Abstract. We extend the Stone duality between topological spaces and locales to include order: there is an adjunction between the category of preordered topological spaces satisfying the so-called open cone condition, and the newly defined category of ordered locales. The adjunction restricts to an equivalence of categories between spatial ordered locales and sober $T_0$-ordered spaces with open cones.

1. Introduction

There are many reasons why one might want to equip (the underlying set of) a topological space with a preorder. In topology itself, ordered spaces were first used to study fixed point theorems [40] and compactifications [32]. But they also give algebraic means to study topological spaces [35], providing models for (modal) logic. This is especially useful in computability- and domain theory [11], where the order indicates levels of knowledge about an ongoing computation. Similarly, ordered spaces are used in the study of concurrent computing [16, 9], where topological invariants can recognise phenomena such as deadlock. A related use of ordered spaces is in directed algebraic topology [14], an abstract setting for homotopy theory where paths need not be reversible. Finally, ordered spaces have been used in the physics of spacetime [25] and quantum gravity [3], where the order models whether there can be a causal influence of one point on another [33].

There are also many reasons why one might want to work with locales instead of topological spaces. Working with the family of open sets while disregarding any possible points leads to better behaved spaces [19], which is intimately related to constructive proofs [20]. It allows considering rigorously the topological intuition inherent to spaces without points [34] that occur naturally throughout mathematics. This is most pronounced in topos theory [21]: pragmatically, any theorem proved locale-theoretically automatically also applies to for examples sheaves [29]. Similarly, synthetic topology [8] can extract computational algorithms from locale-theoretic continuity, and locale-theoretic methods make logical methods available to reason about topology [39]. Finally, there are unresolved foundational discussions about the nature of physical spacetime, which may not be continuous [3] or may not have an empirically accessible notion of point [10].

Stone duality famously links topological spaces and locales, giving an adjunction between the two categories [19]: given a topological space, its partially ordered family of open sets forms a locale, and conversely, given a locale, one can consider the topological space of its points, suitably defined. In this article, we extend Stone duality to take preorders into account. To what structure on its locale does
a preorder on a topological space correspond? Our answer is an axiomatisation of the appropriate notion of ordered locale. We lift the preorder \( x \leq y \) on points of a topological space to the Egli-Milner preorder \( U \leq V \) on the open subsets of the topological space, and axiomatise the latter. For a working theory, we restrict to preorders on topological spaces satisfying the open cone condition: the up- and downsets of an open set are again open. We establish an adjunction between the following two categories: preordered topological spaces with open cones and enough points, and continuous monotone functions; and ordered locales and monotone locale morphisms. The adjunction restricts to an equivalence between the full subcategories of sober \( T_0 \)-ordered spaces with open cones and spatial ordered locales. This could also be regarded as a natural variation on the study of Boolean algebras with additional operations [22].

The article builds up towards that adjunction. Section 2 starts by discussing ordered spaces, and Section 3 introduces ordered locales and their morphisms. Then, Section 4 studies the functor that takes opens, turning an ordered space into an ordered locale. Vice versa, Section 5 studies the functor that takes points, turning an ordered locale into an ordered space. Putting it all together, Section 6 proves the adjunction, and investigates its fixed points to establish Stone duality in the ordered setting. Finally, Section 7 concludes by raising directions for further research.

2. Ordered spaces

We start with basic properties of preordered sets. A preorder, which we also simply call order, is a binary relation on a set that is reflexive \((x \leq x)\) and transitive \((x \leq y \leq z \implies x \leq z)\). A function \(g: S \to T\) between preordered sets is monotone if \(x \leq y\) implies \(g(x) \leq g(y)\). Preordered sets and monotone functions form a category \(\text{Ord}\). If \((S, \leq)\) is a preordered set, we write \(\uparrow A = \{y \in S \mid \exists x \in A: x \leq y\}\) and \(\downarrow A = \{x \in S \mid \exists y \in A: x \leq y\}\) for the up- and downset of a subset \(A \subseteq S\). Thinking of \(\leq\) as a causality relation, we may also call these future- and past cones.

**Lemma 2.1.** If \((S, \leq)\) is a preordered set, then:

(a) \(A \subseteq \uparrow A\) and \(A \subseteq \downarrow A\) for any subset \(A\);
(b) \(\uparrow A = \uparrow \uparrow A\) and \(\downarrow A = \downarrow \downarrow A\) for any subset \(A\);
(c) \(\uparrow A \subseteq \uparrow B\) and \(\downarrow A \subseteq \downarrow B\) for any subsets \(A \subseteq B\);
(d) \(\bigcup \uparrow A_i = \uparrow \left( \bigcup A_i \right)\) and \(\bigcap \downarrow A_i = \downarrow \left( \bigcap A_i \right)\) for any family \(A_i\) of subsets.

**Proof.** The first property follows from reflexivity, the second from transitivity, and the third holds by construction. For the fourth property, note that \(\uparrow \emptyset = \emptyset = \downarrow \emptyset\), so we may assume the indexing set is not empty. Then \(A_j \subseteq \bigcup A_i\) for any index \(j\), and so \(\bigcup (A_i) \subseteq \bigcup (\bigcup A_i)\) by the third property. For the converse inclusion, pick \(y \in \uparrow \left( \bigcup A_i \right)\). Then there exist an index \(j\) and an element \(x \in A_j\) such that \(x \leq y\). But this just means that \(y \in \uparrow A_j\), and by extension \(y \in \bigcup \uparrow A_i\). The argument for downsets is analogous. \(\square\)

Next we consider how inverse images of monotone functions interact with cones.

**Proposition 2.2.** For a function \(g: S \to T\) between preordered sets, the following are equivalent:

(a) \(g\) is monotone;
(b) \( \uparrow g^{-1}(B) \subseteq g^{-1}(\uparrow B) \) for all \( B \subseteq T \);
(c) \( \downarrow g^{-1}(B) \subseteq g^{-1}(\downarrow B) \) for all \( B \subseteq T \).

Proof. First assume (a); we will show (b). Take \( y \in \uparrow g^{-1}(B) \). Then there exists \( x \in g^{-1}(B) \) with \( x \leq y \). Monotonicity of \( g \) gives \( g(x) \leq g(y) \). But \( g(x) \in B \), so \( g(y) \in \uparrow B \), and therefore \( y \in g^{-1}(\uparrow B) \).

Next, assume (b); we will show (a). Suppose \( x \leq y \) in \( S \), that is, \( y \in \uparrow \{x\} \). Furthermore \( x \in g^{-1}\{g(x)\} \). Hence, by assumption:
\[
y \in \uparrow \{x\} \subseteq \uparrow g^{-1}\{g(x)\} \subseteq g^{-1}(\uparrow \{g(x)\}).
\]
But this just means \( g(y) \in \uparrow \{g(x)\} \), that is, \( g(x) \leq g(y) \).

