Categorical quantum mechanics

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informatics

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  Primitive notion: forming compound systems

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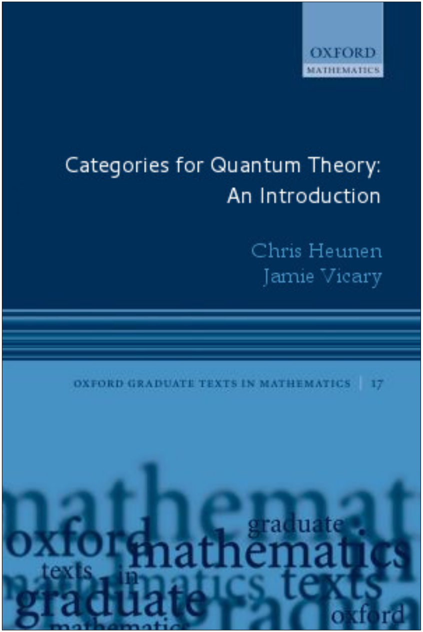
- Powerful graphical calculus

- Allows different interpretation in many different fields
  - Physics: quantum theory, quantum information theory
  - Computer science: logic, topology
  - Mathematics: representation theory, quantum algebra


OXFORD
MATHEMATICS
Categories for Quantum Theory:
An Introduction
Chris Heunen
Jamie Vicary
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Outline

Lecture 1:

- monoidal categories: graphical calculus
- dual objects: entanglement
- (co)monoids: no-cloning

Lecture 2:

- Frobenius structures: observables
- bialgebras: complementarity
- complete positivity: mixed states

Part I

Monoidal categories

Category = systems and processes:

- physical systems, and physical processes governing them;
- data types, and algorithms manipulating them;
- algebraic structures, and structure-preserving functions;
- logical propositions, and implications between them.

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Monoidal category = category + parallelism:

- independent physical systems evolve simultaneously;
- running computer algorithms in parallel;
- products or sums of algebraic or geometric structures;
- using proofs of P and Q to prove conjunction (P and Q).

A category $\mathbf{C}$ is monoidal when equipped with:

- a *tensor product* functor

$$\otimes\colon \mathbf{C}\times\mathbf{C}\longrightarrow\mathbf{C}$$

- a *unit object*

$$I\in\mathrm{Ob}(\mathbf{C})$$

- an *associator* natural isomorphism

$$(A\otimes B)\otimes C\xrightarrow{\alpha_{A,B,C}}A\otimes(B\otimes C)$$

- a *left unitor* natural isomorphism

$$I\otimes A\xrightarrow{\lambda_A}A$$

- a *right unitor* natural isomorphism

$$A\otimes I\xrightarrow{\rho_A}A$$

This data must satisfy the triangle and pentagon equations:

$$\begin{tikzcd}[column sep=small]
(A \otimes I) \otimes B \arrow[rr, "\alpha_{A,I,B}"] \arrow[dr, "\rho_A \otimes \mathrm{id}_B"'] & & A \otimes (I \otimes B) \arrow[dl, "\mathrm{id}_A \otimes \lambda_B"] \\
& A \otimes B &
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
& \big(A \otimes (B \otimes C)\big) \otimes D \arrow[rr, "\alpha_{A,B\otimes C,D}"] & & A \otimes \big((B \otimes C) \otimes D\big) \arrow[dr, "\mathrm{id}_A \otimes \alpha_{B,C,D}"] & \\
\big((A \otimes B) \otimes C\big) \otimes D \arrow[ur, "\alpha_{A,B,C} \otimes \mathrm{id}_D"] \arrow[drr, "\alpha_{A\otimes B,C,D}"'] & & & & A \otimes \big(B \otimes (C \otimes D)\big) \\
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\end{tikzcd}$$

Theorem (coherence for monoidal categories): If the pentagon and triangle equations hold, then so does any well-typed equation built from α, λ, ρ and their inverses using $\otimes$, $\circ$, and id.

Example: Hilbert spaces

Hilbert spaces and bounded linear maps form a monoidal category:

- *tensor product* is the tensor product of Hilbert spaces
- *unit object* is one-dimensional Hilbert space $\mathbb{C}$
- *left unitors* $\mathbb{C} \otimes H \xrightarrow{\lambda_H} H$ are unique linear maps with $1 \otimes u \mapsto u$
- *right unitors* $H \otimes \mathbb{C} \xrightarrow{\rho_H} H$ are unique linear maps with $u \otimes 1 \mapsto u$
- *associators* $(H \otimes J) \otimes K \xrightarrow{\alpha_{H,J,K}} H \otimes (J \otimes K)$ are unique linear maps with $(u \otimes v) \otimes w \mapsto u \otimes (v \otimes w)$

Example: sets and functions

Sets and functions form a monoidal category **Set**:

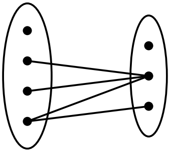
- *tensor product* is Cartesian product of sets
- *unit object* is a chosen singleton set $\{\bullet\}$
- *left unitors* $I \times A \xrightarrow{\lambda_A} A$ are $(\bullet, a) \mapsto a$
- *right unitors* $A \times I \xrightarrow{\rho_A} A$ are $(a, \bullet) \mapsto a$
- *associators* $(A \times B) \times C \xrightarrow{\alpha_{A,B,C}} A \times (B \times C)$ are functions $\big((a,b),c\big) \mapsto \big(a,(b,c)\big)$

(Other tensor products exist.)

Example: sets and relations

A relation $A \xrightarrow{R} B$ between sets is a subset $R \subseteq A \times B$

$$A \xrightarrow{R} B$$



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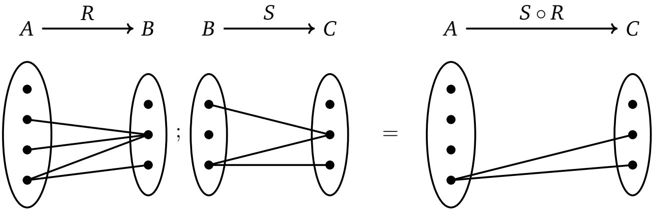

Different notion of process: nondeterministic evolution of states

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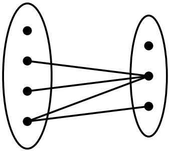

$$\leftrightsquigarrow \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Composition is matrix multiplication, with OR and AND for $+$ and $\times$.

Example: sets and relations

Sets and relations form a monoidal category **Rel**:

- *tensor product* is Cartesian product of sets,
  acting on relations as: $(a,c)(R \times S)(b,d)$ iff aRb and cSd
- *unit object* is a chosen singleton set $= \{\bullet\}$
- *associators* $(A \times B) \times C \xrightarrow{\alpha_{A,B,C}} A \times (B \times C)$ are
  catRelations $\big((a,b),c\big) \sim \big(a,(b,c)\big)$
- *left unitors* $I \times A \xrightarrow{\lambda_A} A$ are given by $(\bullet, a) \sim a$
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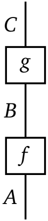
Cartesian product is not a categorical product in **Rel**:
If **Set** is classical, and **Hilb** is quantum, **Rel** is 'in the middle'

Graphical calculus

For $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$, draw their composition $A \xrightarrow{g \circ f} B$ as



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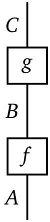
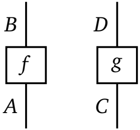


For $A \xrightarrow{f} B$ and $C \xrightarrow{g} D$, draw their tensor product $A \otimes C \xrightarrow{f \otimes g} B \otimes D$ as



"Time" runs upwards, "space" runs sideways

Graphical calculus

The tensor unit is drawn as the empty diagram:

Graphical calculus

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Unitors are also not drawn:

λ_A $\qquad$ ρ_A $\qquad$ $\alpha_{A,B,C}$

Coherence is essential: as there can only be a single morphism built from associators and unitors of given type, it doesn't matter that their depiction encodes no information

Graphical calculus

For example, interchange law:

$$(g \circ f) \otimes (j \circ h) \quad = \quad (g \otimes j) \circ (f \otimes h)$$

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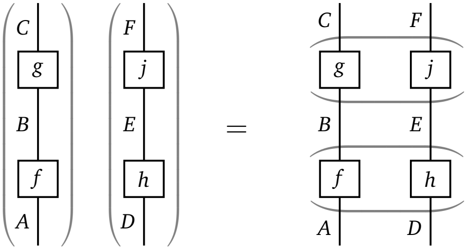

Graphical calculus: sound and complete

Think of diagram as within rectangular region of $\mathbb{R}^n$, with wires terminating at upper and lower boundaries only, morphisms as points. Two diagrams are isotopic when one can be deformed continuously into the other, keeping the boundaries fixed.

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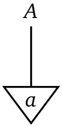
Theorem (correctness): A well-formed equation between morphisms in a monoidal category follows from the axioms iff it holds in the graphical calculus up to planar isotopy.

Soundness: algebraic equality $\Rightarrow$ graphical isotopy
Completeness: algebraic equality $\Leftarrow$ graphical isotopy

States

A state of an object A in a monoidal category is a morphism $I \to A$.



Tensor unit is trivial system; state is way to bring A into existence

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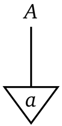
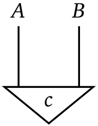


Tensor unit is trivial system; state is way to bring A into existence

- in **Hilb**: linear functions $\mathbb{C} \to H$, correspond to elements of H
- in **Set**: functions $\{\bullet\} \to A$, correspond to elements of A
- in **Rel**: relations $\{\bullet\} \xrightarrow{R} A$, correspond to subsets of A

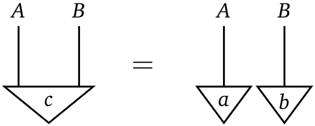
Joint states

A morphism $I \xrightarrow{c} A \otimes B$ is a joint state.



