AXIOMS FOR THE CATEGORY OF HILBERT SPACES AND LINEAR CONTRACTIONS

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Abstract. The category of Hilbert spaces and linear contractions is characterised by elementary categorical properties that do not refer to probabilities, complex numbers, norm, continuity, convexity, or dimension.

1. Introduction

Quantum mechanics, in its traditional formulation, is based on Hilbert space [18]. More precisely, it is based on mappings of Hilbert spaces. In the simplest telling, states are unit vectors, evolving along certain linear maps between Hilbert spaces. Taking into account probabilities, states become operators, and dynamics certain linear maps between the resulting algebras. Either way, it is fair to say that the category of Hilbert spaces and linear maps is crucial to quantum theory.

Foundationally, this is unsatisfactory [15]. Although it gives accurate results, the framework of Hilbert spaces is hard to interpret physically from first principles. Many reconstruction programmes try to reformulate the framework mathematically, so that the primitive assumptions rely on fewer physically unjustified details [4]. For example, instead of starting with Hilbert space, they start from operator algebras [11], orthomodular lattices [14], generalised probabilistic theories [16], or from categories [7].

In this final case, the situation comes down to the following. Somebody hands you a category. How do you know it is (equivalent to) that of Hilbert spaces? The answer depends on which morphisms between Hilbert spaces you choose exactly. Previous work [6] settles the case of bounded linear maps, by giving axioms characterising that category precisely. Importantly, those axioms are purely categorical, and do not mention probabilities, complex numbers, or other analytical details that you may think are fundamental to Hilbert space.

Nevertheless, the story does not end with that answer, because not all bounded linear maps are physical. Quantum systems may only evolve along unitary linear maps. Taking quantum measurement into account allows the larger class of linear contractions [17, 1]. But no quantum-theoretical process can be described by a bounded linear map that is not a contraction. This article settles the case for the physical choice of morphisms being linear contractions.

The main idea is the following: The subcategory of Hilbert spaces and linear contractions generates the category of Hilbert spaces and all bounded linear maps as a monoidal category with invertible nonzero scalars (where scalars in a monoidal category are endomorphism on the tensor unit). Specifically, if you formally adjoin

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inverses for all nonzero scalars to the former category, you get the latter. We find properties of the former category that guarantee that its completion satisfies the axioms of [6] and hence is the category of Hilbert spaces and bounded linear maps. Finally, using the nature of the completion, we show that the former category must consist of exactly the linear contractions.

In addition to being an important characterisation in its own right, this theorem is also a stepping stone towards trying to characterise related categories: Hilbert spaces and completely positive morphisms, going towards foundations of quantum information theory; Hilbert spaces and unitaries, going towards foundations of quantum computing; and Hilbert modules and adjointable morphisms, going towards unitary representations and foundations of quantum field theory [2].

The rest of this article is structured as follows. We start by stating the axioms in Section 2. Their meaning is discussed in Section 3. In particular, the category of Hilbert spaces and linear contractions is defined there, and it is shown how it satisfies the axioms. Next, Section 4 derives basic properties from the axioms. The real work begins in Section 5, which defines a completion by which we can abstractly recognise linear contractions among all bounded linear maps. Section 6 puts everything together to prove the main theorem.

2. The axioms

We will consider the following properties of a category that is equipped with a contravariant endofunctor $\dagger$ and two symmetric monoidal structures $(\otimes, I)$ and $(\oplus, 0)$:

(1) The contravariant endofunctor $\dagger$ satisfies $\mathrm{id}_H^\dagger = \mathrm{id}_H$ for all objects H and $t^{\dagger\dagger} = t$ for all morphisms t. A category equipped with such a dagger is also called a *dagger category*. If $t^\dagger \circ t = \mathrm{id}$, we call t a *dagger monomorphism*, and if additionally $t \circ t^\dagger = \mathrm{id}$, we call it a *dagger isomorphism*.

(2) The category is a *dagger rig* category: the unitors, braidings, and associators of the monoidal structures $(\otimes, I)$ and $(\oplus, 0)$ are dagger isomorphisms, and they satisfy $(s \otimes t)^\dagger = s^\dagger \otimes t^\dagger$ and $(s \oplus t)^\dagger = s^\dagger \oplus t^\dagger$ for all morphisms s and t, and there are natural dagger isomorphisms

$$H \otimes (K \oplus L) \to (H \otimes K) \oplus (H \otimes L)$$
$$(H \oplus K) \otimes L \to (H \otimes L) \oplus (K \otimes L)$$

that satisfy the coherence conditions (I)–(XXIV) of [9].

(3) The unit 0 is initial, and thus a zero object. We write $0_{H,K}$ for the unique morphism $H \to K$ that factors through the object 0. It follows that there are natural morphisms:

$$\mathrm{inl}_{H,K} = \Big(H \simeq H \oplus 0 \xrightarrow{\mathrm{id} \oplus 0} H \oplus K \Big)$$
$$\mathrm{inr}_{H,K} = \Big(K \simeq 0 \oplus K \xrightarrow{0 \oplus \mathrm{id}} H \oplus K \Big)$$

This property of $\oplus$ makes the category so-called *affine* or *semicartesian*.

(4) The injections $\mathrm{inl}_{H,K}$ and $\mathrm{inr}_{H,K}$ are *jointly epic*: $s = t \colon H \oplus K \to L$ as soon as $s \circ \mathrm{inl} = t \circ \mathrm{inl}$ and $s \circ \mathrm{inr} = t \circ \mathrm{inr}$.

(5) *Mixture* occurs: there exists a morphism $s \colon I \to I \oplus I$ with $\mathrm{inl}^\dagger \circ s \neq 0 \neq \mathrm{inr}^\dagger \circ s$.

(6) The unit I is *dagger simple*: any dagger monomorphism $S \to I$ is invertible or 0.
(7) The unit I is a *monoidal separator*: $s = t \colon H \otimes K \to L$ as soon as $s \circ (x \otimes y) = t \circ (x \otimes y)$ for all $x \colon I \to H$ and $y \colon I \to K$.
(8) Any two parallel morphisms have a *dagger equaliser*, that is, an equaliser that is a dagger monomorphism.
(9) Any dagger monomorphism $N \to H$ is a *dagger kernel*, that is, a dagger equaliser of some morphism $H \to K$ and $0 \colon H \to K$.
(10) Subobjects are determined by *positive* maps: monomorphisms $r \colon H \rightarrowtail L$ and $s \colon K \rightarrowtail L$ satisfy $r = s \circ t$ for some isomorphism t if and only if $r \circ r^\dagger = s \circ s^\dagger$.
(11) Any directed diagram has a colimit.

Observe that these axioms are all elementary properties of categories: none of them refer to notions such as complex amplitudes, probabilities, norm, convexity, continuity, or metric completeness.

3. The category

We define our main model.

Definition 1. A linear function $T \colon \mathcal{H} \to \mathcal{K}$ between Hilbert spaces is *bounded* when there exists a constant $N \in [0, \infty)$ such that $\|Tx\| \leq N \cdot \|x\|$ for all $x \in \mathcal{H}$; write $\|T\|$ for the infimum of such N. The function T is called a *contraction* when $\|T\| \leq 1$, that is, when $\|Tx\| \leq \|x\|$ for all $x \in \mathcal{H}$. Such linear functions are also known as *nonexpansive* or *short* maps. We write $\mathbf{Hilb}_{\mathbb{R}}$ and $\mathbf{Hilb}_{\mathbb{C}}$ for the categories of Hilbert spaces and bounded linear maps, and $\mathbf{Con}_{\mathbb{R}}$ and $\mathbf{Con}_{\mathbb{C}}$ for the categories of Hilbert spaces and short linear maps, respectively over the real and complex numbers. When the distinction does not matter, we will simply write **Hilb** and **Con**.

Let us discuss how **Con** satisfies the axioms.