The equivalence between (a) and (c) is proved similarly. \( \square \)

Now we move from ordered sets to ordered spaces.

**Definition 2.3.** An ordered topological space, or ordered space for short, is a topological space equipped with a preorder on its underlying set. Ordered spaces and continuous monotone functions form a category \( \text{OrdTop} \).

The category \( \text{Ord} \) is the (coreflective) subcategory of \( \text{OrdTop} \) of discrete ordered spaces.

**Definition 2.4.** An ordered space \( S \) has open cones if \( \uparrow U \) and \( \downarrow U \) are open whenever \( U \subseteq S \) is open. Write \( \text{OrdTop}_{OC} \) for the full subcategory of \( \text{OrdTop} \) of ordered spaces with open cones.

We will see in Section 4 that the open cone condition provides exactly the right connection between order and topology to obtain a well-defined adjunction.

Any preordered set, regarded as a discrete topological space, has open cones. On the other extreme, on a codiscrete topological space \( S \), any preorder has open cones, because \( \downarrow S = \downarrow S = S \) and \( \uparrow \emptyset = \downarrow \emptyset = \emptyset \) are always open. The next examples are less trivial, and show that the open cone condition is different from the more well known order-separation axioms.

**Example 2.5.** Any topological space becomes an ordered space with equality as the preorder. This ordered space has open cones because \( \downarrow U = U \) and \( \uparrow U = U \). Thus there are examples of topological spaces with poorly behaved separation properties that nevertheless have open cones. This also tells us that the open cone condition does not imply any of the order-separation axioms [30].

**Example 2.6.** Not every ordered space has open cones. Consider the real line \( \mathbb{R} \) with the standard Euclidean topology. Imagine there is a discrete point living “before” the origin of this line. Formally, we get the space \( \{*\} \sqcup \mathbb{R} \), with the preorder generated by \( * \leq 0 \). Now \( \uparrow \{*\} = \{*\} \sqcup \{0\} \), but of course the singleton \( \{0\} \) is not open in \( \mathbb{R} \), so this cone is not open. See also Figure 1. (For another example see [36, Example 10.1.1].)

On the other hand, it is easy to see that the graph of this preorder is closed. Namely, it is the union of the diagonals of \( \mathbb{R} \) and \( \{*\} \) (which are both closed since they are both Hausdorff), together with the singleton \( \{(*,0)\} \), which is also closed. Hence this is an example of a \( T_2 \)-ordered space [30, Theorem 2], also known as a pospace. This, together with the previous example, shows that our condition of having open cones really is different than imposing (order-)separation axioms.
Example 2.7. Any preorder induces the upper topology on its carrier set, where a subset $U$ is open if and only if it is upward closed: if $x \in U$ and $x \leq y$, then also $y \in U$. In this way any preordered set becomes an ordered space. It has open cones if and only if $\downarrow \uparrow A = \uparrow \downarrow \uparrow A$ for any subset $A$. Any monotone function is automatically continuous for the upper topology.

Any lattice, that is, any partially ordered set with binary least upper bounds and binary greatest lower bounds, induces the interval topology on its carrier set, where a subset is open if and only if it is a union of intervals $\langle x, z \rangle = \{ y \mid x \leq y \leq z \}$. The resulting ordered space has open cones, because

$$\uparrow \bigcup (x_i, z_i) = \bigcup \uparrow \langle x_i, z_i \rangle = \bigcup \{ \langle x_i, z \rangle \mid x_i \leq z \},$$

and similarly for past cones.

Any topological space has a specialisation preorder, where $x \leq y$ if and only if $x \in \{ y \}$. In this way any topological space becomes an ordered space, that always has open future cones, but need not have open past cones. Any continuous function is automatically monotone for the specialisation preorders.

Before moving on to physically motivated examples, we record an important characterisation of the open cone condition.

**Proposition 2.8.** An ordered space has open cones if and only if: if $x \leq y$, then $y$ is in the interior $(\uparrow U)°$ of the upset for any open neighbourhood $U$ of $x$, and similarly $x \in (\downarrow V)°$ for any open neighbourhood $V$ of $y$.

**Proof.** Suppose the ordered space has open cones, and let $x \leq y$. We need to show that $y \in (\uparrow U)°$ for any open neighbourhood $U$ of $x$. But since $x \leq y$ we have $y \in \uparrow \{ x \}$, so if $U \ni x$ is open we get $y \in \uparrow U = (\uparrow U)°$, as desired. Similarly, $x \in \downarrow \{ y \} \subseteq \downarrow V = (\downarrow V)°$ for any open neighbourhood $V$ of $y$.

Now for the converse. Let $U$ be an open set. We need to show that $\uparrow U \subseteq (\uparrow U)°$. Take an element $y \in \uparrow U$, meaning that there exists $x \in U$ with $x \leq y$. Now the assumption gives precisely that $y \in (\uparrow U)°$. Hence $\uparrow U = (\uparrow U)°$. That the past cones preserve opens follows similarly. \qed

**Example 2.9.** A consequence of Proposition 2.8 is that $S$ has open cones if and only if

$$U \cap \uparrow \{ x \} \subseteq \uparrow (\downarrow U)° \cap \{ x \} \quad \text{and} \quad U \cap \downarrow \{ x \} \subseteq \downarrow (\uparrow U)° \cap \{ x \}$$

for all $x \in S$ and $U \in \mathcal{O}S$. The natural localic generalisation of this condition is to say that

$$U \cap \uparrow V \subseteq \uparrow ((\downarrow U)° \cap V) \quad \text{and} \quad U \cap \downarrow V \subseteq \downarrow ((\uparrow U)° \cap V)$$

**Figure 1.** An ordered space that satisfies (1) but does not have open cones: $U = \{(0,0)\}$ is open, but $\uparrow U = \{(0,0), (0,1)\}$ is not.
holds for all \( U, V \in \mathcal{O}S \). The following is an example of an ordered space that satisfies this localic law, but does not have open cones.

Recall the ordered space in Example 2.6, which we now exhibit as the subspace
\[
S = (\{0\} \times \{0\}) \cup (\mathbb{R} \times \{1\}) \subseteq \mathbb{R}^2,
\]
where the order \( \leq \) on \( \mathbb{R}^2 \) is given by \((x, y) \leq (a, b)\) if and only if \( x = a\) and \( y \leq b\). Take the topology on \( \mathbb{R}^2 \) generated by the basis consisting of subsets of the form \((a, b) \times (-\infty, y) \subseteq \mathbb{R}^2\), for arbitrary \( a, b, y \in \mathbb{R}\). With this new topology, \( \mathbb{R}^2 \) has open cones, but \( S \) does not. However, after some elementary checks, it can be seen that \( S \) does satisfy (1). See also Figure 1.