Joint states

A morphism $I \xrightarrow{c} A \otimes B$ is a joint state. It is a product state when:



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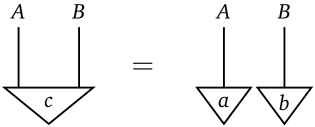

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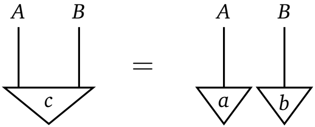


It is entangled when it is not a product state

- entangled states in **Hilb**: vectors of $H \otimes K$ with Schmidt rank > 1
- entangled states in **Set**: don't exist
- entangled states in **Rel**: non-square subsets of $A \times B$

Braiding

A monoidal category is braided when equipped with natural iso

$$A \otimes B \xrightarrow{\sigma_{A,B}} B \otimes A$$

satisfying the hexagon equations

$$\begin{tikzcd}
A \otimes (B \otimes C) \arrow[r, "\sigma_{A,B\otimes C}"] \arrow[d, "\alpha^{-1}_{A,B,C}"] & (B \otimes C) \otimes A \\
(A \otimes B) \otimes C \arrow[d, "\sigma_{A,B} \otimes \mathrm{id}_C"] & B \otimes (C \otimes A) \arrow[u, "\alpha^{-1}_{B,C,A}"'] \\
(B \otimes A) \otimes C \arrow[r, "\alpha_{B,A,C}"'] & B \otimes (A \otimes C) \arrow[u, "\mathrm{id}_B \otimes \sigma_{A,C}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
(A \otimes B) \otimes C \arrow[r, "\sigma_{A\otimes B,C}"] \arrow[d, "\alpha_{A,B,C}"] & C \otimes (A \otimes B) \\
A \otimes (B \otimes C) \arrow[d, "\mathrm{id}_A \otimes \sigma_{B,C}"] & (C \otimes A) \otimes B \arrow[u, "\alpha_{C,A,B}"'] \\
A \otimes (C \otimes B) \arrow[r, "\alpha^{-1}_{A,C,B}"'] & (A \otimes C) \otimes B \arrow[u, "\sigma_{A,C} \otimes \mathrm{id}_B"']
\end{tikzcd}$$

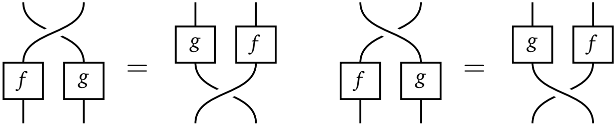
Braiding

Draw σ as  and its inverse as 

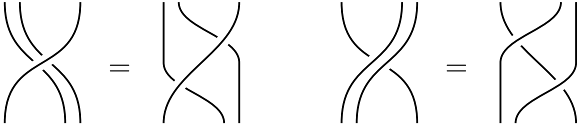
Braiding

Invertibility:

Naturality:



Hexagon:



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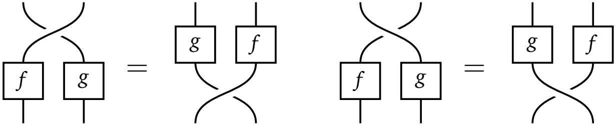

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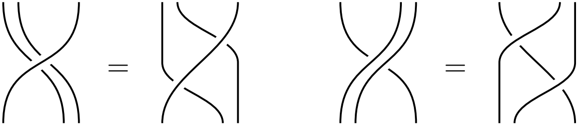


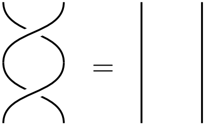
Theorem (correctness): A well-formed equation between morphisms in a braided monoidal category follows from the axioms iff it holds in the graphical language up to 3-dimensional isotopy.

Symmetry

A braided monoidal category is symmetric when

$$\sigma_{B,A} \circ \sigma_{A,B} = \mathrm{id}_{A \otimes B}$$

Graphically: no knots



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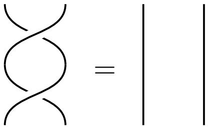

Theorem (correctness): A well-formed equation between morphisms in a symmetric monoidal category follows from the axioms iff it holds in graphical language up to 4-dimensional isotopy.

- in **Hilb**: linear extension of $a \otimes b \mapsto b \otimes a$
- in **Set**: function $(a, b) \mapsto (b, a)$
- in **Rel**: relation $(a, b) \sim (b, a)$

Scalars

A scalar in a monoidal category is a morphism $I \xrightarrow{a} I$

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$$a$$

Lemma: in a monoidal category, scalar composition is commutative
Proof: either algebraically:

$$\begin{tikzcd}
I \arrow[rr, "a"] \arrow[dr, "b"] \arrow[dd, "\lambda_I^{-1}"', "\rho_I^{-1}"] & & I \arrow[dr, "b"] \arrow[dd, "\lambda_I^{-1}"', "\rho_I^{-1}"] & \\
& I \arrow[rr, "a"] & & I \\
I \otimes I \arrow[rr, "a \otimes \mathrm{id}_I"'] \arrow[dr, "\mathrm{id}_I \otimes b"'] & & I \otimes I \arrow[dr, "\mathrm{id}_I \otimes b"'] & \\
& I \otimes I \arrow[uu, "\lambda_I", "\rho_I"'] \arrow[rr, "a \otimes \mathrm{id}_I"'] & & I \otimes I \arrow[uu, "\lambda_I", "\rho_I"']
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Scalars

A scalar in a monoidal category is a morphism $I \xrightarrow{a} I$

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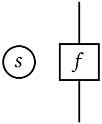
Lemma: in a monoidal category, scalar composition is commutative
Proof: or graphically:

$$\begin{array}{c} b \\ a \end{array} \quad = \quad \begin{array}{c} a \\ b \end{array}$$

Scalar multiplication

Scalar multiplication $A \xrightarrow{a \bullet f} B$ of scalar $I \xrightarrow{a} I$ and morphism $A \xrightarrow{f} B$



satisfies many familiar properties in *any* monoidal category:

- $\mathrm{id}_I \bullet f = f$
- $a \bullet b = a \circ b$
- $a \bullet (b \bullet f) = (a \bullet b) \bullet f$
- $(b \bullet g) \circ (a \bullet f) = (b \circ a) \bullet (g \circ f)$

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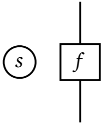

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- $(b \bullet g) \circ (a \bullet f) = (b \circ a) \bullet (g \circ f)$

In our examples:

- in **Hilb**: $a \bullet f$ is the morphism $x \mapsto af(x)$
- in **Set**: $\mathrm{id}_1 \bullet f = f$ is trivial
- in **Rel**: true $\bullet R = R$ and false $\bullet R = \emptyset$

Dagger

A dagger on a category $\mathbf{C}$ is a contravariant functor $\dagger\colon \mathbf{C} \to \mathbf{C}$ satisfying $A^\dagger = A$ on objects and $f^{\dagger\dagger} = f$ on morphisms.

- **Hilb** is a dagger category using adjoints
- **Rel** is a dagger category using converse: $bR^\dagger a$ iff aRb
- **Set** is not a dagger category

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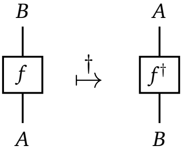
Graphically: flip about horizontal axis



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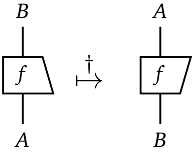

Dagger

A morphism f in a dagger category is:

- self-adjoint when $f = f^\dagger$
- unitary when $f^\dagger \circ f = \mathrm{id}$ and $f \circ f^\dagger = \mathrm{id}$
- positive when $f = g^\dagger \circ g$ for some g

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In a monoidal dagger category:

- $(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger$
- the associators and unitors are unitary

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- positive when $f = g^\dagger \circ g$ for some g

In a monoidal dagger category:

- $(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger$
- the associators and unitors are unitary

In a braided/symmetric monoidal dagger category,
the braiding is additionally unitary

Part II

Dual objects

Dual objects

An object L is left-dual to an object R, and R is right-dual to L, written $L \dashv R$, when there are morphisms $I \xrightarrow{\eta} R \otimes L$ and $L \otimes R \xrightarrow{\varepsilon} I$ with:

$$\begin{tikzcd}
L \arrow[r, "\rho_L^{-1}"] \arrow[d, "\mathrm{id}_L"'] & L \otimes I \arrow[r, "\mathrm{id}_L \otimes \eta"] & L \otimes (R \otimes L) \arrow[d, "\alpha_{L,R,L}^{-1}"] \\
L & I \otimes L \arrow[l, "\lambda_L"] & (L \otimes R) \otimes L \arrow[l, "\varepsilon \otimes \mathrm{id}_L"]
\end{tikzcd}$$

$$\begin{tikzcd}
R \arrow[r, "\lambda_R^{-1}"] \arrow[d, "\mathrm{id}_R"'] & I \otimes R \arrow[r, "\eta \otimes \mathrm{id}_R"] & (R \otimes L) \otimes R \arrow[d, "\alpha_{R,L,R}"] \\
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where we draw η and ε as:

Dual objects: examples

Every finite-dimensional Hilbert space H is both right dual and left dual to its dual Hilbert space H^*:

- cap $H \otimes H^* \to \mathbb{C}$ is evaluation: $|\phi\rangle \otimes \langle\psi| \mapsto \langle\psi|\phi\rangle$
- cup $\mathbb{C} \to H^* \otimes H$ is maximally entangled state:
  $1 \mapsto \sum_i \langle i| \otimes |i\rangle$ for any orthonormal basis $\{|i\rangle\}$

Infinite-dimensional Hilbert spaces do not have duals

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Infinite-dimensional Hilbert spaces do not have duals

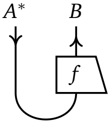
In **Rel**, every object is self-dual:

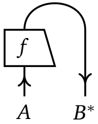
- cap $A \times A \to 1$ is $\bullet \sim (a, a)$ for all $a \in A$
- cup $1 \to A \times A$ is $(a, a) \sim \bullet$

Map-state duality

The category **Set** only has duals for singletons.