(1) The dagger is provided by adjoints: for any bounded linear function $T \colon \mathcal{H} \to K$ there is a unique linear map $T^\dagger \colon \mathcal{K} \to \mathcal{H}$ satisfying $\langle Tx \mid y \rangle_{\mathcal{K}} = \langle x \mid T^\dagger y \rangle_{\mathcal{H}}$ for all $x \in \mathcal{H}$ and $y \in \mathcal{K}$. This is a well-defined functor on **Con** because $\|T^\dagger\| = \|T\|$. The dagger monomorphisms are the isometries, and the dagger isomorphisms are the unitaries.
(2) The monoidal structures $\otimes$ and $\oplus$ are produced by tensor products and direct sums of Hilbert spaces, respectively. This is well-defined in **Con** because $\|S \otimes T\| = \|S\| \cdot \|T\|$ and $\|S \oplus T\| = \max\{\|S\|, \|T\|\}$. The unit I is the one-dimensional Hilbert space given by the base field, and the unit 0 is the zero-dimensional Hilbert space $\{0\}$.
(3) Linear maps must preserve the 0 vector, so the object 0 is indeed initial.
(4) The injections $\mathrm{inl}_{\mathcal{H},\mathcal{K}}(x) = (x, 0)$ and $\mathrm{inr}_{\mathcal{H},\mathcal{K}}(y) = (0, y)$ have images that linearly span $\mathcal{H} \oplus \mathcal{K}$, so are clearly jointly epic.
(5) The vector $s = (\frac{1}{2}, \frac{1}{2})$ in the two-dimensional Hilbert space satisfies (5), so this axiom may be read as saying that nontrivial superpositions of qubits exist.
(6) The only vector subspaces of the base field are the zero-dimensional one and the whole space itself.

(7) Linear contractions are continuous, and the linear span of pure tensor elements $x \otimes y$ is dense in $\mathcal{H} \otimes \mathcal{K}$, so morphisms in **Con** out of a tensor product are indeed determined by their action on pure tensor elements.
(8) If $S, T \colon \mathcal{H} \to \mathcal{K}$ are linear contractions, then $\{x \in \mathcal{H} \mid Sx = Tx\}$ is a closed subspace of $\mathcal{H}$ and hence an object of **Con**, and the inclusion is automatically an isometry.
(9) Any closed subspace $\mathcal{N} \subseteq \mathcal{H}$ is a kernel of the orthogonal projection $\mathrm{inr}^\dagger \colon \mathcal{H} \simeq \mathcal{N} \oplus \mathcal{N}^\perp \to \mathcal{N}^\perp$ onto its orthocomplement, which is a contraction.

The last two axioms are perhaps a bit less standard, and we spell out proofs.

Lemma 2. *The category* **Con** *satisfies* (10).

Proof. Observe that any isomorphism in **Con** is a dagger isomorphism: if $\|T\| \leq 1$ and $\|T^{-1}\| \leq 1$, then $\|x\| = \|T^{-1}Tx\| \leq \|Tx\| \leq \|x\|$ for all vectors x, so T is an isometry and hence a dagger isomorphism. This makes one direction of (10) clear: if $R = ST$ for unitary T, then $RR^\dagger = STT^\dagger S^\dagger = SS^\dagger$.

For the other direction, suppose that injective linear contractions $R \colon \mathcal{H} \to \mathcal{L}$ and $S \colon \mathcal{K} \to \mathcal{L}$ satisfy $RR^\dagger = SS^\dagger$. Regard them as operators $\mathcal{H} \oplus \mathcal{K} \oplus \mathcal{L} \to \mathcal{H} \oplus \mathcal{K} \oplus L$ using inl, inr, and their adjoints. Then the absolute values $(RR^\dagger)^{1/2}$ and $(SS^\dagger)^{1/2}$ are equal; call this operator P. Now R and S have polar decompositions $R = PU$ and $S = PV$. Here U and V are partial isometries such that $UU^\dagger$ and $VV^\dagger$ are the range projections of R and S, and $U^\dagger U$ and $V^\dagger V$ are the range projections of $R^\dagger$ and $S^\dagger$. Hence we may regard them as linear contractions $U \colon \mathcal{H} \to \mathcal{L}$ and $V \colon \mathcal{K} \to \mathcal{L}$, and $P \colon \mathcal{L} \to \mathcal{L}$.

The operators R and S are injective, so U and V are injective partial isometries. Thus, U and V are isometries. Furthermore, the range projections of U and V are equal to the range projections of R and S, respectively, which are equal to each other because $RR^\dagger = SS^\dagger$. Therefore, $T = V^\dagger U$ is a unitary map $\mathcal{H} \to \mathcal{K}$. Finally, $R = PU = PUU^\dagger U = PVV^\dagger U = ST$ because $UU^\dagger$ and $VV^\dagger$ are simply the range projections of U and V, respectively. $\square$

Lemma 3. *The category* **Con** *satisfies* (11).

Proof. See also [12, Proposition 3.1]. Let $(J, \leq)$ be a directed partially ordered set, and $\mathcal{H}_{i \leq j} \colon \mathcal{H}_i \to \mathcal{H}_j$ be a diagram $J \to$ **Con**. Let V be the vector space $\bigcup_{j \in J} \mathcal{H}_j$, where $x \in \mathcal{H}_i$ and $y \in H_j$ are identified when $\mathcal{H}_{i \leq k}(x) = \mathcal{H}_{j \leq k}(y)$ for some $k \in J$. Writing $[x_i]$ for the equivalence class of $x_i \in \mathcal{H}_i$, define $\|[x_i]\| = \lim_j \|\mathcal{H}_{i \leq j}(x_i)\|$; it is routine to verify that this limit exists and defines a seminorm on V. This seminorm satisfies the parallelogram law. So if $N = \{x \in V \mid \|x\| \neq 0\}$, then V/N is an inner product space. Call its completion $\mathcal{H}_\infty$, define $\mathcal{H}_{j<\infty} \colon \mathcal{H}_j \to \mathcal{H}_\infty$ by $x_j \mapsto [x_j]$, and observe that this is a cocone.

$$
\begin{tikzcd}
\mathcal{H}_i \arrow[r, "\mathcal{H}_{i \leq j}"] \arrow[drr, "\mathcal{H}_{i<\infty}"'] \arrow[ddrr, bend right, "T_i"'] & \mathcal{H}_j \arrow[dr, "\mathcal{H}_{j<\infty}"] \arrow[ddr, "T_j"] & \\
& & \mathcal{H}_\infty \arrow[d, dashed, "T_\infty"] \\
& & \mathcal{K}
\end{tikzcd}
$$

Suppose that $T_j \colon \mathcal{H}_j \to \mathcal{K}$ is another cocone. It defines a function $\bigcup_{j \in J} \mathcal{H}_j \to \mathcal{K}$ that respects the equivalence, and so lifts to a linear function $T \colon V \to \mathcal{K}$. Furthermore, if $x_j \in \mathcal{H}_j$ satisfies $\|[x_j]\| = 0$, then $\|T[x_j]\| = \|T[\mathcal{H}_{j \leq k}(x_j)]\| \leq \|\mathcal{H}_{j \leq k}(x_j)\| \to 0$. Thus T lifts to a linear function $V/N \to \mathcal{K}$. Similarly, this function is a contraction, and so extends to a linear contraction $T_\infty \colon \mathcal{H}_\infty \to \mathcal{K}$. By construction $T_\infty(\mathcal{H}_{j<\infty}(x_j)) = T_\infty[x_j] = T_j(x_j)$ for all $x_j \in \mathcal{H}_j$, and so T_∞ is a mediating map from the cocone $\{\mathcal{H}_{j<\infty}\}_{j \in J}$ to the cocone $\{T_j\}_{j \in J}$. Finally, this mediating map is unique, because the $\mathcal{H}_{j<\infty}$ have jointly dense range in $\mathcal{H}_\infty$ by construction. □

Axioms (10) and (11) are the only ones that hold in **Con** but not in **Hilb**. The behaviour following from these two axioms accounts for the difference between the two categories.