We end this section by considering examples from relativity theory, which were our original motivation. In this setting, the upset \( \uparrow\{x\} \) of a point \( x \) is typically called its \textit{causal future} and denoted \( J^+(x) \), and \( \downarrow\{x\} \) is called the \textit{causal past} and denoted \( J^-(x) \). Similarly, the interior \( \mathring{\uparrow}\{x\} \) is called the \textit{chronological future} and written \( I^+(x) \), and \( \mathring{\downarrow}\{x\} \) is called the \textit{chronological past} and written \( I^-(x) \). Extending this notation with \( \mathring{\uparrow}\{U\} = \bigcup_{x \in U} J^+(x) \) and \( \mathring{\downarrow}\{U\} = J^-(U) \), the open cone condition says \( I^\pm(U) = J^\pm(U) \) for all opens \( U \). In the setting of relativity theory, \( I^\pm \) are not intrinsically defined as the interior of \( J^\pm \), but rather this is a derived property [31, Theorem 2.27], that we here lift to a definition.

We will illustrate our results with diagrams taken from relativity theory intuition: one time dimension runs upwards, and one space dimension is drawn horizontally. Future and past cones of points and open sets then look as in Figure 2.

The following lemma shows that the open cone condition can be regarded as abstracting the \textit{“push-up” principle} from Lorentzian causality theory [15, 25].

**Lemma 2.10.** An ordered space has open cones if and only if \( \mathring{\uparrow}\{U\} \subseteq \mathring{\uparrow}\{U\} \) and \( \mathring{\downarrow}\{U\} \subseteq \mathring{\downarrow}\{U\} \) for all open subsets \( U \).

**Proof.** Suppose that the stated condition holds, and let \( U \) be an open subset. By assumption \( \mathring{\uparrow}\{U\} \subseteq \mathring{\uparrow}U \) and \( \mathring{\downarrow}\{U\} \subseteq \mathring{\downarrow}U \). We need to show the reverse inclusions. From Lemma 2.1(a) we know that \( U \subseteq \mathring{\uparrow}U \), so openness of \( U \) gives \( U = \mathring{\uparrow}U \subseteq \mathring{\uparrow}\{U\} \). Because upsets are monotone with respect to subset inclusion by Lemma 2.1(c), this gives \( \mathring{\uparrow}U \subseteq \mathring{\uparrow}\{U\} \subseteq \mathring{\uparrow}U \), and similarly for downsets, so the open cone condition holds.

Conversely, if the open cone condition holds, then \( \mathring{\uparrow}\{U\} = \mathring{\uparrow}\uparrow\mathring{\uparrow}U = \mathring{\uparrow}U = \mathring{\uparrow}\{U\} \) follows from Lemma 2.1(b), and similarly for downsets. \( \square \)

A \textit{smooth spacetime} is a four-dimensional connected smooth Lorentzian manifold that is time oriented [27, Definition 5.3]. We may define the \textit{causality relation} \( x \leq y \)
on points of a spacetime when there exists a future directed causal curve from \( x \) to \( y \): that is, a smooth curve whose tangent vector is future directed and causal at every point (see e.g. 
[27, Section 5.3] or [31] for more details).

**Corollary 2.11.** Any smooth spacetime has open cones.

**Proof.** Combine Lemma 2.10 with [27, Proposition 5.4]. \( \square \)

We expect that the previous corollary continues to hold for a large class of lower-regularity spacetimes; essentially those where the push-up principle continues to hold. However, extremely low-regularity spacetimes potentially exhibit more degenerate causal structure, see e.g. [5, 15] and references therein. The previous corollary shows that \( \text{OrdTop}_{\text{OC}} \) may serve as a viable category of abstract (smooth) spacetimes, and also provides many interesting examples.

## 3. Ordered locales

This section discusses the notion of ordered locales. These axiomatise how the structure of an ordered space transfers to its lattice of open subsets, as will be discussed in the next section. The main idea is to introduce a new order \( \sqsubseteq \) between open subsets, where \( U \sqsubseteq V \) if and only if every element of \( U \) lies in the past of some element of \( V \), and every element of \( V \) lies in the future of some element of \( U \). In formulas: \( U \subseteq (\downarrow V)^{\circ} \) and \( V \subseteq (\uparrow U)^{\circ} \); in pictures: see Figure 3. The behaviour of \( \sqsubseteq \) transfers to properties of \( \sqsubseteq \), which we will capture as the axioms for ordered locales below.

Let us first briefly recall basic properties of the category of locales; see [34] for more details. A frame is a complete lattice \( L \) satisfying infinite distributivity:

\[
U \land \bigvee V_i = \bigvee (U \land V_i).
\]

A frame morphism is a function that preserves all suprema and finite meets, so in particular preserves the least and greatest elements. Write \( \text{Frm} \) for the category of frames and frame morphisms. The category of locales is the opposite:

\[
\text{Loc} = \text{Frm}^{\text{op}}.
\]

Thus a locale \( X \) is formally defined by its frame of opens \( \mathcal{O}X \), and a locale morphism \( f \colon X \to Y \) corresponds a frame morphism \( f^{-1} \colon \mathcal{O}Y \to \mathcal{O}X \), interpreted as the ‘preimage’ map of \( f \). We will write \( \sqsubseteq \) for the intrinsic order of a frame \( \mathcal{O}X \) (to prevent a notation clash with the order \( \sqsubseteq \) on a topological space).
Now we come to the main definition.

**Definition 3.1.** An ordered locale is a locale $X$ together with a preorder $\preceq$ on its frame $\mathcal{O}X$ of opens, satisfying

\[
\forall i: U_i \preceq V_i \implies \bigvee U_i \preceq \bigvee V_i
\]

where $i$ ranges over a non-empty indexing set.

**Lemma 3.2.** In an ordered locale:

(a) if $U \preceq U'$ and $U \subseteq V$, then $V \preceq V'$ and $U' \subseteq V'$ for some $V'$;

(b) if $U \preceq U'$ and $U' \subseteq V'$, then $U \subseteq V$ and $V \preceq V'$ for some $V$.

Diagrammatically:

\[
\begin{array}{ccc}
U & \preceq & U' \\
\sqcap & \sqcap & \sqcap \\
V & \preceq & \exists V' \\
\end{array} \quad \begin{array}{ccc}
U & \preceq & U' \\
\sqcap & \sqcap & \sqcap \\
\exists V' & \preceq & V' \\
\end{array}
\]

See also Figure 4.