The name $I \xrightarrow{\ulcorner f \urcorner} A^* \otimes B$ and coname $A \otimes B^* \xrightarrow{\llcorner f \lrcorner} I$ of a morphism $A \xrightarrow{f} B$, given dual objects $A \dashv A^*$ and $B \dashv B^*$, are





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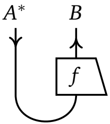
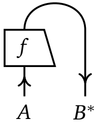
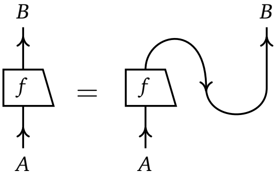




Conversely,



Proof: There is only one function $A \to 1$, so all conames $A \otimes B^* \to 1$ are equal, so all functions $A \to B$ are equal.

Dual objects: properties

Robustly defined:

- Suppose $L \dashv R$. Then $L \dashv R'$ iff $R \simeq R'$.
- If $(L, R, \eta, \varepsilon)$ and $(L, R, \eta, \varepsilon')$ are dualities, then $\varepsilon = \varepsilon'$.

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- Always $I \dashv I$.
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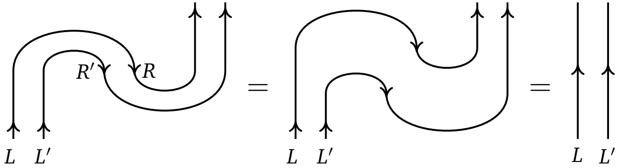

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Monoidal:

- Always $I \dashv I$.
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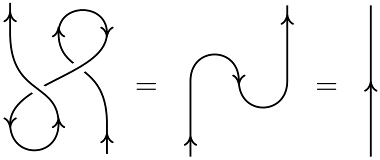
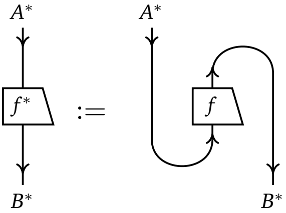
Symmetric:

- If $L \dashv R$ in a braided monoidal category, then also $R \dashv L$.



Duals functor

For $A \xrightarrow{f} B$ and $A \dashv A^*$, $B \dashv B^*$, the right dual $B^* \xrightarrow{f^*} A^*$ is defined as:



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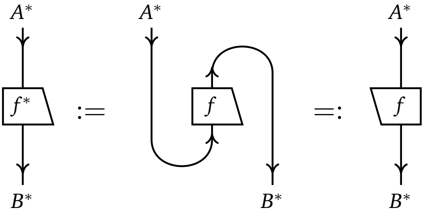

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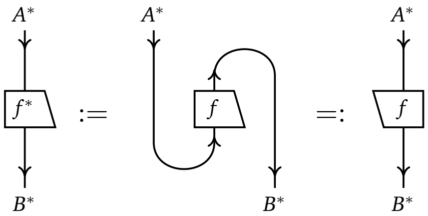
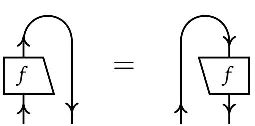


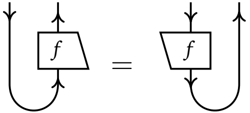
Examples:

- in **FHilb**: usual dual $f^* : K^* \to H^*$ given by $f^*(e) = e \circ f$
- in **Rel**: $R^* = R^\dagger$

Duals functor: properties

Lemma: $(g \circ f)^* = f^* \circ g^*$, and





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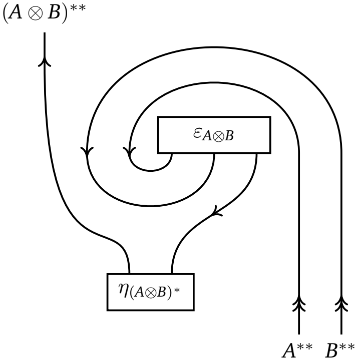
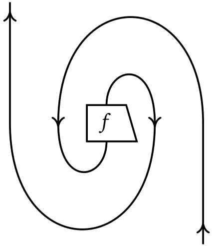
Lemma: $A^{**} \otimes B^{**} \simeq (A \otimes B)^{**}$



Compact categories

A symmetric monoidal category is compact when every object has a (simultaneously left and right) dual.

Theorem (correctness): A well-formed equation between morphisms in a compact category follows from the axioms if and only if it holds in the graphical language up to spatial *oriented* isotopy.



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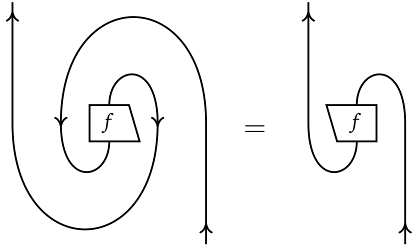

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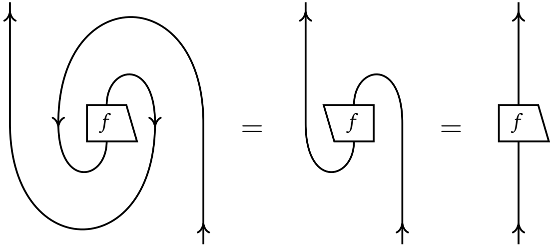

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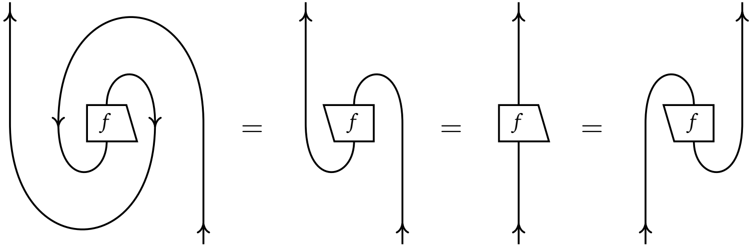

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Lemma: In a monoidal dagger category, $L \dashv R \Leftrightarrow R \dashv L$.

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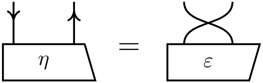

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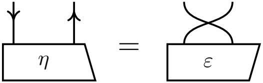
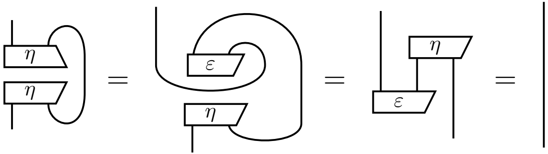


Lemma: Dagger dualities correspond to maximally entangled states



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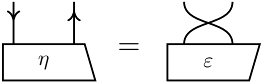

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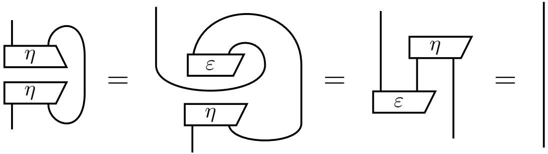

Dagger duals, and hence maximally entangled states,
are unique up to unique unitary

Compact dagger categories

A compact dagger category is both compact and dagger, and duals are dagger duals.

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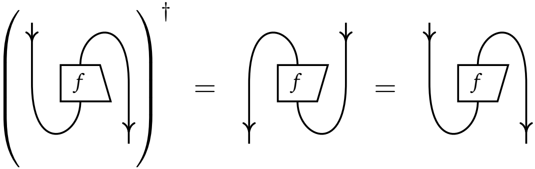
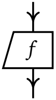
Lemma: Duals and daggers commute

$(f^*)^\dagger =$  $= (f^\dagger)^*$

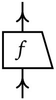
Conjugation is the functor $(-)_* := (-)^{*\dagger} = (-)^{\dagger *}$

Conjugation

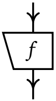


$f_* : A^* \longrightarrow B^*$

conjugation

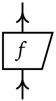


$f : A \longrightarrow B$



$f^* : B^* \longrightarrow A^*$

dual

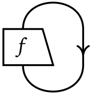


$f^\dagger : B \longrightarrow A$

dagger

Traces

The trace of $A \xrightarrow{f} A$ is the scalar



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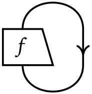

Examples:

- in **FHilb**, it is the ordinary trace
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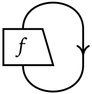

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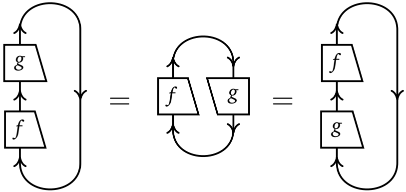
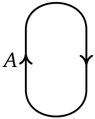
Lemma: Trace is cyclic, $\mathrm{Tr}(f \otimes g) = \mathrm{Tr}(f) \circ \mathrm{Tr}(g)$, and $\mathrm{Tr}(f^\dagger) = \mathrm{Tr}(f)^\dagger$



