form a complete ortholattice with $p^\perp = \text{id}_H - p$.*

Proof (sketch). Projections on H are in one-to-one correspondence with *dagger subobjects* of H , that is, dagger isomorphism classes of dagger monomorphisms $f: N \rightarrow H$. These are also partially ordered, by $f \leq g$ when f factors through g . Axiom (K) provides an order isomorphism between the two, see [8, Lemma 5 and Proposition 12]. By Axiom (C), these partially ordered sets furthermore have least upper bounds of directed subsets.

The orthocomplement of a dagger subobject $f: N \rightarrow H$ is given by $\ker(f^\dagger)$. Axioms (D) and (E) ensure that it indeed satisfies $f^{\perp\perp} = f$, and $f \leq g$ if and only if $g^\perp \leq f^\perp$ [8, Lemma 1].

By axioms (B) and (E), pullbacks exist, and so any two dagger subobjects have a greatest lower bound [8, Lemma 2]. But because of the orthocomplementation any two dagger subobjects then have a least upper bound. Hence any finite number of projections have a least upper bound.

Now, the least upper bound of an arbitrary subset of projections is the least upper bound of the directed family of least upper bounds of finite subsets, and hence exists. Finally, it is clear that $p^\perp = \text{id}_H - p$ is an orthocomplement. \square

Notation. Morphisms out of I will play a dual role, as we will also regard them as vectors in a Hilbert space. To distinguish between these two roles we will use the following notation. For an object H , we write \mathcal{H} for the homset $\mathbf{C}(I, H)$. For a morphism $f: H \rightarrow K$, we write $F: \mathcal{H} \rightarrow \mathcal{K}$ for the function that postcomposes with f .

We have already established that \mathcal{H} is a vector space (over \mathcal{I}), and that F is a linear function. In fact, axiom (D) makes \mathcal{H} into a Hermitian space [11, Definition 1.1]: the Hermitian form $\langle f | g \rangle$ of $f, g \in \mathcal{H}$ is $g^\dagger \circ f \in \mathcal{I}$; nondegeneracy follows from axiom (E) [20,

Lemma 2.5]. Hence we can speak about the orthocomplement $V^\perp = \{f \in \mathcal{H} \mid \forall v \in V: \langle f \mid v \rangle = 0\}$ of a subspace $V \subseteq \mathcal{H}$. A subspace V is called *closed* when $V^{\perp\perp} = V$. We now justify using the same notation for this orthocomplementation, which ostensibly differs from that of Lemma 2.

Lemma 3. *The function $p \mapsto p \circ \mathcal{H} = \{p \circ h \mid h \in \mathcal{H}\}$ is an isomorphism from the ortholattice of projections on H to the ortholattice of closed subspaces of \mathcal{H} .*

Proof. For a projection $p : H \rightarrow H$ and a vector $h \in \mathcal{H}$:

$$\begin{aligned} h \in p^\perp \circ \mathcal{H} &\iff (\text{id}_H - p) \circ h = h \iff p \circ h = 0 \\ &\iff \forall f \in \mathcal{H}: (p \circ f)^\dagger \circ h = f^\dagger \circ p \circ h = 0 \iff h \in (p \circ \mathcal{H})^\perp. \end{aligned}$$

The third equivalence uses that I is a monoidal separator as in axiom (T). Thus $p^\perp \circ \mathcal{H} = (p \circ \mathcal{H})^\perp$, and $(p \circ \mathcal{H})^{\perp\perp} = p \circ \mathcal{H}$ is indeed a closed subspace of \mathcal{H} .

Observe that $0_{H,H} \circ \mathcal{H} = \{0_{H,H}\}$, that $\text{id}_H \circ \mathcal{H} = \mathcal{H}$, and that $p \circ \mathcal{H} \subseteq q \circ \mathcal{H}$ when $p \leq q$. Hence $p \mapsto p \circ \mathcal{H}$ is a monotone function that respects orthocomplementation. It remains to show that it is an order embedding and that it is surjective.

Let p and q be projections on H . If $p \circ \mathcal{H} \subseteq q \circ \mathcal{H}$, then $q \circ p \circ f = p \circ v$ for all $f \in \mathcal{H}$ by axiom (T), and so $q \circ p = p$, that is, $p \leq q$. Therefore, the function $p \mapsto p \circ \mathcal{H}$ is an order embedding, and in particular injective.