**Proof.** For (a), since $U \preceq U'$ and $U \subseteq V$, axiom $(\forall)$ implies $V = U \lor V \preceq U' \lor V$, and so $V' := U' \lor V$ satisfies the statement. Point (b) is proved similarly. □

Morphisms of ordered locales are more intricate to define than in the setting of topological spaces, because a locale morphism $f: X \to Y$ is a frame morphism $f^{-1}: \mathcal{O}Y \to \mathcal{O}X$ in the other direction. To ask that $f^{-1}$ preserves order makes little sense, as it corresponds to asking that the would-be function $f$ reflects order. To remedy this we move from functions to relations. For a locale morphism $f: X \to Y$, define the relation:

\[
R_f := \{(U,V) \mid U \subseteq f^{-1}V\} \subseteq \mathcal{O}X \times \mathcal{O}Y.
\]

Denoting the category of sets and relations by $\text{Rel}$, we can similarly interpret a preorder $\preceq$ on $\mathcal{O}X$ as a morphism $\mathcal{O}X \to \mathcal{O}X$. In fact, $\text{Rel}$ is a 2-category, with a unique 2-morphism $R \Rightarrow S$ precisely when $R \subseteq S$. Finally, $\text{Rel}$ has a dagger, where every relation $R: A \to B$ gives rise to another relation $R^\dagger: B \to A$ defined by $(b,a) \in R^\dagger$ if and only if $(a,b) \in R$. 
Definition 3.3. A morphism of ordered locales is a locale morphism \( f : X \to Y \) that is monotone, in the sense that the following two 2-cells in Rel exist:

\[
\begin{array}{c}
\emptyset X \cong \emptyset X \\
\downarrow R_f \\
\emptyset Y \cong \emptyset Y
\end{array}
\quad
\begin{array}{c}
\emptyset X \cong \emptyset X \\
\downarrow R_f \\
\emptyset Y \cong \emptyset Y
\end{array}
\]

The proof of the following lemma gives a convenient way to recognise monotonicity of a locale morphism.

Lemma 3.4. Let \( X \) be a locale. A preorder \( \subseteq \) on \( OX \) satisfies the properties of Lemma 3.2 if and only if \( \text{id}_X \) is monotone in the sense of Definition 3.3.

Proof. Unpacking what it means for a locale morphism \( f \) to be monotone gives:

- if \( U \subseteq U' \) and \((U',V') \in R_f\), then \((U,V) \in R_f\) and \( V \subseteq V' \); for some \( V' \);
- if \( U \subseteq U' \) and \((U,V) \in R_f\), then \((U',V') \in R_f\) and \( V \subseteq V' \) for some \( V \);

Diagrammatically:

\[
\begin{array}{c}
U \subseteq U' \\
R_f \\
\exists V' \ni V \subseteq \exists V'
\end{array} =
\begin{array}{c}
U \subseteq U' \\
R_f \\
\exists V' \ni V \subseteq \exists V'
\end{array}
\]

See also Figure 5. The statement now follows immediately.

Proposition 3.5. Ordered locales and their morphisms form a category \textbf{OrdLoc}.

Proof. Lemma 3.4 tells us identity morphisms of frames are monotone, so we are left to show that if \( f : X \to Y \) and \( g : Y \to Z \) are monotone locale morphisms, then their composition \( g \circ f \) is again monotone. This follows easily from \( R_{g \circ f} = R_g \circ R_f \) and using the interchange law for horizontal and vertical composition in Rel.

Section 2 showed that an ordered space \( S \) has well-behaved cones \( U \mapsto \uparrow U \) and \( U \mapsto \downarrow U \) on its powerset. These completely determine the order \( \leq \) because \( x \leq y \) if and only if \( y \in \uparrow\{x\} \) if and only if \( x \in \downarrow\{y\} \). This relies heavily on the availability of singleton sets. Can we define a pointless analogue of cones that captures some of the properties of \( \subseteq ? \) The following results answer this question positively.

\[
\begin{array}{c}
f^{-1}(V') \\
\exists V'
\end{array}
\quad
\begin{array}{c}
f^{-1}(V) \\
V
\end{array}
\]

Figure 5. Illustration of monotonicity of locale morphisms.
Suppose $X$ is a locale with a preorder $\sqsubseteq$ on its frame of opens, not necessarily satisfying axiom $(\lor)$ of an ordered locale. For $U \in \mathcal{O}X$, define localic cones:

$$\uparrow U = \bigvee \{V \in \mathcal{O}X \mid U \sqsubseteq V\},$$
$$\downarrow U = \bigvee \{W \in \mathcal{O}X \mid W \sqsubseteq U\}.$$  

**Lemma 3.6.** For a locale $X$ and a preorder $\sqsubseteq$ on its frame of opens:

(a) if $U \sqsubseteq V$ then $U \sqsubseteq \downarrow V$ and $V \sqsubseteq \uparrow U$;
(b) $U \sqsubseteq \uparrow U$ and $U \sqsubseteq \downarrow U$.

If axiom $(\lor)$ is satisfied, then furthermore:

(c) $U \sqsubseteq \uparrow U$ and $\downarrow U \sqsubseteq U$;
(d) $\uparrow \uparrow U = \uparrow U$ and $\downarrow \downarrow U = \downarrow U$;
(e) if $U \sqsubseteq V$ then $\uparrow U \sqsubseteq \uparrow V$ and $\downarrow U \sqsubseteq \downarrow V$.

**Proof.** Points (a) and (c) follow immediately from the definition of localic cones, and point (b) from reflexivity of $\sqsubseteq$. For (d): applying (c) twice gives $U \sqsubseteq \uparrow U \sqsubseteq \uparrow \uparrow U$; and then applying (a) gives $\uparrow \uparrow U \sqsubseteq \uparrow U$; a similar argument holds for $\downarrow$. Lastly, for (e), if $U \sqsubseteq V$ and $U \sqsubseteq W$ then from Lemma 3.2(a) we get $V \sqsubseteq V'$ for some $W \sqsubseteq V'$, and hence $W \sqsubseteq \uparrow V$. Thus we get $\uparrow U \sqsubseteq \uparrow V$; again, a similar argument holds for localic downsets. □

In direct analogy to Proposition 2.2, we can now determine the monotonicity of morphisms using the localic cones, justifying Definition 3.3.