We end this section by determining the subobjects of I in **Con**. Contrast the following lemma to the situation **Hilb**, where the subobjects of I are $\{0, 1\}$; the crucial difference is that scalars $0 < z < 1$ are invertible in **Hilb** but not in **Con**.

Lemma 4. *There is an order isomorphism between the subobjects of I in* **Con** *and the real unit interval* $[0, 1]$.

Proof. Consider a monomorphism of **Con** into the base field I. It is an injective linear contraction $T \colon \mathcal{H} \to I$. The kernel of T is zero, and so $\dim \mathcal{H} = 0$ or $\dim \mathcal{H} = 1$. Hence a subobject of I is represented by the unique morphism $0 \to I$ or by an injective contraction $I \to I$. Up to isomorphism of subobjects, the latter morphisms are scalars in the interval $(0, 1]$. Hence the subobjects of I in **Con** are canonically in bijection with the closed unit interval $[0, 1]$. This bijection is clearly an order isomorphism. □

4. The basic lemmas

From now on, we assume a (locally small) category $\mathbf{D}$ that satisfies the axioms (1)–(11), and set out to prove that $\mathbf{D} \simeq \mathbf{Con}$. In this section we derive some basic properties of $\mathbf{D}$. We start by recalling a factorisation that already follows from (8) and (9) alone [5]. We summarise a proof here for convenience.

Lemma 5. *Any morphism $t \colon H \to K$ factors as an epimorphism $e \colon H \to E$ followed by a dagger monomorphism $k = \ker(\ker(t^\dagger)^\dagger) \colon E \to K$.*

Proof. By (1), $\ker(t^\dagger)^\dagger$ is a cokernel of t, so that $\ker(t^\dagger)^\dagger \circ t = 0$, and t always factors through k via some morphism e. It remains to show that e is an epimorphism.

We first prove that if $s \circ e = 0$, then $s = 0 \colon E \to L$. Observe that $t^\dagger \circ k \circ s^\dagger = e^\dagger \circ s^\dagger = 0$. Hence $k \circ s^\dagger$ factors through $\ker(t^\dagger)$ via some $r \colon L \to F$.

$$\begin{tikzcd}
\bullet \arrow[r, "\ker(t^\dagger)"] & K \arrow[r, shift left, "t^\dagger"] \arrow[r, shift right, "0"'] & H \\
L \arrow[u, dashed, "r"] \arrow[ur, "k \circ s^\dagger"'] & &
\end{tikzcd}$$

But $r = \ker(t^\dagger)^\dagger \circ k \circ s^\dagger = 0$, so $k \circ s^\dagger = 0$, and so $s = 0$.

Next, we prove that e is an epimorphism. Suppose that $s \circ e = s' \circ e$ for morphisms $s, s' \colon E \to L$. Let m be a dagger equaliser of s and s'. Then e factors through m, and, writing $\mathrm{coker}(m)$ for a cokernel of m, we infer $\mathrm{coker}(m) \circ e = 0$.

$$\begin{tikzcd}
H \arrow[rr, "e"] \arrow[d, dashed] & & E \arrow[r, shift left, "s"] \arrow[r, shift right, "s'"'] \arrow[d, "\mathrm{coker}(m)"] & L \\
\bullet \arrow[urr, "m"'] & & \bullet &
\end{tikzcd}$$

But by the property we proved earlier, that means $\mathrm{coker}(m) = 0$, and so m is invertible. We conclude that $s = s'$. $\square$

Next we notice a consequence of axiom (6). A *scalar* in a monoidal category is a morphism $z\colon I \to I$, and we can multiply any morphism $t\colon H \to K$ by it to obtain a morphism $H \simeq I \otimes H \xrightarrow{z \otimes t} I \otimes K \simeq K$ that we denote $z \bullet t$ [7, 2.1]. For a scalar $z\colon I \to I$ and morphisms $s\colon H \to K$ and $t\colon K \to L$, it follows from the bifunctoriality of the tensor product that $z \bullet (g \circ f) = (z \bullet g) \circ f = g \circ (z \bullet f)$. In particular, $w \bullet z = w \circ z$ for scalars $w, z\colon I \to I$. Instead of id_I we may write 1 for the identity scalar.

Lemma 6. *Every nonzero scalar is monic and epic.*

Proof. Let $z\colon I \to I$ be nonzero. Lemma 5 factors it as an epimorphism followed by a dagger monomorphism. Axiom (6) guarantees that that dagger monomorphism is either 0 or invertible. By assumption it cannot be zero, and so z is epic. Similarly, $z^\dagger$ is epic, and so z is also monic. $\square$

It follows that every dagger monic scalar is invertible, for it cannot be zero, so must be epic as well as split monic.

Lemma 7. *For all nonzero scalars z and morphisms $s, t\colon H \to K$, if $z \bullet s = z \bullet t$, then $s = t$.*

Proof. By two applications of axiom (7), it suffices to prove that $y^\dagger \circ s \circ x = y^\dagger \circ t \circ x$ for all $x\colon I \to H$ and $y\colon I \to K$. But $z \circ (y^\dagger \circ s \circ x) = z \bullet (y^\dagger \circ s \circ x) = y^\dagger \circ (z \bullet s) \circ x = y^\dagger \circ (z \bullet t) \circ x = z \bullet (y^\dagger \circ t \circ x) = z \circ (y^\dagger \circ t \circ x)$, and the scalar z is monic by Lemma 6. $\square$

The following two lemmas are consequences of axiom (10).

Lemma 8. *Any isomorphism is a dagger isomorphism. Any split monomorphism is a dagger monomorphism.*

Proof. Let $r\colon H \to L$ be an isomorphism. Taking $s = \mathrm{id}_L$ and $t = r$ in (10) shows that $r \circ r^\dagger = \mathrm{id} \circ \mathrm{id}^\dagger = \mathrm{id}$. Hence $r^{-1} = r^{-1} \circ \mathrm{id}_L = r^{-1} \circ r \circ r^\dagger = r^\dagger$.

Now suppose that $t\colon H \to K$ is a split monomorphism. Factor $t = k \circ e$ for a dagger monomorphism $k\colon E \to K$ and an epimorphism $e\colon H \to E$. Then the epimorphism e is itself split monic. But in any category, a split monic epimorphism is an isomorphism, and hence e is a dagger isomorphism. Therefore, t is a dagger monomorphism. $\square$

Lemma 9. *If a scalar z and a morphism $t\colon H \to K$ satisfy $t^\dagger \circ t = z^\dagger \bullet z \bullet \mathrm{id}_H$, then $t = z \bullet s$ for a dagger monomorphism $s\colon H \to K$.*

Proof. Factor $t = k \circ e$ for a dagger monomorphism $k\colon E \to K$ and an epimorphism $e\colon H \to E$. Then $e^\dagger \circ e = t^\dagger \circ t = (z \bullet \mathrm{id}_H)^\dagger \circ (z \bullet \mathrm{id}_H)$. Now (10) and Lemma 8 provide a dagger isomorphism $u\colon H \to E$ such that $e = z \bullet u$, and so $t = z \bullet s$ for the dagger monomorphism $s = k \circ u$. $\square$

5. The completion

This section contains the main construction of the proof. Lemmas 4, 6, and 8 show that scalars in **Con** correspond to the unit disc:

$$\mathbf{Con}_{\mathbb{R}}(I,I) \simeq [-1,1] \qquad \mathbf{Con}_{\mathbb{C}}(I,I) \simeq \{z \in \mathbb{C} \mid |z| \leq 1\}$$

Observe that nonzero scalars that are not on the unit circle have no inverse in **Con**. But they do in **Hilb**. In fact, **Hilb** is the localisation of **Con** at nonzero scalars. We now abstractly construct the localisation of a monoidal category at its nonzero scalars in a way that respects the axioms (1)–(11). Write $\mathcal{D} = \mathbf{D}(I,I)$ for the scalars of **D**.