To establish that it is surjective, let V be a closed subspace of \mathcal{H} . For a nonzero vector $h \in \mathcal{H}$, define a projection

$$p_h = (h \circ h^\dagger) \cdot (h^\dagger \circ h)^{-1} : H \rightarrow H$$

where the dot denotes the scalar multiplication of (3). Set $p = \bigvee_{v \in V \setminus \{0\}} p_v$. For each vector $v \in V \setminus \{0\}$ then $p_v \circ v = v$ and so $p \circ v = p \circ p_v \circ v = p_v \circ v = v$. Thus $V \subseteq p \circ \mathcal{H}$.

Let h be a nonzero vector in \mathcal{H} , and assume that $h \in V^\perp$. Then $p_h \circ p_v = 0$ for all $v \in V \setminus \{0\}$; in other words, $p_v \leq p_h^\perp$. It follows that $p = \bigvee_{v \in V \setminus \{0\}} p_v \leq p_h^\perp$, and so $p_h \circ p = 0$. Hence $p_h \circ \mathcal{H} \perp p \circ \mathcal{H}$, and in particular $h \perp p \circ \mathcal{H}$. Therefore any vector $h \in \mathcal{H}$ that is orthogonal to V is also orthogonal to $p \circ \mathcal{H}$. Since both V and $p \circ \mathcal{H}$ are closed subspaces of \mathcal{H} , we conclude that $p \circ \mathcal{H} \subseteq V$. Altogether, $p \circ \mathcal{H} = V$, and the function $p \mapsto p \circ \mathcal{H}$ is surjective. \square

4. ORTHOGONALITY

A Hermitian space \mathcal{H} is called *orthomodular* when $\mathcal{H} = V \oplus V^\perp$ for all closed subspaces $V \subseteq \mathcal{H}$ [11, Definition 1.2].

Lemma 4. *\mathcal{H} is an orthomodular space.*

Proof. Let V be a closed subspace of \mathcal{H} , and let v be a vector in V . Lemma 3 provides a projection p on H such that $V = p \circ \mathcal{H}$. Trivially $p \circ v \in V$. Also $p^\perp \circ v \in V^\perp$ because for all $w \in V$:

$$w^\dagger \circ (p^\perp \circ v) = ((\text{id}_H - p) \circ w)^\dagger \circ v = (w - (p \circ w))^\dagger \circ v = 0.$$

We conclude that $v = (p \circ v) + (p^\perp \circ v) \in V + V^\perp$ and more generally that $\mathcal{H} = V \oplus V^\perp$. \square

The next step uses orthogonal subsets of \mathcal{H} that can be infinite. This requires some preparation. If R is a finite set, axiom (B) shows that a coproduct $\bigoplus_R I$ of copies of I indexed by R exists; denote it by I^R . We may assume that $I^R = I$ when R is a singleton.

Write $i_{R,S} : I^R \rightarrow I^S$ for the canonical morphism induced by an inclusion $R \subseteq S$ of finite sets. Then $i_{S,T} \circ i_{R,S} = i_{R,T}$ for all finite sets $R \subseteq S \subseteq T$. Moreover, using the matrix calculus for morphisms afforded by axiom (B) [9, Section 2.2.4], it is clear that $i_{R,S}$ is a dagger monomorphism.

Now let A be an arbitrary set. Write $R \Subset A$ if R is a finite subset of A . The morphisms $i_{R,S}$ form a diagram of dagger monomorphisms in \mathbf{C} indexed by the category of finite subsets $R \Subset A$ and inclusions between them. By axiom (C) this diagram has a colimit I^A , with universal dagger monomorphisms $i_{R,A} : I^R \rightarrow I^A$. (If A is finite, the diagram has a terminal object, so both definitions of I^A agree.)