**Proposition 3.7.** A locale morphism $f : X \to Y$ between ordered locales is monotone if and only if for all $V \in \mathcal{O}Y$:

$$\uparrow f^{-1}(V) \sqsubseteq f^{-1}(\uparrow V) \quad \text{and} \quad \downarrow f^{-1}(V) \sqsubseteq f^{-1}(\downarrow V).$$

**Proof.** Assume first that $f$ is monotone. Using Lemma 3.6(c) we get the relation $f^{-1}(V) \sqsubseteq \uparrow f^{-1}(V)$, and $(f^{-1}(V), V) \in R_f$ trivially holds, so there exists $V' \in \mathcal{O}Y$ such that $V \sqsubseteq V'$ and $(\uparrow f^{-1}(V), V') \in R_f$. Using Lemma 3.6(a) to unpack this, we get

$$\uparrow f^{-1}(V) \sqsubseteq f^{-1}(V') \sqsubseteq f^{-1}(\uparrow V).$$

The inclusion for the localic downsets follows analogously.

Conversely, suppose that the stated inclusions hold. We show that $f$ is monotone. For that, take $U \sqsubseteq U'$ and $(U', V') \in R_f$. Then $\downarrow V' \sqsubseteq V$ by Lemma 3.6(c), and further

$$U \sqsubseteq \downarrow U' \sqsubseteq \downarrow f^{-1}(V) \sqsubseteq f^{-1}(\downarrow V)$$

follows by Lemma 3.6(a) and (e), together with the assumption, so $(U, \downarrow V) \in R_f$. This proves the first square in Definition 3.3. The second one is analogous, so we conclude that $f$ is monotone. □

**Example 3.8.** If $j : Y \hookrightarrow X$ is a sublocale [29, Section IX.4] of an ordered locale $X$, then $Y$ gets the structure of an ordered locale by endowing $\mathcal{O}Y$ with the largest preorder $\sqsubseteq_j$ that makes $j$ monotone. Using $(\lor)$, we can explicitly write:

$$A \sqsubseteq_j B \iff \forall U \in R_j(A) : \uparrow U \in R_j(B) \quad \text{and} \quad \forall V \in R_j(B) : \downarrow V \in R_j(A).$$

Here $R_j(A) = \{U \in \mathcal{O}X : (A, U) \in R_j\}$ is the image of $A$ under the relation $R_j$. The preorder $\sqsubseteq_j$ inherits axiom $(\lor)$ from $X$. 

**ORDERED LOCALES 9**
If the frame map \( j^{-1} \) admits a left adjoint \( j_! : j^{-1} \to j_! \), e.g. when \( j \) is open, then:

\[
A \preceq_j B \quad \iff \quad j_! A \preceq j_! B
\]

**Example 3.9.** The general construction of Example 3.8 generates many specific examples of ordered locales that have no points. For instance, take any ordered space \( S \) that is Hausdorff and has no isolated points, that is, none of whose singletons are open. In that case the double negation (Boolean) sublocale \((\mathcal{O}S)_{\neg\neg}\) of \( \mathcal{O}S \) has no points, but it becomes an ordered locale with the preorder inherited from \( S \).

4. **Opens**

This section discusses turning an ordered space into an ordered locale by taking its opens. Given a topological space \( S \), the locale \( \mathcal{O}S \) is defined to be the underlying frame of open subsets of \( S \). Given a continuous function \( g : S \to T \), a locale morphism \( \mathcal{O}g : \mathcal{O}S \to \mathcal{O}T \), that is, a frame morphism \( \mathcal{O}T \to \mathcal{O}S \), is defined by \( \mathcal{O}g(V) = g^{-1}(V) \in \mathcal{O}S \) for \( V \in \mathcal{O}T \). This constitutes the functor:

\[
\mathcal{O} : \text{Top} \to \text{Loc}.
\]

We investigate how this functor interacts with order structures on spaces. First, we extend it on objects.

**Definition 4.1.** For open subsets \( U \) and \( V \) of an ordered space \( S \), define:

\[
U \preceq V \quad \iff \quad U \subseteq \downarrow V \quad \text{and} \quad V \subseteq \uparrow U.
\]

This is the Egli-Milner order on the powerset of the space [39, Section 11.1]; see Figure 3 for intuition.

**Lemma 4.2.** If \((S, \preceq)\) is an ordered space, then \((\mathcal{O}S, \preceq)\) is an ordered locale.

**Proof.** That \( \preceq \) is a preorder on \( \mathcal{O}S \) follows from Lemma 2.1. To verify axiom \((\lor)\), suppose that \( U_i \preceq V_i \) for all \( i \) ranging over some index set. That means \( U_i \subseteq \downarrow V_i \) and \( V_i \subseteq \uparrow U_i \) for all indices \( i \). For a fixed index \( j \), then \( U_j \subseteq \downarrow V_j \subseteq \downarrow \bigcup V_i = \downarrow \bigcup V_i \) by Lemma 2.1. Therefore \( \bigcup U_i \subseteq \downarrow \bigcup V_i \). Similarly \( \bigcup V_i \subseteq \uparrow \bigcup U_i \). \( \square \)

This defines the functor \( \mathcal{O} : \text{OrdTop} \to \text{OrdLoc} \) on objects. Before moving on to morphisms, let us look at some more detail at its image.

**Lemma 4.3.** For an open subset \( U \) of an ordered space \( S \):

\[
\downarrow U = (\uparrow U)^\circ \quad \text{and} \quad \downarrow U = (\uparrow U)^\circ.
\]

Therefore, in \( \mathcal{O}S \):

\[
U \preceq V \quad \iff \quad U \subseteq \downarrow V \quad \text{and} \quad V \subseteq \uparrow U.
\]

**Proof.** First, we show that:

\[
(\downarrow U)^\circ \preceq U \quad \text{and} \quad U \preceq (\uparrow U)^\circ.
\]

By Definition 4.1, \( U \preceq (\uparrow U)^\circ \) if and only if \( (\uparrow U)^\circ \subseteq \uparrow U \), which is vacuously true, and \( U \subseteq \downarrow (\uparrow U)^\circ \). The latter follows from two applications of Lemma 2.1(a). Similarly \( (\downarrow U)^\circ \preceq U \).

Next, we prove that \( \downarrow U = (\uparrow U)^\circ \); the downset-version is similar. First, take an element \( x \in \downarrow U \), meaning there exists an open neighbourhood \( W \in \mathcal{O}S \) of \( x \) such that \( U \preceq W \). By Definition 4.1 this just means that \( x \in W \subseteq (\uparrow U)^\circ \), and
Proposition 4.4. The following are equivalent for an ordered space $T$:

(a) $T$ has open cones;
(b) $\mathcal{O}g$ is monotone for any continuous monotone function $g: S \to T$;
(c) $\mathcal{O}g$ is monotone for any monotone $g: \{0 < 1\} \to T$.