Proposition 10. *There is a category $\mathbf{D}[\mathcal{D}^{-1}]$ with the same objects as $\mathbf{D}$, where a morphism $[t/z]$ consists of a nonzero scalar z and a morphism t in $\mathbf{D}$ modulo the following equivalence relation:*

$$[t/z] \sim [t'/z'] \iff z' \bullet t = z \bullet t'$$

The identity on H is $[\mathrm{id}_H/1]$, and composition is given by:

$$[t/z] \circ [s/w] = [t \circ s/z \bullet w]$$

There is an embedding $\mathbf{D} \to \mathbf{D}[\mathcal{D}^{-1}]$ defined by $t \mapsto [t/1]$.

Proof. The relation $\sim$ is clearly reflexive and symmetric. To see that it is transitive, suppose that $[r/x] \sim [s/y]$ and $[s/y] \sim [t/z]$. Then $r \bullet y = s \bullet x$ and $s \bullet z = t \bullet y$. It follows that $s \bullet r \bullet z = r \bullet t \bullet y = s \bullet t \bullet x$. Lemma 7 now shows that $r \bullet z = t \bullet x$, that is, $[r/x] \sim [t/z]$. Hence $\sim$ is an equivalence relation.

To see that the composition is well-defined, suppose that $[t/z] \sim [t'/z']$ and $[s/w] \sim [s'/w']$. Then $z' \bullet w' \bullet (t \circ s) = (z' \bullet t) \circ (w' \bullet s) = (z \bullet t') \circ (w \bullet s') = z \bullet w \bullet (t' \circ s')$, so $[t/z] \circ [s/w] \sim [t'/z'] \circ [s'/w']$. It is clearly associative and satisfies the identity laws. If z and w are nonzero scalars, then $z \bullet w = z \circ w$ is nonzero because w is monic by Lemma 6.

The assignment $t \mapsto [t/1]$ is functorial, injective on objects, and faithful. □

The previous proposition concretely describes a quotient category [10, II.8]: writing **I** for the one-object category of non-zero scalars of **D**, take the quotient of $\mathbf{D} \times \mathbf{I}$ under the congruence relation $(s,w) \sim (t,z) \iff z \bullet s = w \bullet t$.

Example 11. There is an isomorphism $F\colon \mathbf{Con}[\mathcal{D}^{-1}] \to \mathbf{Hilb}$ of categories. It is defined as the identity $F(\mathcal{H}) = \mathcal{H}$ on objects, and as $F[T/z] = z^{-1}T$ on morphims. It is faithful by construction, because $F[S/w] = F[T/z]$ implies $zS = wT$ and so $[S/w] \sim [T/z]$. To see that it is full, let $T\colon \mathcal{H} \to \mathcal{K}$ be a bounded linear function. If $\|T\| \leq 1$, then $T = F([T/1])$. If $\|T\| \geq 1$, then $\|T\|^{-1} \in (0,1]$ is a scalar in **Con** and $T/\|T\|$ is a contraction, so $F\big([\|T\|^{-1}T/\|T\|^{-1}]\big) = T$.

Lemma 12. *The category $\mathbf{D}[\mathcal{D}^{-1}]$ inherits monoidal structure $\otimes$ from $\mathbf{D}$, and the functor $\mathbf{D} \to \mathbf{D}[\mathcal{D}^{-1}]$ is strict monoidal for $\otimes$.*

Proof. Observe that the equivalence relation of Proposition 10 is a monoidal congruence: if $[s/w] \sim [s'/w']$ and $[t/z] \sim [t'/z']$, then $[s \otimes t/w \bullet z] \sim [s' \otimes t'/w' \bullet z']$. □

Before we show that it satisfies the axioms of [6] one by one, let us first establish the universal property that justifies the notation $\mathbf{D}[\mathcal{D}^{-1}]$. Notice that any morphism $[t/z]$ in $\mathbf{D}[\mathcal{D}^{-1}]$ can be factored as $[t/z] = [\mathrm{id}/z] \circ [t/1]$.

Proposition 13. *Any functor $F\colon \mathbf{D} \to \mathbf{C}$ that is strong monoidal for $\otimes$ such that $F(z)$ is invertible for all nonzero scalars z factors through a unique functor $\mathbf{D}[\mathcal{D}^{-1}] \to \mathbf{C}$ that is strong monoidal for $\otimes$ via the functor $\mathbf{D} \to \mathbf{D}[\mathcal{D}^{-1}]$.*

$$\begin{tikzcd} \mathbf{D} \arrow[r] \arrow[dr, "F"'] & \mathbf{D}[\mathcal{D}^{-1}] \arrow[d, dashed] \\ & \mathbf{C} \end{tikzcd}$$

Proof. Define the functor $G\colon \mathbf{D}[\mathcal{D}^{-1}] \to \mathbf{C}$ by $G(H) = F(H)$ on objects, and by $G[t/z] = F(z)^{-1} \bullet F(t)$ on morphisms. This is the only functor making the triangle commute, because it is completely determined by its values at $[t/1]$ and $[\mathrm{id}/z]$ for all morphisms t and all scalars z in $\mathbf{D}$. □

Thus $\mathbf{D}[\mathcal{D}^{-1}]$ is the localisation of $\mathbf{D}$ at all nonzero scalars: it formally adjoins inverses for all nonzero scalars to $\mathbf{D}$. The concrete description of Proposition 10 simplifies the general construction for localisation [8].

Lemma 14. *The category $\mathbf{D}[\mathcal{D}^{-1}]$ has a dagger, and the embedding $\mathbf{D} \to \mathbf{D}[\mathcal{D}^{-1}]$ preserves it. The embedding restricts to an isomorphism between the wide subcategories of dagger monomorphisms in $\mathbf{D}$ and $\mathbf{D}[\mathcal{D}^{-1}]$.*

Proof. Set $[t/z]^\dagger = [t^\dagger/z^\dagger]$, which is clearly well-defined. By Proposition 10, the embedding $\mathbf{D} \to \mathbf{D}[\mathcal{D}^{-1}]$ sends an object A to itself, and sends a morphism t to $[t/1]$. It is faithful by Lemma 7. The embedding clearly preserves daggers, hence dagger monomorphisms, and so restricts to the wide subcategories of dagger monomorphisms. This restricted functor is still bijective on objects and faithful. To see that it is full, let $[t/z]$ be a dagger monomorphism in $\mathbf{D}[\mathcal{D}^{-1}]$. The assumption $[\mathrm{id}/1] = [t/z]^\dagger \circ [t/z] = [t^\dagger \circ t/z^\dagger \bullet z]$ means $t^\dagger \circ t = z^\dagger \bullet z \bullet \mathrm{id}$, so Lemma 9 implies $t = z \bullet s$ for a dagger monic s in $\mathbf{D}$. But then $[t/z] = [s/1]$ is the image of a dagger monic in $\mathbf{D}$ under the embedding. □

An object is *simple* when any monomorphism into it must be 0 or invertible. This requirement on I is stronger than axiom (6), which only concerns dagger monomorphisms.

Lemma 15. *The tensor unit I in $\mathbf{D}[\mathcal{D}^{-1}]$ is simple.*

Proof. First, we will show that a monomorphism $[s/z]\colon S \to I$ represents the same subobject of I as $[0/1]$ if and only if $s = 0$. Suppose $[s/z]\colon S \to I$ is a monomorphism in $\mathbf{D}[\mathcal{D}^{-1}]$. It is always true that $[0/1] \leq [s/z]$ as subobjects, so $[s/z]$ is the minimum subobject if and only if $[s/z] \leq [0/1]$. This happens when there is a morphism $[0, w]\colon S \to 0$ such that $[s/z] \sim [0/1] \circ [0/w] = [0/w]$, that is, $0_{S,I} = z \bullet 0 = w \bullet s$. Because $w \bullet 0_{S,I} = 0_{S,I}$, by Lemma 7, this means exactly that $s = 0$. Thus $[s/z] = 0$ is the minimum subobject if and only if $s = 0$.