Dimension

The dimension of A is the scalar $\dim(A) := \mathrm{Tr}(\mathrm{id}_A)$



Lemma:

- $\dim(I) = \mathrm{id}_I$
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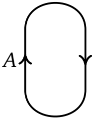

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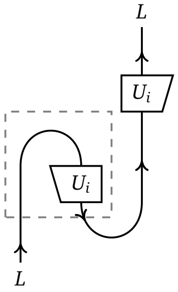
Teleportation

In **FHilb**:

- begin with a single system L
- prepare a joint system $R \otimes L$ in a maximally entangled state
- perform a joint measurement on the first two systems
- perform a unitary operation on the remaining system



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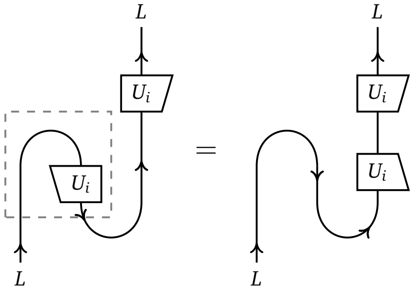

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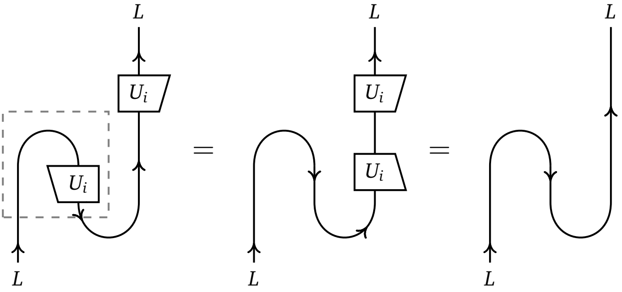

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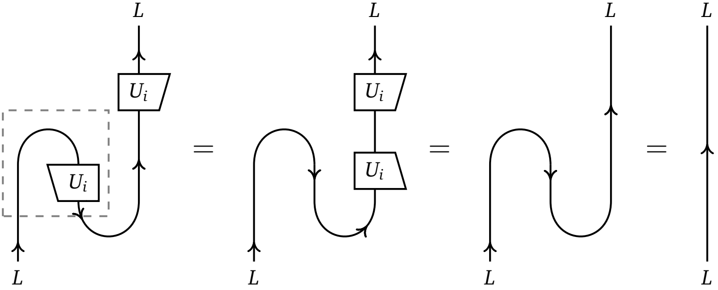

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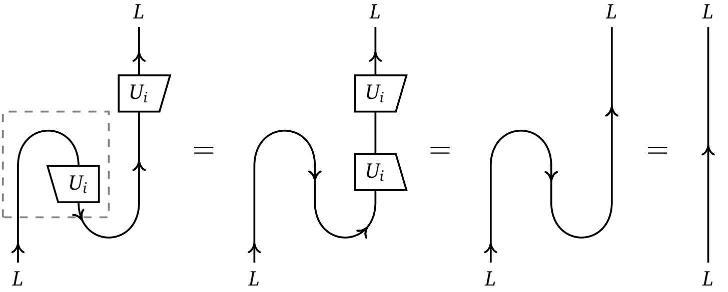


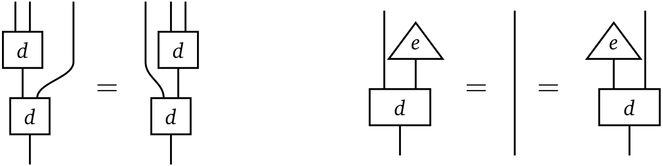
In **Rel**: encrypted communication using one-time pad

Part III

(Co)monoids

Comonoids

A comonoid in a monoidal category is an object A with *comultiplication* $A \xrightarrow{d} A \otimes A$ and *counit* $A \xrightarrow{e} I$ satisfying



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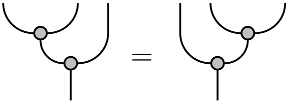
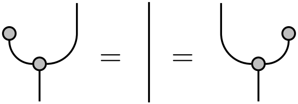




Examples:

- in **Set**, any object has unique cocommutative comonoid with comultiplication $a \mapsto (a, a)$ and counit $a \mapsto \bullet$
- in **Rel**, any group forms a comonoid with comultiplication $g \sim (h, h^{-1}g)$ and counit $1 \sim \bullet$
- in **FHilb**, any choice of basis $\{e_i\}$ gives cocommutative comonoid with comultiplication $e_i \mapsto e_i \otimes e_i$ and counit $e_i \mapsto 1$

Monoids

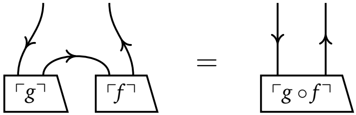
A monoid in a monoidal category consists of maps $I \rightarrow A \leftarrow A \otimes A$ satisfying associativity and unitality.

Lemma: In braided monoidal category, two (co)monoids combine

Lemma: In monoidal dagger category, monoid gives comonoid

Pair of pants

Map-state duality: composition $A \xrightarrow{g \circ f} A$ becomes $I \xrightarrow{\ulcorner g \circ f \urcorner} A^* \otimes A$.



Pair of pants

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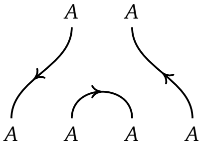
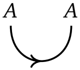
Lemma: If $A \dashv A^*$, then $A^* \otimes A$ is pair of pants monoid





Pair of pants

Map-state duality: composition $A \xrightarrow{g \circ f} A$ becomes $I \xrightarrow{\ulcorner g \circ f \urcorner} A^* \otimes A$.
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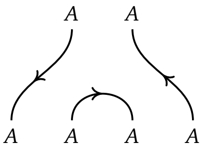

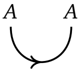

Example: Pair of pants on $\mathbb{C}^n$ in **FHilb** is n-by-n matrices $\mathbb{M}_n$
Proof: define $(\mathbb{C}^n)^* \otimes \mathbb{C}^n \to \mathbb{M}_n$ by $\langle j| \otimes |i\rangle \mapsto e_{ij}$

Pair of pants: one size fits all

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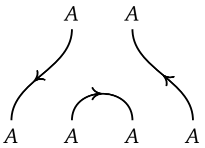

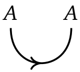

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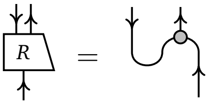
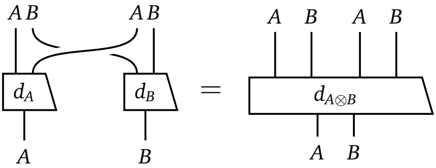
Proposition: Any monoid $(A, ,)$ embeds into $(A^* \otimes A, , \smile)$



Cloning

A braided monoidal category has cloning if there is natural $A \xrightarrow{d_A} A \otimes A$ with cocommutativity, coassociativity, $d_I = \rho_I$, and



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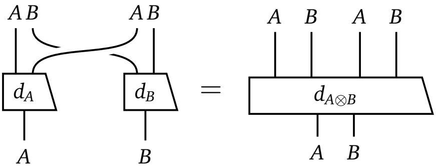
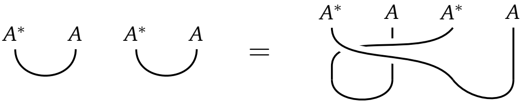


Set has cloning, but compact categories like **Rel** or **FHilb** cannot

Lemma: If compact category has cloning, then



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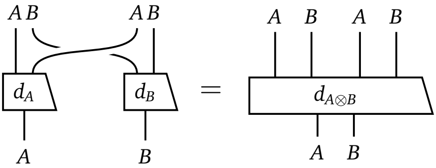

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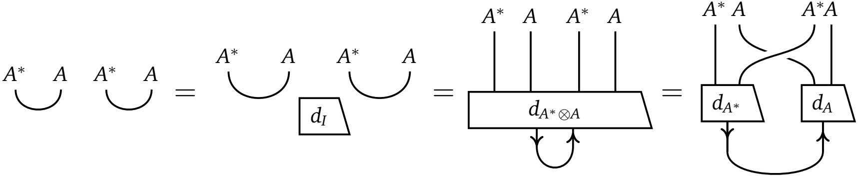

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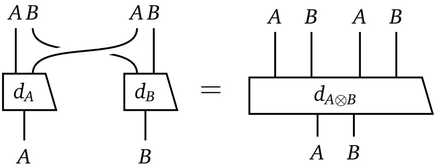

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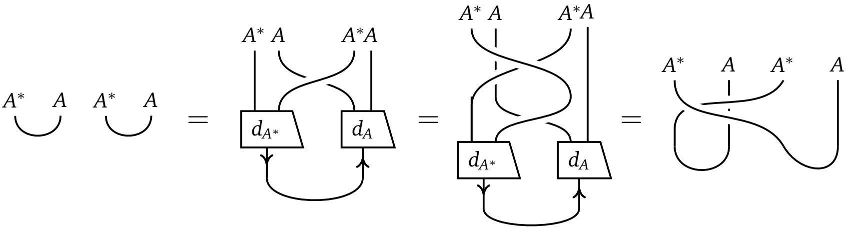

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Theorem: If braided monoidal category with duals has cloning, then $f = \mathrm{Tr}(f) \bullet \mathrm{id}_A$ for any $A \xrightarrow{f} A$