Proposition 5. *The involutive field \mathcal{I} is isomorphic to \mathbb{R} (with the trivial involution) or \mathbb{C} , and \mathcal{H} is a Hilbert space for any object H .*

Proof. In a first step, we show that $e_a = i_{\{a\},A} : I \rightarrow I^A$ is an orthonormal subset of $\mathbf{C}(I, I^A)$, where a ranges over an arbitrary set A . Because e_a is a dagger monomorphism, $\langle e_a | e_a \rangle = e_a^\dagger \circ e_a = \text{id}_{I^A}$. Similarly, for distinct $a, b \in A$:

$$\langle e_a | e_b \rangle = e_a^\dagger \circ e_b = i_{\{a\},\{a,b\}}^\dagger \circ i_{\{a,b\},A}^\dagger \circ i_{\{a,b\},A} \circ i_{\{a,b\},A} \circ i_{\{b\},\{a,b\}} = i_{\{a\},\{a,b\}}^\dagger \circ i_{\{b\},\{a,b\}} = 0,$$

because $i_{\{a\},\{a,b\}} : I^{\{a\}} \rightarrow I^{\{a,b\}}$ and $i_{\{b\},\{a,b\}} : I^{\{b\}} \rightarrow I^{\{a,b\}}$ are inclusions of distinct summands of $I^{\{a,b\}} = I^{\{a\}} \oplus I^{\{b\}}$.

Thus $\mathbf{C}(I, I^{\mathbb{N}})$, which is an orthomodular space over \mathcal{I} by Lemma 4, has an infinite orthonormal subset. It follows from Solèr's Theorem that \mathcal{I} is isomorphic to \mathbb{R} or \mathbb{C} , see [19] or [11, Theorem 1.3]. It is not isomorphic to \mathbb{H} because it is commutative [2].

Now let H be an object of \mathbf{C} . The Hermitian space $\mathbf{C}(I, H \oplus I^{\mathbb{N}})$ is an orthomodular space by Lemma 4. It has an infinite orthonormal subset because the inclusion morphism $I^{\mathbb{N}} \rightarrow H \oplus I^{\mathbb{N}}$ is a dagger monomorphism, and thus postcomposition by this inclusion morphism is a function $\mathbf{C}(I, I^{\mathbb{N}}) \rightarrow \mathbf{C}(I, H \oplus I^{\mathbb{N}})$ that preserves the Hermitian form. Appealing again to Solèr's theorem, we conclude that $\mathbf{C}(I, H \oplus I^{\mathbb{N}})$ is a Hilbert space over \mathcal{I} .

Similarly, postcomposing with the inclusion morphism $H \rightarrow H \oplus I^{\mathbb{N}}$ gives a function $\mathcal{H} \rightarrow \mathbf{C}(I, H \oplus I^{\mathbb{N}})$ that preserves the Hermitian form. It follows that the Hermitian form on \mathcal{H} is an inner product. If \mathcal{H} has an infinite orthonormal subset, then it is a Hilbert space over \mathcal{I} by Lemma 4 and Solèr's Theorem. If \mathcal{H} does not have an infinite orthonormal subset, then it is simply a finite-dimensional inner product space over \mathcal{I} and thus also a Hilbert space over \mathcal{I} . \square

We could insist on complex rather than real numbers by adding a further axiom stating that there exists a morphism $z : I \rightarrow I$ such that $z^\dagger \neq z$.

Corollary 6. *The assignment of $F : \mathcal{H} \rightarrow \mathcal{K}$ to $f : H \rightarrow K$ is a functor $\mathbf{C}(I, -) : \mathbf{C} \rightarrow \mathbf{Hilb}$.*

Proof. If $f : H \rightarrow K$ is a morphism in \mathbf{C} , we have to show that the function $F : \mathcal{H} \rightarrow \mathcal{K}$ is continuous and linear. Because $f^\dagger \circ f : H \rightarrow H$ is a morphism, postcomposition with it is a linear operator $T : \mathcal{H} \rightarrow \mathcal{H}$ by axioms (B) and (T) [9, Lemma 2.6 and Proposition 2.23]. For all $g, h \in \mathcal{H}$, we find $\langle h | T(g) \rangle = h^\dagger \circ (f^\dagger \circ f \circ g) = (f^\dagger \circ f \circ h)^\dagger \circ g = \langle T(h) | g \rangle$, so T is self-adjoint. It follows from the Hellinger-Toeplitz theorem that T is continuous [6]. Because

$$\|f \circ g\|^2 = g^\dagger \circ f^\dagger \circ f \circ g = \langle g | T(g) \rangle \leq \|g\|^2 \|T\|,$$

F has norm at most $\|T\|^{1/2}$ and is therefore continuous. \square

5. BASES

We continue the analysis of $\mathbf{C}(I, I^A)$ by showing that the orthonormal set in the proof of Proposition 5 is in fact an orthonormal basis.