Next, we will show that a monomorphism $[s/z]\colon S \to I$ represents the same subobject of I as $[1/1]$ if and only if $s \neq 0$. Similarly to the last paragraph, $[s/z] \leq [\mathrm{id}/1]$ as subobjects, so $[s/z]$ is the maximum subobject if and only if $[\mathrm{id}/1] \leq [s/z]$. This happens when there is a morphism $[t/w]\colon I \to S$ such that $[\mathrm{id}/\mathrm{id}] \sim [s/z] \circ [t/w] = [s \circ t/z \bullet w]$, that is, $s \circ t = z \bullet w$. We may choose $w = s \circ s^\dagger$

and $t = z \bullet s^\dagger$ to see that this is true as soon as $s \circ s^\dagger \neq 0$. This final inequality is equivalent to $s^\dagger \neq 0$ because s is monic in this case.

Now, either $s = 0$ or $s \neq 0$. If $s = 0$ then $[s/z]$ is the minimum subobject, and if $s \neq 0$ then $[s/z]$ is the maximum subobject. Therefore I is simple in $\mathbf{D}[\mathcal{D}^{-1}]$. $\square$

Lemma 16. *The tensor unit I is a monoidal separator in $\mathbf{D}[\mathcal{D}^{-1}]$.*

Proof. Let $[t/z], [t'/z'] \colon H \otimes K \to L$, and suppose $[t/z] \circ [x \otimes y/w] \sim [t'/z'] \circ [x \otimes y/w]$ for all $0 \neq w \colon I \to I$ and $x \colon I \to H$ and $y \colon I \to K$ in $\mathbf{D}$. Then $(w \bullet z \bullet t') \circ (x \otimes y)$ is equal to $(w \bullet z' \bullet t) \circ (x \otimes y)$. By (7), $w \bullet z \bullet t' = w \bullet z' \bullet t$. Taking $w = 1$ gives $z \bullet t' = z' \bullet t$, that is, $[t/z] \sim [t'/z']$. $\square$

A *dagger biproduct* of H and K in a dagger category with a zero object 0 is a product $H \overset{\pi_1}{\leftarrow} H \oplus K \overset{\pi_2}{\rightarrow} K$ such that $\pi_1^\dagger$ and $\pi_2^\dagger$ are dagger monomorphisms and $\pi_2 \circ \pi_1^\dagger = 0$.

Lemma 17. *The category $\mathbf{D}[\mathcal{D}^{-1}]$ has dagger biproducts, given by $\oplus$ and 0, and the functor $\mathbf{D} \to \mathbf{D}[\mathcal{D}^{-1}]$ is strict monoidal for $\oplus$.*

Proof. For all morphisms $[t/z] \colon H \to K$ and $[t'/z'] \colon H' \to K'$, define a morphism $H \oplus H' \to K \oplus K'$ by $[t/z] \oplus [t'/z'] = [z' \bullet t \oplus z \bullet t'/z \bullet z']$. This produces a functor $\oplus \colon \mathbf{D}[\mathcal{D}^{-1}] \times \mathbf{D}[\mathcal{D}^{-1}] \to \mathbf{D}[\mathcal{D}^{-1}]$. The embedding $\mathbf{D} \to \mathbf{D}[\mathcal{D}^{-1}]$ induces coherence isomorphisms, making $(\mathbf{D}[\mathcal{D}^{-1}], \oplus, 0)$ a symmetric monoidal category. To show that it is cartesian monoidal, it suffices to prove that $K \oplus L$ is the product of K and L with projections $[\mathrm{inl}^\dagger/1]$ and $[\mathrm{inr}^\dagger/1]$, because 0 is clearly terminal [3]. It then follows that $K \oplus L$ is a dagger biproduct of K and L because inl and inr are dagger monomorphisms.

Axiom (5) provides a map $s \colon I \to I \oplus I$ in $\mathbf{D}$ such that the scalars $x = \mathrm{inl}^\dagger \circ s$ and $y = \mathrm{inr}^\dagger \circ s$ are nonzero. We will use this to show that $\oplus$ forms a product in $\mathbf{D}[\mathcal{D}^{-1}]$ with projections $[\mathrm{inl}^\dagger/1]$ and $[\mathrm{inr}^\dagger/1]$. Let $[r/w] \colon H \to K$ and $[t/z] \colon H \to L$ in $\mathbf{D}[\mathcal{D}^{-1}]$. Define $q \colon H \to K \oplus L$ to be the following composite:

$$\begin{tikzcd}
H \arrow[rrr, dashed, "{[p/1]}"] \arrow[d, phantom, "\wr"] & & & H \oplus H \arrow[rr, "{\left[z \bullet y \bullet r \oplus w \bullet x \bullet t \middle/ w \bullet z \bullet x \bullet y\right]}"] \arrow[d, phantom, "\wr"] & & K \oplus L \\
I \otimes H \arrow[r, "{[s \otimes \mathrm{id}/1]}"'] & (I \oplus I) \otimes H \arrow[rr, phantom, "\simeq"] & & (I \otimes H) \oplus (I \otimes H) & &
\end{tikzcd}$$

Here p is made up from s and isomorphisms provided by axiom (2). We will prove that the following diagram commutes in $\mathbf{D}[\mathcal{D}^{-1}]$:

$$\begin{tikzcd}
 & H \arrow[dl, "{[r/w]}"'] \arrow[d, dashed, "q"] \arrow[dr, "{[t/z]}"] & \\
K & K \oplus L \arrow[l, "{[\mathrm{inl}^\dagger/1]}"] \arrow[r, "{[\mathrm{inr}^\dagger/1]}"'] & L
\end{tikzcd}$$

The left triangle commutes because the following diagram commutes in $\mathbf{D}$:

$$\begin{tikzcd}
H \arrow[r, phantom, "\simeq"] \arrow[dd, equal] & I\otimes H \arrow[r, "s\otimes\mathrm{id}"] \arrow[d, equal] & (I\oplus I)\otimes H \arrow[r, phantom, "\simeq"] \arrow[d, "\mathrm{inl}^\dagger\otimes\mathrm{id}"] & (I\otimes H)\oplus(I\otimes H) \arrow[r, phantom, "\simeq"] \arrow[d, "\mathrm{inl}^\dagger"] & H\oplus H \arrow[r, "z\bullet y\bullet r\oplus w\bullet x\bullet t"] \arrow[dd, "\mathrm{inl}^\dagger"] & K\oplus L \arrow[dd, "\mathrm{inl}^\dagger"] \\
& I\otimes H \arrow[r, "x\otimes\mathrm{id}"'] & I\otimes H \arrow[r, equal] & I\otimes H \arrow[dr, "\simeq"] & & \\
H \arrow[ur, "\simeq"] \arrow[rrrr, "x\bullet\mathrm{id}"'] & & & & H \arrow[r, "z\bullet y\bullet r"'] & K
\end{tikzcd}$$

The right triangle commutes similarly.

Moreover, the mediating map q is unique. For suppose $[t/z],[t'/z']\colon H\to K\oplus L$ both satisfy $[\mathrm{inl}^\dagger/1]\circ[t/z]=[\mathrm{inl}^\dagger/1]\circ[t'/z']$ and $[\mathrm{inr}^\dagger/1]\circ[t/z]=[\mathrm{inr}^\dagger/1]\circ[t'/z']$. Then $\mathrm{inl}^\dagger\circ(z\bullet t')=\mathrm{inl}^\dagger\circ(z'\bullet t)$ and $\mathrm{inr}^\dagger\circ(z\bullet t')=\mathrm{inr}^\dagger\circ(z'\bullet t)$. Because $\mathrm{inl}^\dagger$ and $\mathrm{inr}^\dagger$ are jointly epic by axiom (4), we have that $z\bullet t'=z'\bullet t$. Thus, $[t/z]=[t'/z']$. $\square$

Lemma 18. *The category $\mathbf{D}[\mathcal{D}^{-1}]$ has dagger equalisers, and the functor $\mathbf{D}\to\mathbf{D}[\mathcal{D}^{-1}]$ preserves them.*