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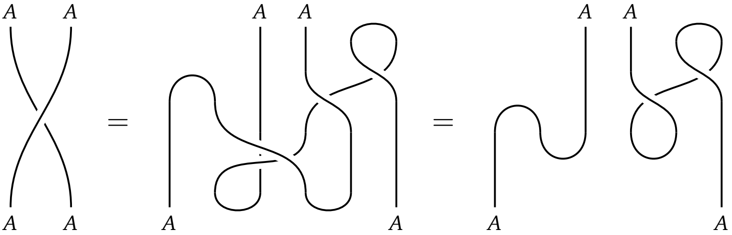

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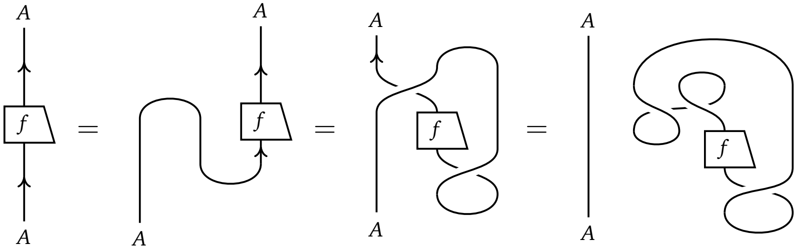

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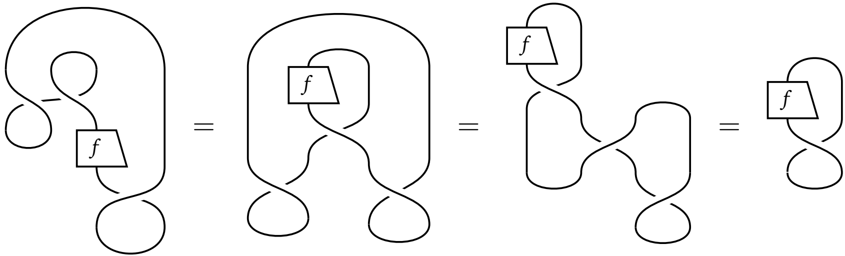


Summary

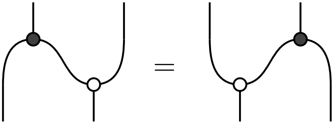
- Monoidal categories
  scalars, sound and complete graphical calculus
- Dual objects
  entanglement, teleportation, encrypted communication
- Monoids
  no cloning

Part IV

Frobenius structures

Frobenius structure

A Frobenius structure in a monoidal category is pair of comonoid $(A, \,, \,)$ and monoid $(A, \,, \,)$ satisfying *Frobenius law*:



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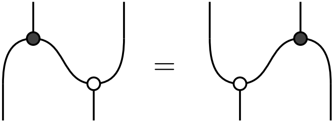

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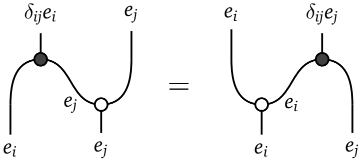
Example in **FHilb**: copying an orthogonal basis



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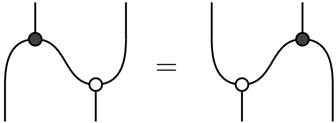

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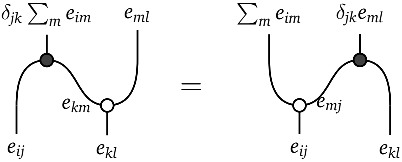
Example in **FHilb**: matrix algebra



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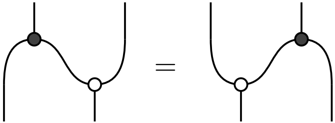

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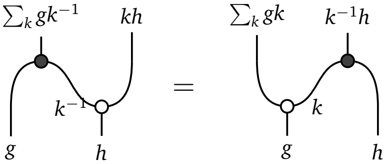
Example in **FHilb**: group algebra of finite group



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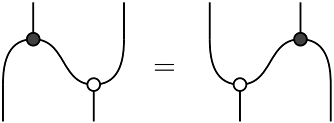

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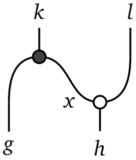
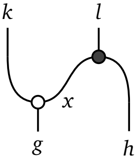
Example in **Rel**: set of morphisms of groupoid



$$\iff \quad h \circ l^{-1} = g^{-1} \circ k \quad \iff$$



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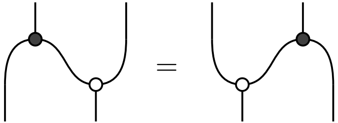

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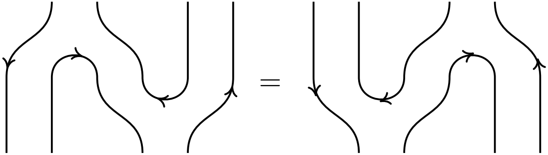
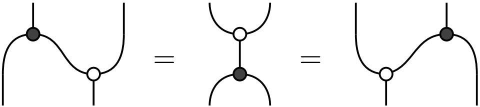
Example in any compact dagger category: pair of pants

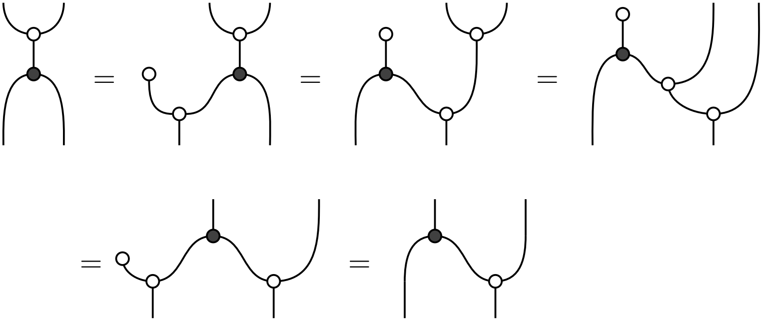


Frobenius law

Lemma: Any Frobenius structure satisfies:

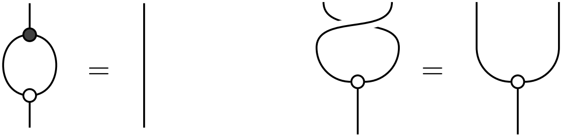


Proof: Suffices to prove one of the equalities



Classical structures

A classical structure is a special and commutative Frobenius structure



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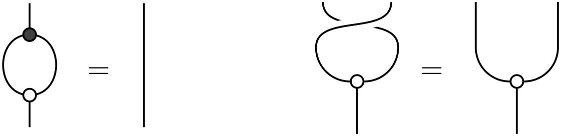
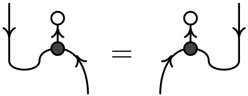


Examples:

- in **FHilb**: copying orthonormal basis is classical structure
- in **FHilb**: matrix algebra only special when trivial
- in **FHilb**: group algebra only special when trivial
- in **Rel**: groupoid always special
- in general: pair of pants only special when trivial

Symmetry

Frobenius structure in monoidal category is symmetric when:



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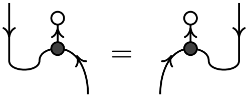
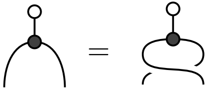


In braided monoidal category, this is equivalent to:



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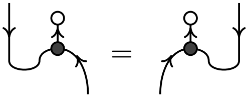

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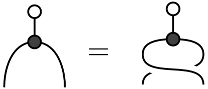

Examples:

- in **FHilb**: copying orthonormal basis is symmetric
- in **FHilb**: matrix algebra symmetric as $\mathrm{Tr}(ab) = \mathrm{Tr}(ba)$
- in **FHilb**: group algebra symmetric as inverses are two-sided
- in **Rel**: groupoid symmetric as inverses are two-sided
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Proposition: If (A, , , ,) Frobenius structure in monoidal category, then $A \dashv A$ is self-dual

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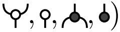
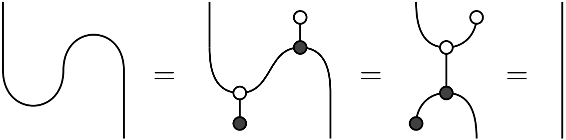
Proposition: If $(A,$ $)$ Frobenius structure in monoidal category, then $A \dashv A$ is self-dual

Proof: Snake equation:



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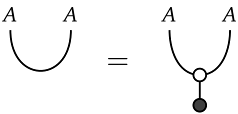
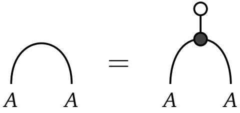
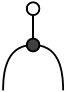




Conversely, monoid $(A, ,)$ forms Frobenius structure with some comonoid $(A, ,)$ iff allows nondegenerate form: map $: A \longrightarrow I$ with



part of self-duality $A \dashv A$.