Proof. To see that (a) implies (b), we need to show that

- if $U \subseteq U'$ and $U' \subseteq g^{-1}V'$, then $V \subseteq V'$ and $U \subseteq g^{-1}V$ for some $V$;
- if $U \subseteq U'$ and $U \subseteq g^{-1}V$, then $V \subseteq V'$ and $U' \subseteq g^{-1}V'$ for some $V'$;

where $U, U'$ are open in $S$ and $V, V'$ are open in $T$. We will prove the first point, as the second is analogous. Take $V = (\downarrow V')^\circ$. Then $V \subseteq V'$ by (the proof of) Lemma 4.3. Also:

$$U \subseteq (\downarrow U')^\circ \subseteq (\downarrow g^{-1}V')^\circ \subseteq \downarrow g^{-1}(V') \subseteq g^{-1}(\downarrow V') = g^{-1}(\downarrow V)^\circ = g^{-1}V$$

where the first inclusion follows from $U \subseteq U'$, the second from $U' \subseteq g^{-1}V'$ and Lemma 2.1, the third one by definition of interiors, the fourth from the fact that $g$ is monotone and Proposition 2.2, the fifth from the assumption that $T$ has open cones, and the final equality by our choice of $V$.

Trivially (b) implies (c).

Finally, we show that (c) implies (a). Take an open subset $U$ of $T$. We need to show that $\uparrow U \subseteq (\uparrow U)^\circ$. Let $y \in \uparrow U$, meaning there exists $x \in U$ with $x \leq y$. Define a function $g: \{0 < 1\} \to T$ by $g(0) = x$ and $g(1) = y$; we endow the domain of $g$ with the discrete topology, so $g$ is clearly monotone and continuous. By assumption, now $\mathcal{O}g$ is a monotone map of ordered locales. In particular, (the proof of) Lemma 3.4 now gives an open neighbourhood $U \subseteq \mathcal{O}T$ of $y$ such that $U \subseteq V$. Therefore $y \in V \subseteq \uparrow U$, and since $V$ is open this means precisely that $y \in (\uparrow U)^\circ$. Downsets are similarly open, so $T$ has open cones. \qed

Corollary 4.5. There is a functor $\mathcal{O}: \text{OrdTop}_{\text{OC}} \to \text{OrdLoc}$ defined on objects by $(S, \leq) \mapsto (\mathcal{O}S, \leq)$ and on morphisms by $g \mapsto \mathcal{O}g$. \qed

5. Points

This section turns an ordered locale into an ordered space by taking its points. A point in a locale $X$ is a completely prime filter, that is, a subset $\mathcal{F} \subseteq \mathcal{O}X$ satisfying the following properties:

- proper: $\mathcal{F}$ is nonempty, and the least element 0 of $\mathcal{O}X$ is not in $\mathcal{F}$;
- upward closed: if $U \in \mathcal{F}$ and $U \subseteq V$, then also $V \in \mathcal{F}$;
- downward directed: if $U, V \in \mathcal{F}$, then also $U \wedge V \in \mathcal{F}$;
- completely prime: if $\bigvee U_i \in \mathcal{F}$, then $U_i \in \mathcal{F}$ for some index $i$.

Write $\text{pt}(X)$ for the set of points of a locale $X$. It becomes a topological space whose open sets are $\{\mathcal{F} \in \text{pt}(X) \mid U \in \mathcal{F}\}$ for $U \in \mathcal{O}X$. If $f: X \to Y$ is a locale morphism, and $\mathcal{F}$ a point of $X$, then $\{V \in \mathcal{O}Y \mid f^{-1}(V) \in \mathcal{F}\}$ is a point of $Y$, and this defines a continuous function $\text{pt}(f): \text{pt}(X) \to \text{pt}(Y)$. Thus there is a functor
pt: \textbf{Loc} \rightarrow \textbf{Top} \cite[Section XI.3]{[29]}. In this section we extend it to the ordered setting, starting with objects.

**Definition 5.1.** For a locale \(X\), a preorder \(\preceq\) on its frame of opens, and points \(\mathcal{F}\) and \(\mathcal{G}\), define:
\[
\mathcal{F} \preceq \mathcal{G} \iff \forall U \in \mathcal{F} \exists V \in \mathcal{G}: U \preceq V \quad \text{and} \quad \forall V \in \mathcal{G} \exists U \in \mathcal{F}: U \preceq V.
\]

It is easy to see that \(\preceq\) is a preorder on pt\((X)\) because \(\preceq\) is a preorder on \(\mathcal{O}X\). Intuitively, no matter how small an open neighbourhood we pick around the (imaginary) point, there is always a neighbourhood around the other point that precedes it.

**Lemma 5.2.** For points \(\mathcal{F}\) and \(\mathcal{G}\) in an ordered locale:
\[
\mathcal{F} \preceq \mathcal{G} \iff \forall U \in \mathcal{F}: \uparrow U \in \mathcal{G} \quad \text{and} \quad \forall V \in \mathcal{G}: \downarrow V \in \mathcal{F}.
\]

*Proof.* If \(\mathcal{F} \preceq \mathcal{G}\) then for any \(U \in \mathcal{F}\) we can find \(V \in \mathcal{G}\) such that \(U \preceq V\), which with Lemma 3.6(a) gives \(V \subseteq \uparrow U\). Since \(\mathcal{G}\) is upwards closed, this implies \(\uparrow U \in \mathcal{G}\). Similarly \(\downarrow V \in \mathcal{F}\) for \(V \in \mathcal{G}\). The converse follows from Lemma 3.6(b) and (c). \(\square\)

Next we move to morphisms.

**Lemma 5.3.** If \(f: X \rightarrow Y\) is a morphism of ordered locales, then the continuous function \(pt(f): pt(X) \rightarrow pt(Y)\) is monotone.

*Proof.* Suppose that \(\mathcal{F} \preceq \mathcal{G}\) for points in \(X\). We will show that \(pt(f)(\mathcal{F}) \preceq pt(f)(\mathcal{G})\) in \(Y\). Let \(V \in pt(f)(\mathcal{F})\), that is, \(f^{-1}V \in \mathcal{F}\). Definition 5.1 then gives a \(U \in \mathcal{G}\) with \(f^{-1}V \preceq U\). Because \(f\) is monotone, there now exists a \(W \in \mathcal{O}Y\) such that \(V \preceq W\) and \(U \preceq f^{-1}W\). As \(\mathcal{G}\) is upwards closed, \(W \in pt(f)(\mathcal{G})\). The symmetric condition follows analogously, thus \(pt(f)(\mathcal{F}) \preceq pt(f)(\mathcal{G})\). \(\square\)

**Corollary 5.4.** There is a functor \(pt: \textbf{OrdLoc} \rightarrow \textbf{OrdTop}\) defined on objects by \((X, \preceq) \mapsto \text{(pt}(X), \preceq)\) and on morphisms by \(f \mapsto pt(f)\). \(\square\)

We would like the functor \(pt: \textbf{OrdLoc} \rightarrow \textbf{OrdTop}\) to land in \textbf{OrdTop}_{\text{OC}}. That is: when does pt\((X)\) have open cones? To force this we will make an additional (but reasonable) assumption.