Lemma 7. *For any set A , the dimension of the Hilbert space $\mathbf{C}(I, I^A)$ is the cardinality of A .*

Proof. Proposition 5 showed that $e_a = i_{\{a\}, A}$, where a ranges over A , is an orthonormal subset of $\mathbf{C}(I, I^A)$. We will show that it is an orthonormal basis. Suppose that a vector $f : I \rightarrow I^A$ is orthogonal to e_a for all $a \in A$. Given $a \in R \subseteq A$, we find $f^\dagger \circ i_{R,A} \circ i_{\{a\}, R} = f^\dagger \circ e_a = 0$. Because $i_{\{a\}, R} : I \rightarrow I^R$ are the coprojections of the coproduct I^R , therefore $f^\dagger \circ i_{R,A} = 0$ for all $R \subseteq A$. The morphism $f^\dagger : I^A \rightarrow I$ has a dagger monomorphism $\ker(f^\dagger) : K \rightarrow I^A$ as kernel. This gives dagger monomorphisms $g_R : I^R \rightarrow K$ such that $\ker(f^\dagger) \circ g_R = i_{R,A}$, for $R \subseteq A$. These form a cocone on the diagram of dagger monomorphisms $i_{R,S}$ for $R, S \subseteq A$: for if $R \subseteq S \subseteq A$, then

$$\ker(f^\dagger) \circ g_S \circ i_{R,S} = i_{S,A} \circ i_{R,S} = i_{R,A} = \ker(f^\dagger) \circ g_R,$$

and because $\ker(f^\dagger)$ is a dagger monomorphism therefore $g_R \circ i_{S,R} = g_R$.

$$\begin{array}{ccccc}
 I^{\{a\}} & \xrightarrow{i_{\{a\}, R}} & I^R & \xrightarrow{i_{R,S}} & \cdots \\
 \downarrow g_{\{a\}} & \searrow e_a & \downarrow i_{R,A} & & \downarrow 0 \\
 K & \xleftarrow{g_R} & I^A & \xrightarrow{f^\dagger} & I \\
 & \xleftarrow{\ker(f^\dagger)} & & & \\
 & \text{---} g \text{---} & & &
 \end{array}$$

Therefore there exists a unique mediating morphism $g : I^A \rightarrow K$ with $g \circ i_{R,A} = g_R$ for all $R \subseteq A$. Thus, $\ker(f^\dagger) \circ g \circ i_{R,A} = \ker(f^\dagger) \circ g_R = i_{R,A}$ for all $R \subseteq A$, and by the universal property of the colimit I^A , we conclude $\ker(f^\dagger) \circ g = \text{id}_{I^A}$. It follows that $\ker(f^\dagger)$ is an epimorphism, and being a dagger monomorphism, it is a dagger isomorphism. Therefore, $f^\dagger = 0$.

Overall, the zero vector is the only one in the Hilbert space $\mathbf{C}(I, I^A)$ that is orthogonal to each vector e_a , for $a \in A$. Thus, $\{e_a\}_{a \in A}$ is an orthonormal basis of $\mathbf{C}(I, I^A)$. \square

6. EQUIVALENCE

We are now ready to prove the main result. A functor is a dagger equivalence when it is an equivalence of categories that preserves the dagger of morphisms.

Theorem 8. *The functor $\mathbf{C}(I, -) : \mathbf{C} \rightarrow \mathbf{Hilb}$ is an equivalence of dagger categories.*

Proof. Let $f : H \rightarrow K$ be a morphism in \mathbf{C} . For all vectors $h \in \mathcal{H}$ and $k \in \mathcal{K}$ axiom (D) shows:

$$\langle f^\dagger \circ k \mid h \rangle = (f^\dagger \circ k)^\dagger \circ h = k^\dagger \circ f \circ h = \langle k \mid f \circ h \rangle.$$

Thus $\mathbf{C}(I, f^\dagger) = \mathbf{C}(I, f)^\dagger$, that is, $\mathbf{C}(I, -)$ is a dagger functor.