Normal forms

Two ways to think about graphical calculus diagram:

- representing morphism; shorthand for e.g. linear map
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Theorem: if $(A, , , ,)$ is Frobenius structure,
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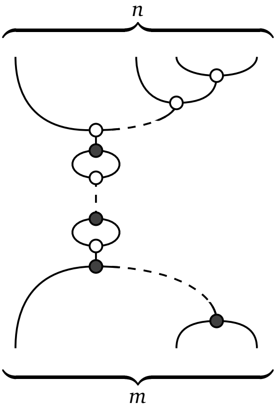

Normal forms

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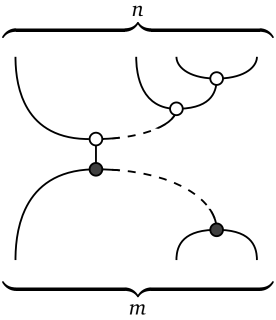

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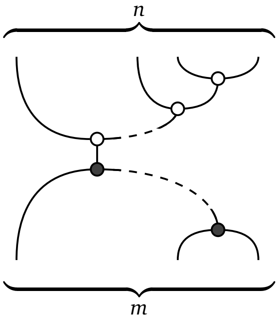

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Map-state duality: $f \mapsto f^\dagger$ is involution on pair of pants $A = H^* \otimes H$

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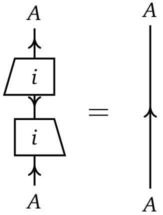

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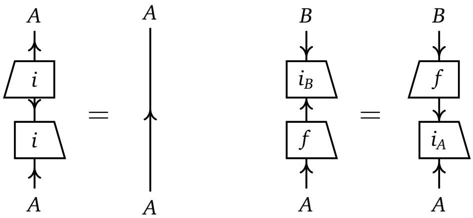

Morphisms $A \xrightarrow{f} B$ of involutive monoids satisfy $i_B \circ f = f_* \circ i_A$

Frobenius law and dagger

set of maps $A \to A$ closed under composition
$=$ submonoid of pair of pants $A^* \otimes A$.

Frobenius law and dagger

set of maps $A \to A$ closed under composition *and dagger*
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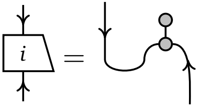

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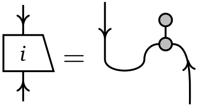
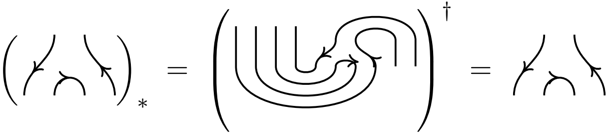


"Frobenius law = coherence law between dagger and closure"

- matrix algebra in **FHilb**: involution $\mathbb{M}_n \to \mathbb{M}_n^*$ is $f \mapsto f^\dagger$
- groupoid in **Rel**: involution $G \to G^*$ is $g \sim g^{-1}$
- pair of pants in general: involution invisible



Classification

Corollary: Dagger Frobenius structures in **FHilb** are C*-algebras

Proof: Correspond to $A \subseteq \mathbb{M}_n$ closed under addition, scalar multiplication, matrix multiplication, adjoint, and contain identity

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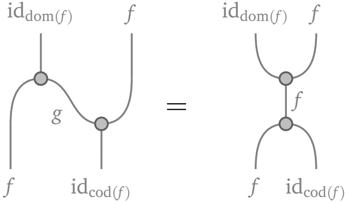

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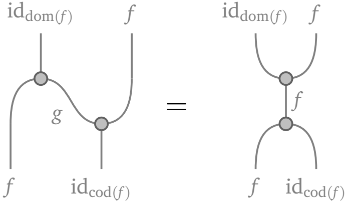
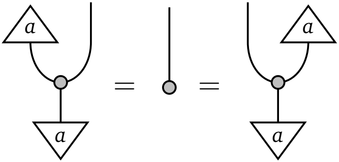


Corollary: Classical structures in **Rel** are abelian groupoids

Phases

A state $I \xrightarrow{a} A$ of a Frobenius structure is a phase when



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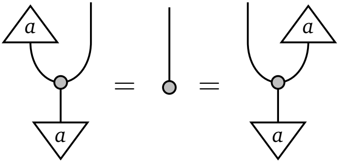

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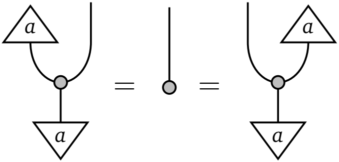
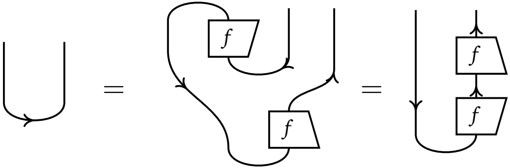


Proposition: phases of dagger Frobenius structure in (braided) monoidal dagger category form (abelian) phase group

- phase group of C*-algebra is its unitary group
- phase group of orthonormal basis are powers of circle group
- phase group of a group is group itself
- phase group of pair or pants are unitary endomorphisms

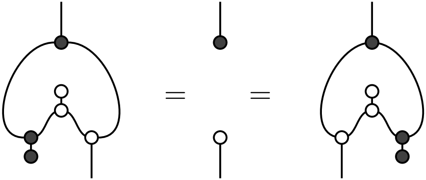


Part V

Complementarity

Complementarity

Symmetric dagger Frobenius structures  and  on the same object in a braided monoidal dagger category are complementary when



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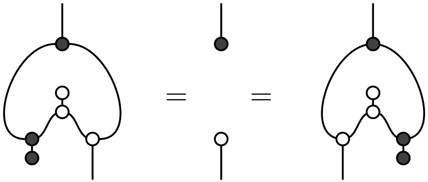
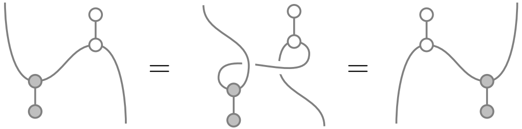


Black and white not obviously interchangeable. But by symmetry



Complementarity: examples

Proposition: classical structures in **FHilb** are complementary iff they copy mutually unbiased orthonormal bases

$$|\langle d_i | e_j \rangle|^2 = \frac{1}{\dim(H)}$$

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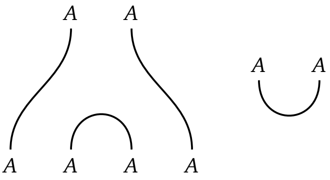
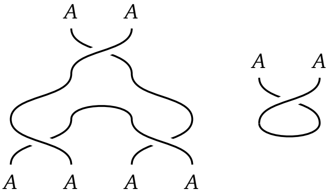
Proposition: classical structures in **FHilb** are complementary iff they copy mutually unbiased orthonormal bases

$$|\langle d_i | e_j \rangle|^2 \;=\; \frac{1}{\dim(H)}$$

Lemma: if A dagger self-dual in braided monoidal dagger category, then pair of pants and twisted knickers on $A \otimes A$ are complementary





Symmetric Frobenius structure A gives complementary pair on $A \otimes A$

Complementarity in **Rel**

Example: Let G and H be nontrivial groups, $A = G \times H$,

- $\mathbf{G}$ totally disconnected groupoid: objects G and $\mathbf{G}(g, g) = H$;
- $\mathbf{H}$ totally disconnected groupoid: objects H and $\mathbf{H}(h, h) = G$.

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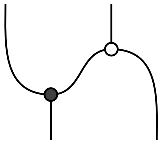
Then $\mathbf{G}$ and $\mathbf{H}$ give rise to complementary Frobenius structures.

Proposition: equivalent for groupoids $\mathbf{G}, \mathbf{H}$ on set A of morphisms:

- give complementary Frobenius structures
- map $A \to \mathrm{Ob}(\mathbf{G}) \times \mathrm{Ob}(\mathbf{H})$, $a \mapsto \big(\mathrm{dom}_{\mathbf{G}}(a), \mathrm{dom}_{\mathbf{H}}(a)\big)$ is bijective

Complementarity and dagger

Proposition: symmetric dagger Frobenius structures in braided category complementary iff the following morphism is unitary



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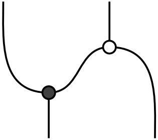
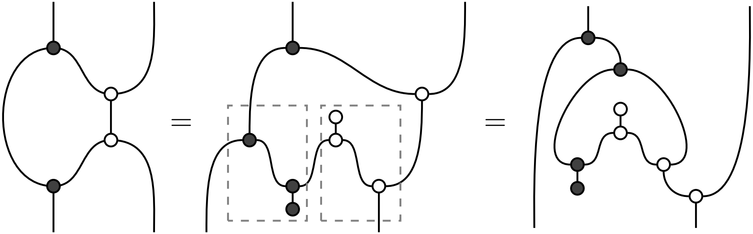
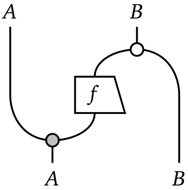


Proof:



Oracles

An oracle is a morphism $A \xrightarrow{f} B$ together with Frobenius structures on A and on B such that the following morphism is unitary:



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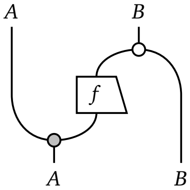
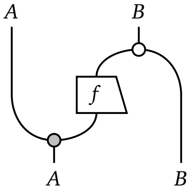


Example: Extension of function between mutually unbiased bases

Oracles

An oracle is a morphism $A \xrightarrow{f} B$ together with Frobenius structures $\tinymult[graydot]$ on A and $\tinymult[whitedot]$ on B such that the following morphism is unitary:

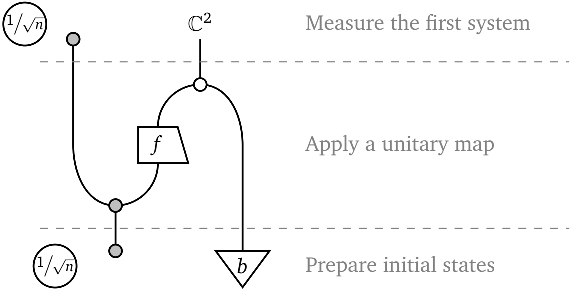


Example: Extension of function between mutually unbiased bases

Proposition: Let $(A, \tinymult[graydot])$, $(B, \tinymult[blackdot])$ and $(B, \tinymult[whitedot])$ be symmetric dagger Frobenius structures. Self-conjugate comonoid morphism $(A, \tinymult[graydot]) \xrightarrow{f} (B, \tinymult[blackdot])$ is oracle $(A, \tinymult[graydot]) \to (B, \tinymult[whitedot])$ iff $\tinymult[blackdot]$ complements $\tinymult[whitedot]$

Deutsch–Josza algorithm

Let function $\{1, \ldots, n\} \xrightarrow{f} \{0, 1\}$ be promised balanced or constant.
Extend to oracle $\mathbb{C}^n \to \mathbb{C}^2$; latter with computational and X bases.
Write $b = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$.