**Lemma 5.5.** If \(X\) is an ordered locale and
\[
\text{(P)} \quad U \preceq V \text{ in } X \implies \text{pt}(U) \preceq \text{pt}(V) \text{ in } \mathcal{O}(\text{pt}(X))
\]
then the ordered space pt\((X)\) has open cones.

*Proof.* Assume (P); we will show that pt\((X)\) has open cones. By Proposition 4.4, it suffices to show that every monotone function \(g: \{0 < 1\} \rightarrow \text{pt}(X)\) induces a monotone map \(\mathcal{O}g\) of locales. That is, we need to show that if \(V \in \mathcal{O}X\) satisfies \(\{1\} \subseteq \text{pt}(V)\), then there exists \(U \in \mathcal{O}X\) such that \(\{0\} \subseteq \text{pt}(U)\) and pt\((U) \subseteq \text{pt}(V)\); the required symmetric condition is analogous. Unpacking this further, with \(\mathcal{F} = g(\{0\}) = \mathcal{G}\), this means that for all \(U \in \mathcal{F}\) there exists \(V \in \mathcal{G}\) with pt\((U) \subseteq \text{pt}(V)\), and for every \(V \in \mathcal{G}\) there exists \(U \in \mathcal{F}\) with pt\((U) \subseteq \text{pt}(V)\). But this follows directly from Definition 5.1 and (P). \(\square\)

Intuitively, condition (P) guarantees that the topology of the locale is rich enough to carry its points into the future and past. In particular, it implies that if \(U \preceq V\)
then pt(U) = ∅ if and only if pt(V) = ∅. Note that any locale without points trivially satisfies (P). We discuss the axiom briefly further in Section 7.

There is a well-known notion of a locale X having enough points, which intuitively says that its points distinguish its opens, and formally that the counit ε_X is an isomorphism. Less well known is the following condition on a space: a topological space S has enough points if the unit η_S: S → pt(OS) is surjective. That is, any completely prime filter on a space S with enough points is of the form F_x for some x ∈ S. Having enough points is halfway to being sober: a topological space is sober if and only if it is T_0 and has enough points. Equivalently, a space has enough points if and only if its T_0-quotient is sober [2]. Ordered spaces with open cones that have enough points always satisfy (P).

Lemma 5.6. If S is an ordered space with open cones that has enough points, then the ordered locale OS satisfies (P).

Proof. Let U and V be opens in the space S satisfying U ⊆ V. We need to show that there are inclusion pt(U) ⊆ ↓ pt(V) and pt(V) ⊆ ↑ pt(U); we will prove the first inclusion, the second one is similar. Take a point F ∈ pt(U). Since S has enough points, there is an element x ∈ U such that F = F_x. Because U ⊆ V now x ∈ U ⊆ ↓ V, so x ≤ y for some y ∈ V. Using the fact that S has open cones, Lemma 6.1 gives that F_x ≤ F_y, and hence F = F_x ∈ ↓ pt(V).

Example 5.7. Any topological space S that does not have enough points becomes an ordered space with open cones by taking the preorder to be equality of elements (Example 2.5). The induced order on the locale OS and on the space pt(OS) are also equality. It is then easily seen that axiom (P) is always satisfied.

6. THE ADJUNCTION

This section puts everything together, proving that the functors of the previous two sections form an adjunction between ordered spaces and ordered locales, and investigating the resulting equivalence. We build on the fundamental adjunction between spaces and locales [19, 29]:

$$
\begin{array}{ccc}
\text{Top} & \overset{O}{\longrightarrow} & \text{Loc} \\
\downarrow \text{pt} & & \downarrow \text{pt} \\
\end{array}
$$

and will extend it to the ordered setting. The unit of this classic adjunction is the natural transformation whose components are continuous functions

$$
\eta_S: S \longrightarrow \text{pt}(OS)
$$

$$
x \longmapsto F_x = \{ U \in OS \mid x \in U \}
$$

for topological spaces S. The counit is the natural transformation ε: O ▷ pt ⇒ Id_{Loc}

whose components are frame morphisms

$$
\varepsilon_X^{-1}: OX \longrightarrow O(\text{pt}(X))
$$

$$
U \longmapsto \text{pt}(U)
$$

for locales X.

We start by investigating how this unit and counit interact with the order structures on pt(OS) and O(\text{pt} X) given by Definitions 4.1 and 5.1.
Lemma 6.1. If \( x \) and \( y \) are points of an ordered space \( S \), then in \( \text{pt}(\mathcal{O}S) \):
\[
\mathcal{F}_x \subseteq \mathcal{F}_y \iff \forall U \in \mathcal{F}_x : (\uparrow U)^\circ \in \mathcal{F}_y \quad \text{and} \quad \forall V \in \mathcal{F}_y : (\downarrow V)^\circ \in \mathcal{F}_x.
\]
Hence the unit \( \eta : S \to \text{pt}(\mathcal{O}S) \) is monotone if and only if \( S \) has open cones.

Proof. The first statement follows immediately from Lemmas 4.3 and 5.2. Using this, the unit being monotone, that is, \( x \preceq y \implies \mathcal{F}_x \subseteq \mathcal{F}_y \), is equivalent to the open cone condition by Proposition 2.8. □

Lemma 6.2. If \( X \) is a locale, \( \preceq \) a preorder on its frame of opens, and \( U \in \mathcal{O}X \) an open, then in \( \text{pt}(X) \):
\[
\uparrow (\text{pt} U) \subseteq \text{pt}(\uparrow U) \quad \text{and} \quad \downarrow (\text{pt} U) \subseteq \text{pt}(\downarrow U).
\]
If \( X \) is an ordered locale, then the counit \( \varepsilon_X : \mathcal{O} (\text{pt} X) \to X \) is monotone.

Proof. For the first statement, take \( \mathcal{G} \in \uparrow (\text{pt} U) \), so that there exists \( \mathcal{F} \in \text{pt}(U) \) with \( \mathcal{F} \leq \mathcal{G} \). In particular, there then exists \( V \in \mathcal{G} \) such that \( U \leq V \). Lemma 3.6(a) now gives \( V \subseteq \uparrow U \), and since \( \mathcal{G} \) is upwards closed this implies \( \uparrow U \in \mathcal{G} \), that is, \( \mathcal{G} \in \text{pt}(\uparrow U) \). The proof for downsets is similar.

That the counit \( \varepsilon_X \) is monotone for an ordered locale now follows immediately from Proposition 3.7. □

We have arrived at our main result.