If morphisms $f, g : H \rightarrow K$ satisfy $\mathbf{C}(I, f) = \mathbf{C}(I, g)$, then $f \circ h = g \circ h$ for all vectors $h \in \mathcal{H}$, so $f = g$ by axiom (T). Therefore, $\mathbf{C}(I, -)$ is faithful.

Assume $\dim(\mathcal{H}) \leq \dim(\mathcal{K})$, and let $U: \mathcal{H} \rightarrow \mathcal{K}$ be a linear isometry. Choose an orthonormal basis $\{h_a\}_{a \in A}$ for the Hilbert space \mathcal{H} . Because U is an isometry, the vectors $k_a = U(h_a)$ form an orthonormal subset of the Hilbert space \mathcal{K} .

Extending the dagger monomorphisms $h_a: I \rightarrow H$ and $k_a: I \rightarrow K$ for finite subsets $R \subseteq A$ to dagger monomorphisms $h_R: I^R \rightarrow H$ and $k_R: I^R \rightarrow K$ using axiom (B) gives two cocones on the diagram $i_{R,S}$. Thus by axiom (C) there are unique mediating dagger monomorphisms $h_A: I^A \rightarrow H$ and $k_A: I^A \rightarrow K$ satisfying $h_A \circ e_a = h_a$ and $k_A \circ e_a = k_a$ for all $a \in A$, where $e_a = i_{\{a\}, A}$ as before.

Define $u = k_A \circ h_A^\dagger: H \rightarrow K$. For each $a \in A$ then:

$$\mathbf{C}(I, u)(h_a) = k_A \circ h_A^\dagger \circ h_a = k_A \circ h_A^\dagger \circ h_a \circ e_a = k_A \circ e_a = k_a = U(h_a).$$

Because $\{h_a\}_{a \in A}$ is an orthonormal basis of the Hilbert space \mathcal{H} , we conclude that $\mathbf{C}(I, u) = U$.

Any continuous linear function $\mathcal{H} \rightarrow \mathcal{K}$ is a linear combination of isometric ones, because we have assumed that $\dim(\mathcal{H}) \leq \dim(\mathcal{K})$. It follows that each continuous linear function $F: \mathcal{H} \rightarrow \mathcal{K}$ is the image of a linear combination of morphisms $H \rightarrow K$ under the functor $\mathbf{C}(I, -)$, which is linear on morphisms by axiom (B). Therefore, there exists a morphism $f: H \rightarrow K$ such that $\mathbf{C}(I, f) = F$. Combining this fact for $\dim(\mathcal{H}) \leq \dim(\mathcal{K})$ with the fact that the functor $\mathbf{C}(I, -)$ preserves daggers, we conclude that the functor is full.

Finally, the functor $\mathbf{C}(I, -)$ is essentially surjective by Lemma 7. Being a full, faithful and essentially surjective dagger functor, it is an equivalence of dagger categories. \square

Because equivalences preserve colimits, it follows directly that the functor $\mathbf{C}(I, -): \mathbf{C} \rightarrow \mathbf{Hilb}$ preserves all the structure concerned in axioms (D), (B), (E), (K), and (C).

7. TENSOR PRODUCTS

Finally, we prove that the equivalence $\mathbf{C}(I, -): \mathbf{C} \rightarrow \mathbf{Hilb}$ also preserves the tensor products of axiom (T).

For objects H and K , the function $\mathcal{H} \times \mathcal{K} \rightarrow \mathbf{C}(I, H \otimes K)$ given by $(h, k) \mapsto h \otimes k$ is bilinear by axioms (B) and (T), see [9, Lemma 3.6, Corollary 3.20, and Lemma 3.22]. It is also bounded as a bilinear map because

$$\begin{aligned} \|h \otimes k\|^2 &= \langle h \otimes k | h \otimes k \rangle = (h \otimes k)^\dagger \circ (h \otimes k) \\ &= (h^\dagger \circ h) \otimes (k^\dagger \circ k) = \langle h | h \rangle \cdot \langle k | k \rangle = \|h\|^2 \cdot \|k\|^2 \end{aligned}$$

where we suppressed the isomorphism $I \otimes I \simeq I$. This defines a continuous linear function:

$$(4) \quad \begin{aligned} M_{H,K}: \mathcal{H} \otimes \mathcal{K} &\rightarrow \mathbf{C}(I, H \otimes K) \\ h \otimes k &\mapsto h \otimes k. \end{aligned}$$

Here the tensor product on the left-hand side is that of Hilbert spaces, whereas the tensor product on the right-hand side is the monoidal structure of \mathbf{C} .