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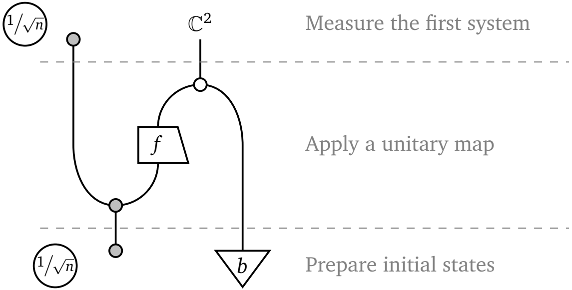
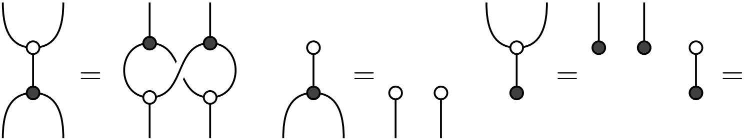


History is *certain* when f constant, *impossible* when f balanced.

Bialgebras

Complementarity related to Hopf algebras



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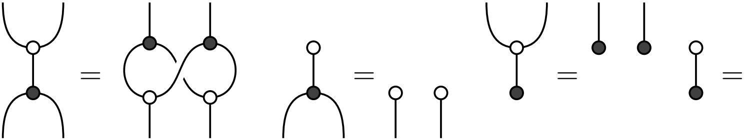

Strong complementarity = complementarity + bialgebra

(in **FHilb** and **Rel**: complementary $\underset{\not\Longrightarrow}{\overset{\text{only if commutative}}{\Longleftarrow}}$ bialgebra)

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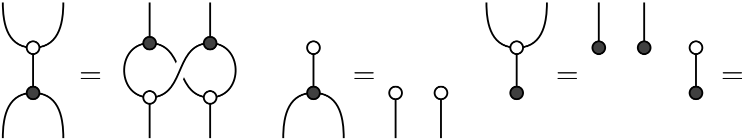

Strong complementarity = complementarity + bialgebra

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Theorem: the strongly complementary classical structures in **FHilb** are the group algebras

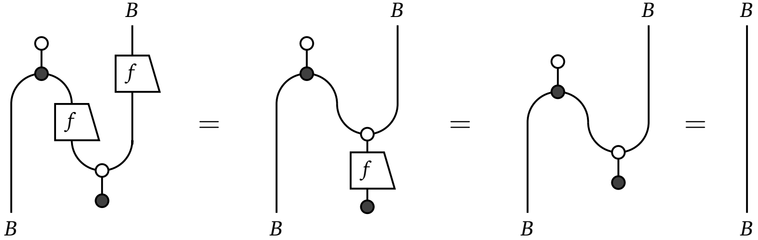
Part VI

Complete positivity

Morphisms of Frobenius structures

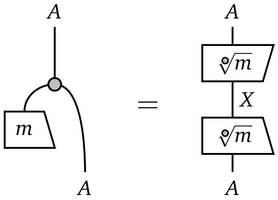
Lemma: If a morphism between Frobenius structures preserves (co)multiplication and (co)unit, then it is an isomorphism.

Proof:



Mixed states

State $I \xrightarrow{m} A$ of dagger Frobenius structure is mixed when



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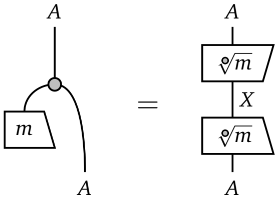

Examples:

- In **FHilb**: mixed state of C*-algebra is positive element $a = b^* b$
- In **Rel**: mixed state of groupoid is inverse-closed set of arrows
- In general: mixed state of pair of pants is name of positive map

Completely positive maps

If $(A, \ , \ )$ and $(B, \ , \ )$ are dagger Frobenius structures, a morphism $A \xrightarrow{f} B$ is completely positive when $I \xrightarrow{(f \otimes \mathrm{id}) \circ m} B \otimes E$ is mixed state for mixed state $I \xrightarrow{m} A \otimes E$ and any dagger Frobenius structure $(E, \ , \ )$.

Examples:

- Unitary evolution $A^* \otimes A \xrightarrow{u_* \otimes u} A^* \otimes A$ for unitary $A \xrightarrow{u} A$

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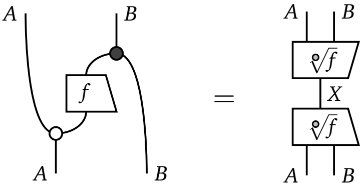
- Measurement: completely positive map $A^* \otimes A \longrightarrow \mathbb{C}^n$ in **FHilb** is precisely *positive-operator valued measure*
- Completely positive maps $\mathbf{G} \longrightarrow \mathbf{H}$ in **Rel** respect inverses: $g \sim h$ implies $g^{-1} \sim h^{-1}$ and $\mathrm{id}_{\mathrm{dom}(g)} \sim \mathrm{id}_{\mathrm{dom}(h)}$

The CP condition

Lemma: Assume $f \otimes \mathrm{id}_E \geq 0 \implies f \geq 0$.
If $A \xrightarrow{f} B$ is completely positive, then CP condition holds



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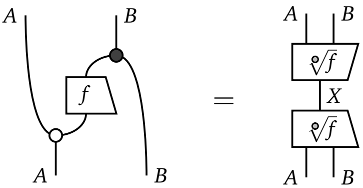
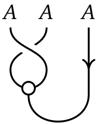


Proof: Let $E = A \otimes A^*$ be pair of pants, define $I \xrightarrow{m} A \otimes E$ as:



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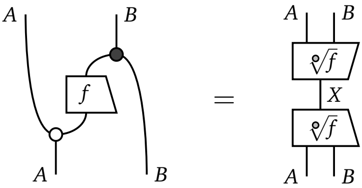
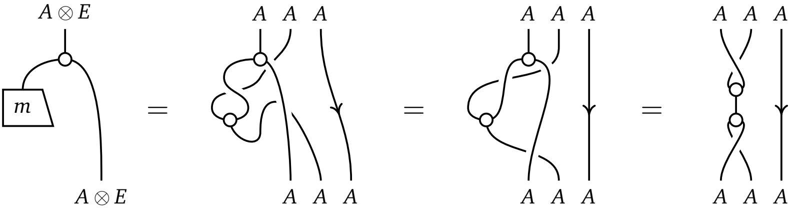


Proof: Let $E = A \otimes A^*$ be pair of pants, define $I \xrightarrow{m} A \otimes E$
Then m is a mixed state



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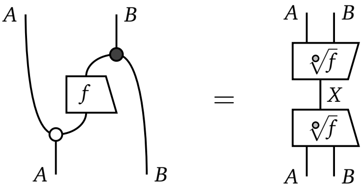
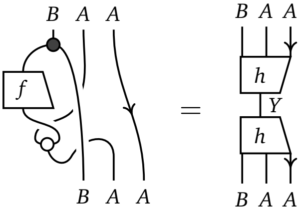


Proof: Let $E = A \otimes A^*$ be pair of pants, define $I \xrightarrow{m} A \otimes E$
Then m is a mixed state, as is $(f \otimes \mathrm{id}_E) \circ m$.



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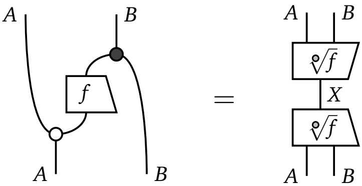
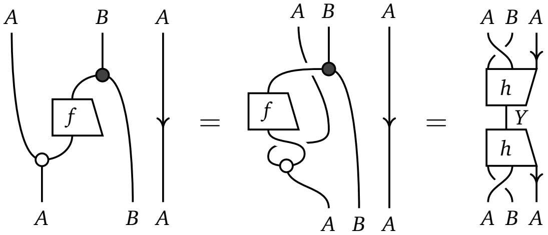


Proof: Let $E = A \otimes A^*$ be pair of pants, define $I \xrightarrow{m} A \otimes E$
Then m is a mixed state, as is $(f \otimes \mathrm{id}_E) \circ m$. Hence:



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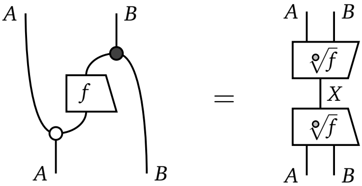
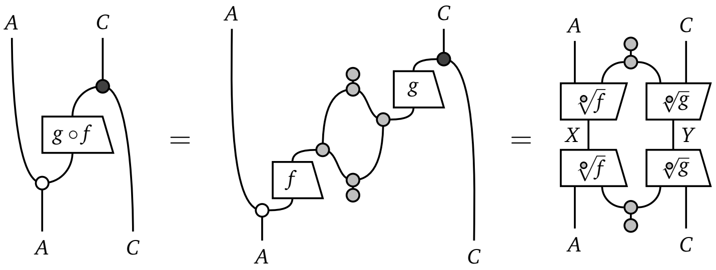


Theorem (Stinespring): If $A \xrightarrow{f} B$ satisfies CP condition, then it is completely positive.