Proof. Let $[t/z],[t'/z']\colon H\to K$. Let $e\colon E\to H$ be a dagger equaliser of $z'\bullet t$ and $z\bullet t'$ in $\mathbf{D}$. Then $[t/z]\circ[e/1]\sim[t'/z']\circ[e/1]$. Suppose that also $[t/z]\circ[e'/w]\sim[t'/z']\circ[e'/w]$ for a nonzero scalar w and $e'\colon E'\to H$. Then $z'\bullet t\circ w\bullet e'=z\bullet t'\circ w\bullet e'$, and hence there is a unique morphism $m\colon E'\to E$ in $\mathbf{D}$ with $w\bullet e'=e\circ m$. Thus, $w\bullet w\bullet e'=w\bullet e\circ m$, that is, $[e'/w]\sim[e/1]\circ[m/w\bullet w]$.

$$\begin{tikzcd}
E \arrow[r, "{[e/1]}"] & H \arrow[r, shift left, "{[t/z]}"] \arrow[r, shift right, "{[t'/z']}"'] & K \\
E' \arrow[u, dashed, "{[m/w\bullet w]}"] \arrow[ur, "{[e'/w]}"] & &
\end{tikzcd}$$

$$\begin{tikzcd}
E \arrow[r, "e"] & H \arrow[r, shift left, "z'\bullet t"] \arrow[r, shift right, "z\bullet t'"'] & K \\
E' \arrow[u, dashed, "m"] \arrow[ur, "w\bullet e'"] & &
\end{tikzcd}$$

To see that $[m/w\bullet w]$ is the unique such morphism, suppose that we also have $[e'/w]\sim[e/1]\circ[n/v]$, that is, $v\bullet e'=w\bullet e\circ n$. We have that $e\circ(v\bullet w\bullet m)=v\bullet w\bullet w\bullet e'=e\circ(w\bullet w\bullet w\bullet n)$. The morphism e is dagger monic because it is a dagger equalizer, so $v\bullet w\bullet m=w\bullet w\bullet w\bullet n$. By Lemma 7, $v\bullet m=w\bullet w\bullet n$, and so $[m/w\bullet w]\sim[n/v]$. $\square$

Lemma 19. *Any dagger monomorphism in $\mathbf{D}[\mathcal{D}^{-1}]$ is a kernel.*

Proof. Consider a dagger monomorphism in $\mathbf{D}[\mathcal{D}^{-1}]$. By Lemma 14, we may assume that it is of the form $[t/1]$, where $t\colon H\to K$ is a dagger monomorphism in $\mathbf{D}$. Now (9) gives $t=\ker(s)$ for some $s\colon K\to L$ in $\mathbf{D}$. We claim that $[t/1]=\ker([s/1])$ in $\mathbf{D}[\mathcal{D}^{-1}]$.

First, clearly $[s/1]\circ[t/1]=0$ because $s\circ t=0$. Next suppose that $[s/1]\circ[t'/z']\sim[0/1]\circ[t'/z']$ for some $[t'/z']\colon H'\to K$. Then $z'\bullet s\circ t'=z'\bullet 0$, so $s\circ t'=0$

by Lemma 7. Thus t' factors through $t = \ker(s)$ via some $m\colon H' \to H$. Then $[t/1] \circ [m/z'] \sim [t'/z']$ because $z' \bullet t \circ m = z' \bullet t'$.

$$\begin{tikzcd}
H \arrow[r, "{[t/1]}"] & K \arrow[r, shift left, "{[s/1]}"] \arrow[r, shift right, "{[0/1]}"'] & L \\
H' \arrow[u, dashed, "{[m/z']}"] \arrow[ur, "{[t'/z']}"'] & &
\end{tikzcd}$$

$$\begin{tikzcd}
H \arrow[r, "t"] & K \arrow[r, shift left, "s"] \arrow[r, shift right, "0"'] & L \\
H' \arrow[u, dashed, "m"] \arrow[ur, "t'"'] & &
\end{tikzcd}$$

Finally, to see that $[m/z']$ is unique, suppose that $[t/1] \circ [r/w] \sim [t'/z']$. Then $w \bullet t \circ m = w \bullet t' = z' \bullet t \circ r$. It follows that $z' \bullet r = w \bullet m$ because t is dagger monic, and hence $[m/z'] \sim [r/w]$. □

Lemma 20. *The wide subcategory of dagger monomorphisms in* $\mathbf{D}[\mathcal{D}^{-1}]$ *has directed colimits.*

Proof. Consider a directed partially ordered set indexing a diagram of dagger monomorphisms in $\mathbf{D}[\mathcal{D}^{-1}]$. By Lemma 14, we may assume that the diagram is represented as $[t_{ij}/1]\colon H_i \to H_j$ for dagger monomorphisms $t_{ij}\colon H_i \to H_j$ in $\mathbf{D}$ for $i \leq j$. Axiom (11) provides a colimiting cocone $c_i\colon H_i \to C$ in $\mathbf{D}$. By Lemma 14, for $[c_i/1]\colon H_i \to C$ to be a colimiting cocone in the wide subcategory of dagger monomorphisms of $\mathbf{D}[\mathcal{D}^{-1}]$, it suffices to prove that each c_i is dagger monic in $\mathbf{D}$.

$$\begin{tikzcd}
H_i \arrow[rr, "t_{ij}"] \arrow[dr, "c_i"] \arrow[ddr, bend right, "\mathrm{id}"'] & & H_j \arrow[dl, "c_j"'] \arrow[ddl, bend left, "t_{ij}^\dagger"] \\
& C \arrow[d, dashed, "m_i"] & \\
& H_i &
\end{tikzcd}$$

Fix an index i, and restrict both the diagram and the cocone to indices $j \geq i$. This restricted cocone is still colimiting. Another cocone on the restricted diagram is given by $t_{ij}^\dagger\colon H_j \to H_i$. Indeed, for $i \leq j \leq k$, we have:

$$t_{ik}^\dagger \circ t_{jk} = (t_{jk} \circ t_{ij})^\dagger \circ t_{jk} = t_{ij}^\dagger \circ t_{jk}^\dagger \circ t_{jk} = t_{ij}^\dagger$$

Thus, we obtain a mediating morphism $m_i\colon C \to H_i$ satisfying $m_i \circ c_j = t_{ij}^\dagger$ for all $j \geq i$. In particular, $m_i \circ c_i = \mathrm{id}_{H_i}$. Therefore, c_i is a dagger monomorphism by Lemma 8. □

Proposition 21. *There is an equivalence* $\mathbf{D}[\mathcal{D}^{-1}] \simeq \mathbf{Hilb}$.

Proof. Use Lemmas 12–20 to apply [6]. □

6. The theorem

The previous section showed that if a category $\mathbf{D}$ satisfies (1)–(11), then the functor $\mathbf{C}(I,-)$ from the completion $\mathbf{C} = \mathbf{D}[\mathcal{D}^{-1}]$ to $\mathbf{Hilb}_{\mathcal{C}}$ is an equivalence of monoidal dagger categories [6, Theorem 8]. Here $\mathcal{C} = \mathbf{C}(I,I)$ is an involutive field that is canonically isomorphic either to $\mathbb{R}$ or to $\mathbb{C}$ up to a choice of imaginary unit [6, Proposition 5]. To complete the proof of the main theorem, in this section we will determine the image of $\mathbf{D}$ in $\mathbf{Hilb} = \mathbf{Hilb}_{\mathcal{C}}$. The same proof applies to both the real and the complex cases. To simplify the presentation, we identify $\mathbf{D}$ with its canonical image in $\mathbf{C}$.

First, we characterize $\mathcal{D} \subseteq \mathcal{C}$.

Lemma 22. *We have that $\mathcal{D} = \{z \in \mathcal{C} \mid |z| \leq 1\}$.*

Proof. Let $z \in [0,1] \subseteq \mathcal{C}$ be an element of the unit interval. The morphism $v\colon I \to I \oplus I$ with components $\sqrt{z}$ and $\sqrt{1-z}$ is a dagger monomorphism and thus in $\mathbf{D}$. It follows that $z = v^\dagger \circ (1 \oplus 0) \circ v$ is also in $\mathbf{D}$. We conclude that $\mathcal{D} \supseteq \{z \in \mathcal{C} \mid 0 \leq z \leq 1\}$. We also know that $\mathcal{D} \supseteq \{z \in \mathcal{C} \mid |z| = 1\}$ because $\mathbf{D}$ contains all isometries. Therefore, $\mathcal{D} \supseteq \{z \in \mathcal{C} \mid |z| \leq 1\}$.