Lemma 9. *The continuous linear functions of (4) are unitary and form the components of a natural transformation from the functor $\mathbf{C}(I, -) \otimes \mathbf{C}(I, -): \mathbf{C} \times \mathbf{C} \rightarrow \mathbf{Hilb}$ to the functor $\mathbf{C}(I, - \otimes -): \mathbf{C} \times \mathbf{C} \rightarrow \mathbf{Hilb}$.*

Proof. If $\{h_a\}_{a \in A}$ is an orthonormal basis of \mathcal{H} and $\{k_b\}_{b \in B}$ is an orthonormal basis of \mathcal{K} , then $\{h_a \otimes k_b\}_{(a,b) \in A \times B}$ is an orthonormal basis of $\mathbf{C}(I, H) \otimes \mathbf{C}(I, K)$. Furthermore, $\{h_a \otimes k_b\}_{(a,b) \in A \times B}$ is an orthonormal subset of $\mathbf{C}(I, H \otimes K)$. So $M_{H,K}$ maps an orthonormal basis to an orthonormal set. It is therefore an isometry, and its range is a closed subspace of $\mathbf{C}(I, H \otimes K)$. Suppose there were a nonzero vector f orthogonal to $h \otimes k$ for all $h \in \mathcal{H}$ and $k \in \mathcal{K}$. In other words, $f^\dagger \circ (h \otimes k) = \langle f | h \otimes k \rangle = 0$. It would then follow from axiom (T) that $f^\dagger = 0$, contradicting our choice of f . Therefore, $M_{H,K}$ is surjective, and hence unitary.

If $f : H_1 \rightarrow H_2$ and $g : K_1 \rightarrow K_2$ are morphisms of \mathbf{C} , then the following diagram commutes:

$$\begin{array}{ccc} \mathbf{C}(I, H_1) \otimes \mathbf{C}(I, K_1) & \xrightarrow{M_{H_1, K_1}} & \mathbf{C}(I, H_1 \otimes K_1) \\ \mathbf{C}(I, f) \otimes \mathbf{C}(I, g) \downarrow & & \downarrow \mathbf{C}(I, f \otimes g) \\ \mathbf{C}(I, H_2) \otimes \mathbf{C}(I, K_2) & \xrightarrow{M_{H_2, K_2}} & \mathbf{C}(I, H_2 \otimes K_2) \end{array}$$

Indeed, for all $h \in \mathcal{H}_1$ and $k \in \mathcal{K}_1$:

$$\begin{aligned} \mathbf{C}(I, f \otimes g)(M_{H_1, K_1}(h \otimes k)) &= \mathbf{C}(I, f \otimes g)(h \otimes k) = (f \circ h) \otimes (g \circ k) \\ &= \mathbf{C}(I, f)(h) \otimes \mathbf{C}(I, g)(k) = M_{H_2, K_2}(\mathbf{C}(I, f)(h) \otimes \mathbf{C}(I, g)(k)) \\ &= M_{H_2, K_2}((\mathbf{C}(I, f) \otimes \mathbf{C}(I, g))(h \otimes k)). \end{aligned}$$

Therefore, the unitary linear functions $M_{H,K}$ are natural in H and K . \square

Recall that a (dagger) functor is symmetric monoidal when it preserves (the dagger and) the tensor product up to a natural (dagger) isomorphism that respects the (dagger) isomorphisms $(H \otimes K) \otimes L \simeq H \otimes (K \otimes L)$ and $H \otimes K \simeq K \otimes H$ and that respects the tensor unit I up to (dagger) isomorphism [3, Section 2.4]. If a symmetric monoidal functor is an equivalence, then its adjoint is automatically also symmetric monoidal.