Theorem 6.3. There is an adjunction:
\[
\begin{array}{ccc}
\text{OrdTop}_{\mathcal{O}C} & \cong & \text{OrdLoc} \\
\downarrow \circ & & \uparrow \text{pt} \\
\end{array}
\]

between the full subcategory of \( \text{OrdTop} \) of ordered spaces with open cones and enough points, and the full subcategory of \( \text{OrdLoc} \) of ordered locales satisfying (P).

Proof. Combine Corollaries 4.5 and 5.4 and Lemmas 5.6, 6.1 and 6.2. □

Finally, we investigate the fixed points of this adjunction. In the unordered setting, the fixed points of the classic adjunction between spaces and locales are precisely the sober spaces and spatial locales [29, Section IX.3]. On the locale side this remains the same in the ordered setting.

Lemma 6.4. If an ordered locale \( X \) is spatial, the inverse \( (\varepsilon_X)^{-1} : X \to \mathcal{O}(\text{pt} X) \) of the counit locale morphism is monotone.

Proof. This follows immediately from Lemma 3.2. □

But the ordered spaces fixed by the adjunction gain an interesting new condition. Recall from Examples 2.5 and 2.6 that there is no relation between the open cone condition and separation axioms.

Definition 6.5. An ordered space is \( T_0 \)-ordered if, whenever \( x \nleq y \), either there exists an open neighbourhood \( U \) of \( x \) such that \( y \nleq \uparrow U \), or there exists an open neighbourhood \( V \) of \( y \) such that \( x \nleq \downarrow V \).

The previous definition is a natural weakening of the \( T_1 \)-order separation axiom from [30]; see also [26]. Indeed, it is easy to see that being \( T_1 \)-ordered implies being \( T_0 \)-ordered. Also, \( T_0 \)-ordered implies \( T_0 \). In particular, any pospace is \( T_0 \)-ordered.
**Example 6.6.** In contrast to Corollary 2.11, not every smooth spacetime is $T_0$-ordered: Minkowski space with one point removed fails this property.

Any causally simple spacetime [31, Definition 4.112] is by definition $T_1$-ordered, and hence $T_0$-ordered. In particular, any globally hyperbolic spacetime is $T_0$-ordered.

**Lemma 6.7.** Let $S$ be an ordered space with open cones that is sober. The inverse $\eta_S^{-1} : \text{pt}(O(S)) \to S$ of the unit is monotone if and only if $S$ is $T_0$-ordered.

**Proof.** Since $\eta_S^{-1}(\mathcal{F}_X) = x$, it follows that $\eta_S^{-1}$ is monotone if and only if $x \preceq y$ implies $\mathcal{F}_x \subseteq \mathcal{F}_y$. Using Lemma 6.1 to unpack the order on $\text{pt}(O(S))$, together with the open cone condition, this is equivalent to $S$ being $T_0$-ordered. □

**Lemma 6.8.** If $X$ is an ordered locale, then $\text{pt}(X)$ is $T_0$-ordered.

**Proof.** If $\mathcal{F} \not\subseteq \mathcal{G}$ then by Lemma 5.2 there exists $U \in \mathcal{F}$ such that $\uparrow U \not\subseteq \mathcal{G}$, or there exists $V \in \mathcal{G}$ such that $\downarrow V \not\in \mathcal{F}$. Following Lemma 6.2, this implies, respectively, that $\mathcal{G} \not\subseteq \uparrow \text{pt}(U)$ or $\mathcal{F} \not\subseteq \downarrow \text{pt}(V)$, as desired. □

This brings us to the ordered version of Stone duality [29, Corollary IX.3.4].

**Theorem 6.9.** The adjunction of Theorem 6.3 restricts to an equivalence of categories between the full subcategory of $\text{OrdLoc}$ of spatial ordered locales satisfying (P) and the full subcategory of $\text{OrdTop}_{OC}$ of sober $T_0$-ordered spaces. □

### 7. Conclusion

To conclude, we raise directions for further research.

- Axiom (P) in Lemma 5.5 is a somewhat ad hoc solution to ensure open cones in $\text{pt}(X)$. This natural condition is unsatisfactory from a localic perspective, since it is stated in terms of points, instead of purely in terms of opens. Note that the definition of an ordered locale actually does not axiomatise ordered spaces with open cones, but instead axiomatises arbitrary ordered spaces: the open cone assumption is never used in the proof of Lemma 4.2. There are several natural candidates (cf. Example 2.9), but currently we are not aware of an additional axiom for ordered locales, purely stated in terms of opens, that could replace the need for axiom (P). One may be similarly dissatisfied with the need to restrict to spaces with enough points (Lemma 5.6). In Example 5.7 we saw examples of ordered spaces without enough points that nevertheless satisfy axiom (P). This shows that the enough points assumption is not necessary. Future work could elucidate both of these restrictions and generalise the adjunction.

- Ordered spaces are an important source of examples in (directed) algebraic topology, the study of the fundamental category of a (directed) topological space, which consists of homotopy classes of (directed) paths between points [14]. In the undirected case, there are good notions of fundamental category geared towards locales rather than spaces [23]. Our results could be used to develop a good notion of fundamental category of an ordered locale.

- Ordered spaces generalise to locally ordered spaces, also called streams [24], which arise often in concurrent computation. Our results could similarly be generalised to ‘localic streams’, which entails lifting the adjunction to categories of cosheaves.
Ordered spaces also generalise to topological categories. Similarly, categorical versions of the Egli-Milner order exist \cite{28}. Our results could be generalised to sites. This also raises the question of what the natural notion of sheaf is on an ordered locale. Given such a notion, our framework could be used to study contextuality \cite{12}.

- Ordered spaces form the basis of structural approaches to causality in quantum gravity \cite{25, 4}. Our results could form the basis for a pointfree approach to causality in quantum gravity.

  Similarly, ordered spaces can satisfy many properties of the so-called causal ladder that make them more well-behaved as models of physical spacetime \cite{31}. For example, antisymmetry of the preorder relates to the existence of closed causal curves. Our results could form a starting point to find pointfree analogues of such properties.

- Quantales are a noncommutative generalisation of locales, which are used in the study of observational logic \cite{1} and resource theories \cite{13}. Our ordered locales could generalise to quantales, to help study causality in those settings.

- Locales can be categorified to monoidal categories via tensor topology \cite{7, 6}. Our results justify looking for a structure on a monoidal category that decategorifies to an ordered locale. Similarly, relations to monoidal topology \cite{38, 18} and promonoidal categories \cite{17} could be investigated.

- Ordered spaces are sometimes also regarded as bitopological spaces, as the order induces a second topology. There is a notion of biframe, and a corresponding categorical duality \cite{37}. The relationship to our results should be investigated.

References


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