For the reverse inclusion, recall that for any set A we have a directed diagram in $\mathbf{C}$ that assigns an object I^R to each finite $R \subseteq A$, namely the biproduct of R many copies of I. The diagram consists of dagger monomorphisms $i_{R,S}\colon I^R \to I^S$ for finite $R \subseteq S$. For its colimit among the dagger monomorphisms of $\mathbf{C}$, write $i_{R,A}\colon I^R \to I^A$.

Because the morphisms of the diagram are dagger monomorphisms, it is also a diagram in $\mathbf{D}$. Write $j_R\colon I^R \to H$ for the colimit of this diagram in $\mathbf{D}$. A priori, the objects H and I^A may not be isomorphic. We now specialize to the case $A = \mathbb{N}$.

Let $z \in \mathcal{D}$. Without loss of generality, assume that the objects I^R, for finite $R \subseteq \mathbb{N}$, are constructed using the canonical monoidal product $\oplus$ on $\mathbf{D}$. Construct a natural transformation $\{t_R\colon I^R \to I^R\}_{R \subseteq \mathbb{N}}$ in $\mathbf{D}$ such that $t_{\{n\}}$ is scalar multiplication by z^n for each $n \in \mathbb{N}$. In the colimit, we obtain a morphism $t\colon H \to H$ such that the following square always commutes:

$$\begin{tikzcd}
I^{\{n\}} \arrow[r, "j_{\{n\}}"] \arrow[d, "t_{\{n\}}"'] & H \arrow[d, "T"] \\
I^{\{n\}} \arrow[r, "j_{\{n\}}"'] & H
\end{tikzcd}$$

As in [6], $I^{\{n\}} = I$ for each $n \in \mathbb{N}$. Thus, $j_{\{n\}}$ is a vector in the Hilbert space $\mathbf{C}(I,H)$. The wide subcategory of $\mathbf{C}$ with dagger monomorphisms is a subcategory of $\mathbf{D}$, and thus $i_{\{n\},\mathbb{N}}$ factors through $j_{\{n\}}$. Therefore the vector $j_{\{n\}}$ is nonzero.

The operator $T = \mathbf{C}(I,t)$ satisfies $T(j_{\{n\}}) = t \circ j_{\{n\}} = j_{\{n\}} \circ t_{\{n\}} = z^n \bullet j_{\{n\}}$. In other words, $j_{\{n\}}$ is an eigenvector of T with eigenvalue z^n. Thus, $\|T\| \geq |z^n| = |z|^n$ for all $n \in \mathbb{N}$. But T is bounded, so we must have $|z| \leq 1$. Therefore, this shows that $\mathcal{D} \subseteq \{z \in \mathcal{C} \mid |z| \leq 1\}$. Altogether, we have equality. □

Our next goal is to show that $\mathbf{D}$ cannot contain morphisms T of $\mathbf{Hilb}$ with $\|T\| > 1$. To engineer a counterexample, we first examine the situation in $\mathbf{Con}$.

Example 23. Let $0 < z_1 < z_2 < z_2 < \ldots < 1$ be an increasing sequence in $(0, 1]$, and let z_∞ be its supremum. The following is a directed diagram in **Con**:

$$\begin{tikzcd}
I \arrow[r, "z_1/z_2"] & I \arrow[r, "z_2/z_3"] & I \arrow[r] & \cdots
\end{tikzcd}$$

This diagram admits two cocones, among others: first, a cocone into I whose edges are simply the scalars $z_1, z_2, z_3, \ldots$; second, a cocone into I whose edges are the scalars $z_1/z_\infty, z_2/z_\infty, z_3/z_\infty, \ldots$. This gives the following diagram:

$$\begin{tikzcd}
I \arrow[r, "z_1/z_2"] \arrow[drrr] \arrow[ddrrr, "z_1"'] & I \arrow[r, "z_2/z_3"] \arrow[drr] \arrow[ddrr, "z_2"'] & I \arrow[r] \arrow[dr, "z_3/z_\infty"] \arrow[ddr, "z_3"'] & \cdots \\
 & & & I \arrow[d, dashed, "z_\infty"] \\
 & & & I
\end{tikzcd}$$

It is routine to verify that $z_1/z_\infty, z_2/z_\infty, z_3/z_\infty, \ldots$ is a colimiting cocone.

The following lemma demonstrates the same phenomenon in **D**.

Lemma 24. *The functor* $\mathbf{C}(I, -) \colon \mathbf{C} \to \mathbf{Hilb}$ *restricts to a functor* $\mathbf{D} \to \mathbf{Con}$. *Furthermore, for all objects H and all sequences* $0 < z_1 < z_2 < \ldots$ *with* $\sup z_n = 1$, *the following is a colimiting cocone in* **D***:*

$$\begin{tikzcd}
H \arrow[r, "(z_1/z_2)\bullet\mathrm{id}"] \arrow[drrr, "z_1\bullet\mathrm{id}"'] & H \arrow[r, "(z_2/z_3)\bullet\mathrm{id}"] \arrow[drr, "z_2\bullet\mathrm{id}"'] & H \arrow[r] \arrow[dr, "z_3\bullet\mathrm{id}"] & \cdots \\
 & & & H
\end{tikzcd}$$

Proof. Let $t \colon H \to K$ be a morphism in **D**. Assume that $\|\mathbf{C}(I, t)\| > 1$, and let $x \in \mathbf{C}(I, H)$ be such that $\|x\| = 1$ and $\|t \circ x\| > 1$. Now $x^\dagger \circ x = \|x\|^2 = 1$, so x is dagger monic and hence in **D**. Furthermore $x^\dagger \circ t^\dagger \circ t \circ x = \|t \circ x\|^2 > 1$. But $x^\dagger \circ t^\dagger \circ t \circ x$ is a scalar in **D**, contradicting Lemma 22. We conclude that $\|\mathbf{C}(I, t)\| \leq 1$, and that $\mathbf{C}(I, -)$ restricts to a functor $\mathbf{D} \to \mathbf{Con}$.

Let $0 < z_1 < z_2 < \ldots$ be an increasing sequence of numbers with supremum 1. The morphisms $(z_n/z_{n+1}) \bullet \mathrm{id}$ are all in **D** by Lemma 22. They form a directed diagram, which has a colimiting cocone $t_n \colon H \to K$. Now $z_n \bullet \mathrm{id} \colon H \to H$ forms another cocone. Thus there is a unique mediating morphism $s \colon K \to H$ in **D**:

$$\begin{tikzcd}
H \arrow[r, "(z_1/z_2)\bullet\mathrm{id}"] \arrow[drrr, "t_1"] \arrow[ddrrr, "z_1\bullet\mathrm{id}"'] & H \arrow[r, "(z_2/z_3)\bullet\mathrm{id}"] \arrow[drr, "t_2"] \arrow[ddrr, "z_2\bullet\mathrm{id}"'] & H \arrow[r] \arrow[dr, "t_3"] \arrow[ddr, "z_3\bullet\mathrm{id}"'] & \cdots \\
 & & & K \arrow[d, dashed, "s"] \\
 & & & H
\end{tikzcd}$$

The functor $\mathbf{C}(I,-)\colon \mathbf{D} \to \mathbf{Con}$ maps the cocone $z_n \bullet \mathrm{id}\colon H \to H$ to a colimiting cocone in $\mathbf{Con}$. Thus there is a mediating bounded operator $T\colon \mathbf{C}(I,H) \to \mathbf{C}(I,K)$:

$$\begin{tikzcd}
\mathbf{C}(I,H) \arrow[r, "(z_1/z_2)\mathrm{Id}"] \arrow[drrrr] \arrow[ddrrrr] \arrow[dddrrrr, "z_1\mathrm{Id}"'] & \mathbf{C}(I,H) \arrow[r, "(z_2/z_3)\mathrm{Id}"] \arrow[drrr] \arrow[ddrrr, "\mathbf{C}(I,t_2)"] \arrow[dddrrr] & \mathbf{C}(I,H) \arrow[rr] \arrow[drr, "z_3\mathrm{Id}"] \arrow[ddrr] \arrow[dddrr] & & \cdots \\
& & & & \mathbf{C}(I,H) \arrow[d, dashed, "T"] \\
& & & & \mathbf{C}(I,K) \arrow[d, dashed, "\mathbf{C}(I,s)"] \\
& & & & \mathbf{C}(I,H)
\end{tikzcd}$$

The universal property now implies that $\mathbf{C}(I,s) \circ T$ is the identity on $\mathbf{C}(I,H)$. Because both $\mathbf{C}(I,s)$ and T are contractions, T must be an isometry. It follows that there is a dagger monomorphism $t\colon H \to K$ in $\mathbf{C}$, and hence in $\mathbf{D}$, with $\mathbf{C}(I,t) = T$.