Theorem 10. *The functor $\mathbf{C}(I, -) : \mathbf{C} \rightarrow \mathbf{Hilb}$ is a symmetric monoidal dagger equivalence.*

Proof. After Theorem 8, it suffices to show that the functor $\mathbf{C}(I, -)$ together with the natural transformation M of Lemma 9 and the identity operator E on \mathcal{I} is a symmetric monoidal functor. By construction, the Hilbert space $\mathbf{C}(I, I)$ is just the field \mathcal{I} considered as a Hilbert space over itself, and so E is a dagger isomorphism from the value of the functor $\mathbf{C}(I, -)$ at I to the tensor unit of $\mathbf{Hilb}_{\mathcal{I}}$.

Write $b_{H,K} : H \otimes K \rightarrow K \otimes H$ for the symmetry of \mathbf{C} , and $B_{\mathcal{H}, \mathcal{K}} : \mathcal{H} \otimes \mathcal{K} \rightarrow \mathcal{K} \otimes \mathcal{H}$ for that of \mathbf{Hilb} . For each pair of objects H and K of \mathbf{C} , the symmetry diagram

$$\begin{array}{ccc} \mathcal{H} \otimes \mathcal{K} & \xrightarrow{B_{\mathcal{H}, \mathcal{K}}} & \mathcal{K} \otimes \mathcal{H} \\ M_{H, K} \downarrow & & \downarrow M_{K, H} \\ \mathbf{C}(I, H \otimes K) & \xrightarrow{\mathbf{C}(I, b_{H, K})} & \mathbf{C}(I, K \otimes H) \end{array}$$

commutes because for all $h \in \mathcal{H}$ and $k \in \mathcal{K}$:

$$\begin{aligned} M_{K, H}(B_{\mathbf{C}(I, H), \mathbf{C}(I, K)}(h \otimes k)) &= M_{K, H}(k \otimes h) = k \otimes h = b_{H, K} \circ (h \otimes k) \\ &= \mathbf{C}(I, b_{H, K})(h \otimes k) = \mathbf{C}(I, b_{H, K})(M_{H, K}(h \otimes k)). \end{aligned}$$

Similarly, writing $a_{H,K,L} : (H \otimes K) \otimes L \rightarrow H \otimes (K \otimes L)$ and $A_{\mathcal{H},\mathcal{K},\mathcal{L}} : (\mathcal{H} \otimes \mathcal{K}) \otimes \mathcal{L} \rightarrow \mathcal{H} \otimes (\mathcal{K} \otimes \mathcal{L})$ for the associators of \mathbf{C} and **Hilb**, the diagram

$$\begin{array}{ccc}
(\mathcal{H} \otimes \mathcal{K}) \otimes \mathcal{L} & \xrightarrow{A_{\mathcal{H},\mathcal{K},\mathcal{L}}} & \mathcal{H} \otimes (\mathcal{K} \otimes \mathcal{L}) \\
M_{H,K} \otimes \text{id}_{\mathcal{L}} \downarrow & & \downarrow \text{id}_{\mathcal{H}} \otimes M_{K,L} \\
\mathbf{C}(I, H \otimes K) \otimes \mathcal{L} & & \mathcal{H} \otimes \mathbf{C}(I, K \otimes L) \\
M_{H \otimes K, L} \downarrow & & \downarrow M_{H, K \otimes L} \\
\mathbf{C}(I, (H \otimes K) \otimes L) & \xrightarrow{\mathbf{C}(I, a_{H,K,L})} & \mathbf{C}(I, H \otimes (K \otimes L))
\end{array}$$

commutes because for all $h \in \mathcal{H}$, $k \in \mathcal{K}$ and $l \in \mathcal{L}$:

$$\begin{aligned}
M_{H,K \otimes L}(\text{id} \otimes M_{K,L}(A_{\mathcal{H} \otimes \mathcal{K}, \mathcal{L}}((h \otimes k) \otimes l))) &= M_{H,K \otimes L}(\text{id} \otimes M_{K,L}(h \otimes (k \otimes l))) \\
&= h \otimes (k \otimes l) = a_{H,K,L} \circ ((h \otimes k) \otimes l) = \mathbf{C}(I, a_{H,K,L})((h \otimes k) \otimes l) \\
&= \mathbf{C}(I, a_{H,K,L})(M_{H \otimes K, L}((M_{H,K} \otimes \text{id})((h \otimes k) \otimes l))).
\end{aligned}$$