The CP construction

Theorem: If **C** is a monoidal dagger category, there is a new category CP[**C**]:

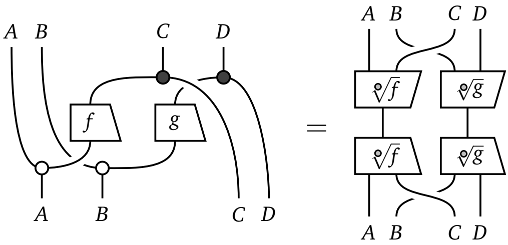
- objects in CP[**C**] are special dagger Frobenius structures in **C**
- morphisms in CP[**C**] are morphisms in **C** satisfying CP condition



The CP construction

Theorem: If **C** is a *braided* monoidal dagger category, there is a new *monoidal* category CP[**C**]:

- objects in CP[**C**] are special dagger Frobenius structures in **C**
- morphisms in CP[**C**] are morphisms in **C** satisfying CP condition
- tensor product in CP[**C**] is as in **C**



The CP construction

Theorem: If **C** is a *symmetric* monoidal dagger category,
there is a new *symmetric monoidal* category CP[**C**]:

- objects in CP[**C**] are special dagger Frobenius structures in **C**
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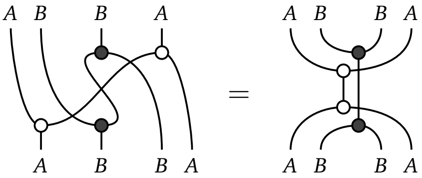

The CP construction

Theorem: If **C** is a *symmetric* monoidal dagger category, there is a new *symmetric monoidal dagger* category CP[**C**]:

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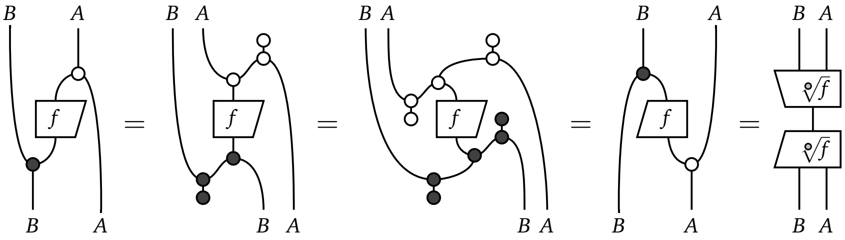

(duality between *Schrödinger* and *Heisenberg* pictures)

The CP construction

Theorem: If **C** is a *symmetric* monoidal dagger category, there is a new *compact dagger* category CP[**C**]:

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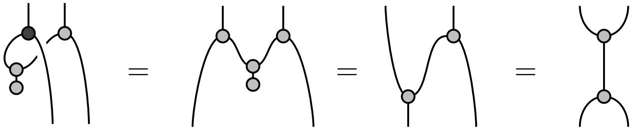

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Examples:

- CP[**FHilb**] $=$ fin-dim C*-algebras and completely positive maps
- CP[**Rel**] $=$ groupoids and inverse-respecting relations

Classical and quantum structures

Consider full subcategory $\mathrm{CP_c}[\mathbf{C}]$ of classical structures.
Lemma: $\mathrm{CP_c}[\mathbf{FHilb}] \simeq$ stochastic matrices (rows sum to 1)

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Lemma ("P is faithful up to phase"):

- if $P(f) = P(g)$, then $s \bullet f = t \bullet g$ and $s^\dagger \bullet s = t^\dagger \bullet t$ for some $I \xrightarrow{s,t} I$
- if $s \bullet f = t \bullet g$ and $s^\dagger \bullet s = \mathrm{id}_I = t^\dagger \bullet t$, then $P(f) = P(g)$

Environment

An environment structure for compact dagger category $\mathbf{C}^{\text{pure}}$ is

- a compact dagger category $\mathbf{C}$ of which $\mathbf{C}^{\text{pure}}$ is subcategory
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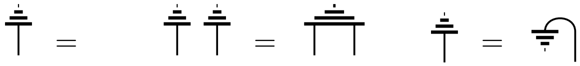
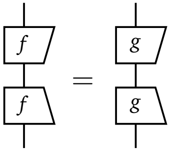
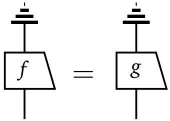
- 
-  in $\mathbf{C}^{\text{pure}}$ $\iff$  in $\mathbf{C}$

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An environment structure with purification for category $\mathbf{C}^{\mathrm{pure}}$ is

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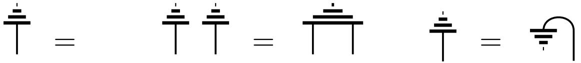
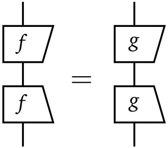
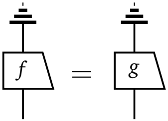
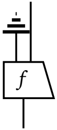
- 
-  in $\mathbf{C}^{\mathrm{pure}}$ $\iff$  in $\mathbf{C}$
- every map in $\mathbf{C}$ is of the form  for f in $\mathbf{C}^{\mathrm{pure}}$

  (Hence $\mathbf{C}$ and $\mathbf{C}^{\mathrm{pure}}$ must have same objects)

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Theorem: If $\mathbf{C}^{\text{pure}}$ has environment structure with purification, there is isomorphism $F\colon \mathrm{CP_q}[\mathbf{C}^{\text{pure}}] \to \mathbf{C}$ with $F(A) = A$ on objects, $F(f \otimes g) = F(f) \otimes F(g)$ on morphisms, that preserves daggers

Decoherence

A decoherence structure for $\mathbf{C}^{\mathrm{pure}}$ is

- an environment structure $\mathbf{C}$ with discarding maps $: A \to I$ in $\mathbf{C}$
- for each special dagger Frobenius structure $(A, ,)$ in $\mathbf{C}^{\mathrm{pure}}$, an object $A_{\circ}$ in $\mathbf{C}$ and a measuring map $: A \to A_{\circ}$ in $\mathbf{C}$, with:

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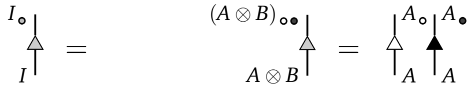
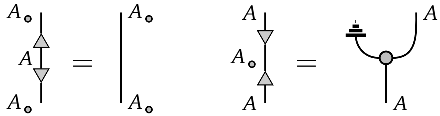
- 

- 

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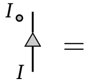
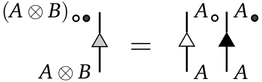
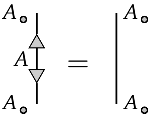
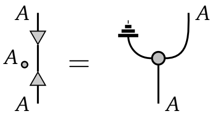
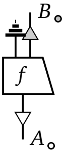
- 



- 



- every map in $\mathbf{C}$ is of the form 

for f in $\mathbf{C}^{\text{pure}}$

(Hence each object in $\mathbf{C}$ is of form $A_\circ$)

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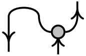
$(A^* \otimes A, ,)_\circ = (A, ,)$ and measuring map 

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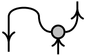
Example: $\mathbf{C}^{\text{pure}}$ has decoherence structure in $\text{CP}[\mathbf{C}^{\text{pure}}]$ with $(A^* \otimes A, , \cup)_{\circ} = (A, ,)$ and measuring map 

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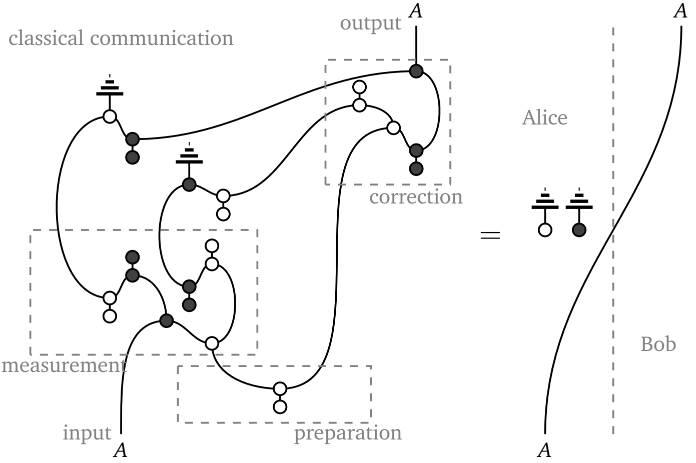
Teleportation

Theorem: If $(A, ⅄, ọ)$ and $(A, ⅄, ●)$ complementary dagger Frobenius in braided monoidal dagger category $\mathbf{C}$, and ⅄ commutative, then in CP[$\mathbf{C}$]:



Teleportation

Theorem: If $(A, \ldots)$ and $(A, \ldots)$ complementary dagger Frobenius in braided monoidal dagger category $\mathbf{C}$, and commutative, then in $\mathrm{CP}[\mathbf{C}]$:

- mixed states
- arbitrary systems
- 'classical communication' only in sense of 'copied' by Frobenius structures, one of which noncommutative
- 'two bits' of classical communication: two channels used, maybe more than two copyable states
- tensor product and composition only

Summary

- Monoidal categories
  scalars, sound and complete graphical calculus
- Dual objects
  entanglement, teleportation, encrypted communication
- Monoids
  no cloning
- Frobenius structures
  normal form, classical structures, observables, classification
- Complementarity
  Deutsch-Josza
- Completely positive maps
  mixed states, axiomatization, teleportation