Thus, the morphisms s and t in $\mathbf{D}$ satisfy $\mathbf{C}(I, s \circ t) = \mathbf{C}(I,s) \circ \mathbf{C}(I,t) = \mathbf{C}(I,\mathrm{id})$. Because the functor $\mathbf{C}(I,-)$ is faithful, $s \circ t = \mathrm{id}$. In the same way, we find that $\mathbf{C}(I,t_n) = \mathbf{C}(I,t) \circ (z_n \mathrm{id}) = \mathbf{C}(I, z_n \bullet t)$, so $t_n = t \circ (z_n \bullet \mathrm{id}) = t \circ s \circ t_n$, concluding that $t \circ s = \mathrm{id}$. Since t is dagger monic, both s and t are dagger isomorphisms that are inverse to each other. Thus the cocone $z_n \bullet \mathrm{id}\colon H \to H$ is colimiting in $\mathbf{D}$. $\square$

Finally, we show that the image of $\mathbf{D}$ in $\mathbf{Hilb}$ contains all contractions.

Lemma 25. *For all objects H and K, $\mathbf{D}(H,K) = \{t \in \mathbf{C}(H,K) \mid \|\mathbf{C}(I,t)\| \leq 1\}$.*

Proof. It remains to show that morphisms t of $\mathbf{C}$ with $\|\mathbf{C}(I,t)\| \leq 1$ are in $\mathbf{D}$.

First, suppose $t\colon H \to H$ in $\mathbf{C}$ satisfies $\|\mathbf{C}(I,t)\| < 1$. The Russo-Dye-Gardner Theorem [13, Proposition 3.2.23] gives unitary operators $U_1, \ldots, U_n$ on the Hilbert space $\mathbf{C}(I,H)$ with $\mathbf{C}(I,t) = \frac{1}{n}(U_1 + \cdots + U_n)$. Now [6] gives dagger isomorphisms $u_1 \ldots, u_n\colon H \to H$ in $\mathbf{C}$ such that $t = \frac{1}{n}(u_1 + \cdots + u_n)$. Each u_i is in $\mathbf{D}$. For any Hilbert space $\mathcal{H}$, the function $W\colon \mathcal{H} \to \mathcal{H} \oplus \cdots \oplus \mathcal{H}$ that is defined by $x \mapsto (x, \cdots, x)/\sqrt{n}$ has the property that all operators $T_1, \ldots, T_n$ on $\mathcal{H}$ satisfy $W^\dagger \circ (T_1 \oplus \cdots \oplus T_n) \circ W = \frac{1}{n}(T_1 + \cdots + T_n)$. Now [6] provides a dagger monomorphism $w\colon H \to H \oplus \cdots \oplus H$ in $\mathbf{C}$ with $w^\dagger \circ (t_1 \oplus \cdots \oplus t_n) \circ w = \frac{1}{n}(t_1 + \cdots + t_n)$. As before, w is in $\mathbf{D}$. Now, we reason that $u_1, \ldots, u_n$ are in $\mathbf{D}$, that $u_1 \oplus \cdots \oplus u_n$ is in $\mathbf{D}$, and that $w^\dagger \circ (u_1 \oplus \cdots \oplus u_n) \circ w$ is in $\mathbf{D}$. We conclude that $t = \frac{1}{n}(u_1 + \cdots + u_n)$ is in $\mathbf{D}$. Therefore, any endomorphism t of $\mathbf{C}$ with $\|\mathbf{C}(I,t)\| < 1$ is in $\mathbf{D}$.

Next, suppose $t\colon H \to K$ in $\mathbf{C}$ satisfies $\|\mathbf{C}(I,t)\| < 1$. Using polar decomposition, any operator T in $\mathbf{Hilb}$ factors as $T = U \circ V^\dagger \circ S$, where U and V are isometries and S is a bounded operator on the domain of T such that $\|S\| = \|T\|$. Because $\mathbf{C}(I,-)\colon \mathbf{C} \to \mathbf{Hilb}$ is an equivalence of monoidal dagger categories, t factors as $t = u \circ v^\dagger \circ s$, where u and v are dagger monomorphisms and s is an endomorphism such that $\|\mathbf{C}(I,s)\| < 1$. Thus, t is in $\mathbf{D}$. Therefore, any morphism t in $\mathbf{C}$ with $\|\mathbf{C}(I,t)\| < 1$ is in $\mathbf{D}$.

Finally, suppose $t\colon H \to K$ in $\mathbf{C}$ satisfies $\|\mathbf{C}(I,t)\| = 1$. Let $0 < z_1 < z_2 < \ldots$ be a sequence of numbers with supremum 1. For each $0 < z < 1$, we calculate that $\|\mathbf{C}(I, z \bullet t)\| = \|z\mathbf{C}(I,t)\| = z\|\mathbf{C}(I,t)\| < 1$; hence $z \bullet t \in \mathbf{D}(H,K)$. Therefore, the following diagram lies entirely in $\mathbf{D}$:

$$\begin{tikzcd}
H \arrow[r, "(z_1/z_2)\bullet\mathrm{id}"] \arrow[drrr, "z_1\bullet\mathrm{id}"] \arrow[ddrrr, bend right, "z_1\bullet t"'] & H \arrow[r, "(z_2/z_3)\bullet\mathrm{id}"] \arrow[drr, "z_2\bullet\mathrm{id}"] \arrow[ddrr, "z_2\bullet t"] & H \arrow[r] \arrow[dr, "z_3\bullet\mathrm{id}"] \arrow[ddr, "z_3\bullet t"] & \cdots \\
 & & & H \arrow[d, dashed, "s"] \\
 & & & K,
\end{tikzcd}$$

Lemma 24 provides a unique morphism $s \in \mathbf{D}(H,K) \subseteq \mathbf{C}(H,K)$ making this diagram commute. Thus, $z_1 \bullet s = s \circ (z_1 \bullet \mathrm{id}) = z_1 \bullet t$. Since $z_1 \neq 0$, we conclude by Lemma 7 that $t = s$. Hence t is in $\mathbf{D}$. Therefore, any morphism t of $\mathbf{C}$ with $\|\mathbf{C}(I,t)\| \leq 1$ is in $\mathbf{D}$. $\square$

We have arrived at the main theorem.

Theorem 26. *A category satisfying axioms* (1)–(11) *is equivalent to* $\mathbf{Con}_{\mathbb{R}}$ *or* $\mathbf{Con}_{\mathbb{C}}$*, and the equivalence preserves daggers and is strong symmetric monoidal for both* $\otimes$ *and* $\oplus$*.*

Proof. Combine Proposition 21 and Lemma 25. $\square$

In fact, all of the isomorphisms of axiom (2) are preserved, making any category satisfying axioms (1)–(11) equivalent to $\mathbf{Con}_{\mathbb{R}}$ or $\mathbf{Con}_{\mathbb{C}}$ as dagger rig categories.

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