Thus $\mathbf{C}(I, -)$ is a symmetric monoidal functor. \square

Recall that H and H^* are *dagger dual* objects when there is a morphism $f : I \rightarrow H^* \otimes H$ making the following diagram commute [9, Definition 3.46]:

$$\begin{array}{ccccc}
H \otimes I & \simeq & H & \simeq & I \otimes H \\
\text{id}_H \otimes f \downarrow & & & & \uparrow f^\dagger \otimes \text{id}_H \\
H \otimes (H^* \otimes H) & \simeq & (H \otimes H^*) \otimes H & \simeq & (H^* \otimes H) \otimes H
\end{array}$$

In **Hilb**, an object has a dagger dual if and only if it is finite-dimensional [9, Corollary 3.65]. Because monoidal dagger functors preserve dagger dual objects [9, Theorem 3.14], it follows from Theorem 10 that the finite-dimensional Hilbert spaces can be categorically axiomatised within \mathbf{C} as the dagger dual objects.

REFERENCES

- [1] S. Abramsky and B. Coecke. *Handbook of Quantum Logic and Quantum Structures*, volume 2, chapter Categorical Quantum Mechanics. Elsevier, 2008.
- [2] J. C. Baez. Division algebras and quantum theory. *Foundations of Physics*, 42:819–855, 2012.
- [3] P. Etingof, S. Gelaki, D. Nikshych, and V. Ostrik. *Tensor Categories*. American Mathematical Society, 2015.
- [4] P. Ghez, R. Lima, and J. E. Roberts. W^* -categories. *Pacific Journal of Mathematics*, 120:79–109, 1985.
- [5] L. Hardy. Quantum theory from five reasonable axioms. arXiv:0101012, 2001.
- [6] E. Hellinger and O. Toeplitz. Grundlagen für eine Theorie der unendlichen Matrizen. *Mathematische Annalen*, 69:289–330, 1910.
- [7] C. Heunen. An embedding theorem for Hilbert categories. *Theory and Applications of Categories*, 13:321–344, 2009.
- [8] C. Heunen and B. Jacobs. Quantum logic in dagger kernel categories. *order*, 27:177–212, 2010.
- [9] C. Heunen and J. Vicary. *Categories for Quantum Theory*. Oxford University Press, 2019.
- [10] J. M. Jauch. *Foundations of Quantum Mechanics*. Addison-Wesley, 1968.
- [11] S. S. Holland Jr. Orthomodularity in infinite dimensions; a theorem of M. Solèr. *Bulletin of the American Mathematical Society*, 32:205–234, 1995.
- [12] A. Kapustin. Is quantum mechanics exact? *Journal of Mathematical Physics*, 54:062107, 2013.

- [13] M. Karvonen. Biproducts without pointedness. *Cahiers de topologie et géométrie différentielle catégoriques*, LXI(3):229–238, 2020.
- [14] F. W. Lawvere. An elementary theory of the category of sets. *Proceedings of the National Academy of Sciences*, 52(6):1506–1511, 1964.
- [15] S. Mac Lane. *Categories for the Working Mathematician*. Springer, 2nd edition, 1971.
- [16] F. J. Murray and J. von Neumann. On rings of operators. *Annals of Mathematics*, 37(1):116–229, 1936.
- [17] C. Piron. *Foundations of Quantum Physics*. W. A. Benjamin, 1976.
- [18] M. Rédei. Why John von Neumann did not like the Hilbert space formalism of quantum mechanics (and what he liked instead). *Studies in the History and Philosophy of Modern Physics*, 27(4):493–510, 1996.
- [19] M. P. Solèr. Characterization of Hilbert spaces by orthomodular spaces. *Communications in Algebra*, 1:219–243, 1995.
- [20] J. Vicary. Completeness of \dagger -categories and the complex numbers. *Journal of Mathematical Physics*, page 082104, 2011.
- [21] J. von Neumann. *Mathematical Foundations of Quantum Mechanics*. Princeton University Press, 1955.

UNIVERSITY OF EDINBURGH, SCHOOL OF INFORMATICS, 10 CRICHTON STREET, EH8 9AB, EDINBURGH, UNITED KINGDOM

Email address: `chris.heunen@ed.ac.uk`

TULANE UNIVERSITY, DEPARTMENT OF COMPUTER SCIENCE, 6823 ST. CHARLES AVENUE, NEW ORLEANS, LA 70118, UNITED STATES

Email address: `akornell@tulane.edu`