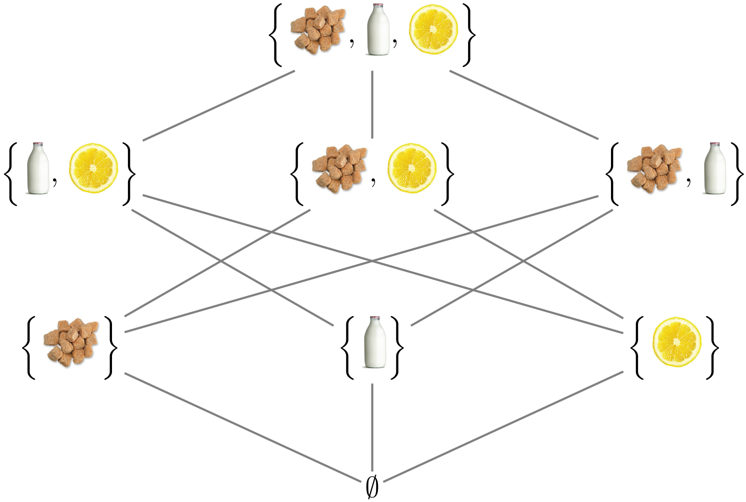
Piecewise Boolean algebra

Chris Heunen

THE UNIVERSITY *of* EDINBURGH
informatics

Boolean algebra: example



Boolean algebra: definition

A Boolean algebra is a set B with:

- a distinguished element $1 \in B$;
- a unary operations $\neg\colon B \to B$;
- a binary operation $\wedge\colon B \times B \to B$;

such that for all $x, y, z \in B$:

- $x \wedge (y \wedge z) = (x \wedge y) \wedge z$;
- $x \wedge y = y \wedge x$;
- $x \wedge 1 = x$;
- $\neg x = \neg(x \wedge \neg y) \wedge \neg(x \wedge y)$

“Sets of independent postulates for the algebra of logic”
Transactions of the American Mathematical Society 5:288–309, 1904

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- $x \wedge y = y \wedge x$;
- $x \wedge 1 = x$;
- $x \wedge x = x$;
- $x \wedge \neg x = \neg 1 = \neg 1 \wedge x$; ($\neg x$ is a complement of x)
- $x \wedge \neg y = \neg 1 \Leftrightarrow x \wedge y = x$ ($0 = \neg 1$ is the least element)

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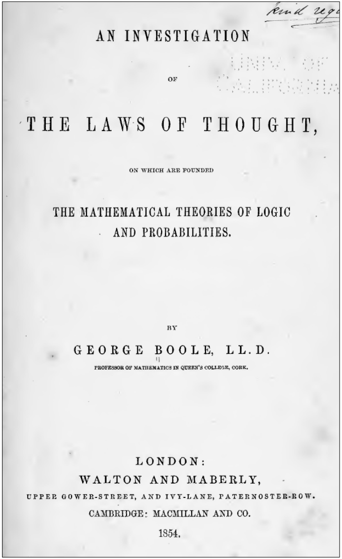
Boole's algebra

Boolean algebra $\neq$ Boole's algebra



AN INVESTIGATION

OF

THE LAWS OF THOUGHT,

ON WHICH ARE FOUNDED

THE MATHEMATICAL THEORIES OF LOGIC AND PROBABILITIES.

BY

GEORGE BOOLE, LL.D.

PROFESSOR OF MATHEMATICS IN QUEEN'S COLLEGE, CORK.

LONDON:
WALTON AND MABERLY,
UPPER GOWER-STREET, AND IVY-LANE, PATERNOSTER-ROW.
CAMBRIDGE: MACMILLAN AND CO.
1854.

Boolean algebra ≠ Boole's algebra

kind regards

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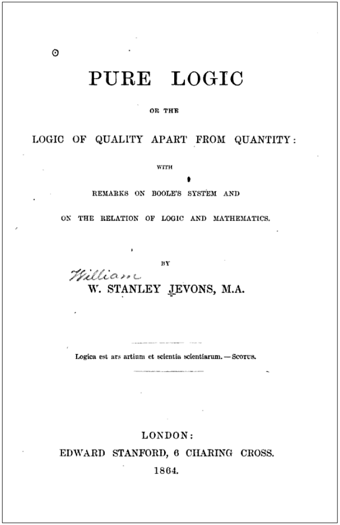
CHAPTER V.

OF THE FUNDAMENTAL PRINCIPLES OF SYMBOLICAL REASONING, AND OF THE EXPANSION OR DEVELOPMENT OF EXPRESSIONS INVOLVING LOGICAL SYMBOLS.

1. THE previous chapters of this work have been devoted to the investigation of the fundamental laws of the operations of the mind in reasoning; of their development in the laws of the symbols of Logic; and of the principles of expression, by which that species of propositions called primary may be represented in the language of symbols. These inquiries have been in the strictest sense preliminary. They form an indispensable introduction to one of the chief objects of this treatise—the construction of a system or method of Logic upon the basis of an exact summary of the fundamental laws of thought. There are certain considerations touching the nature of this end, and the means of its attainment, to which I deem it necessary here to direct attention.

2. I would remark in the first place that the generality of a method in Logic must very much depend upon the generality of its elementary processes and laws. We have, for instance, in the previous sections of this work investigated, among other things, the laws of that logical process of *addition* which is symbolized by the sign +. Now those laws have been determined from the study of instances, in all of which it has been a necessary condition, that the classes or things added together in thought should be mutually exclusive. The expression $x + y$ seems indeed uninterpretable, unless it be assumed that the things represented by x and the things represented by y are entirely separate; that they embrace no individuals in common. And conditions analogous to this have been involved in those acts of conception from the study of which the laws of the other symbolical operations have been ascertained. The question then arises, whether

Boolean algebra = Jevon's algebra



PURE LOGIC

OR THE

LOGIC OF QUALITY APART FROM QUANTITY:

WITH

REMARKS ON BOOLE'S SYSTEM AND

ON THE RELATION OF LOGIC AND MATHEMATICS.

BY

William

W. STANLEY JEVONS, M.A.

Logica est ars artium et scientia scientiarum.—SCOTUS.

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is AB; if it is C, is AC, and it is therefore either AB or AC.

Use of brackets. **67.** Let a plural term enclosed in brackets (. . . . . .), and placed beside another term, mean that it is combined with it, as one single term is with another:

Thus A (B+C) = AB+AC.

Combination of plural terms. **68.** One plural term is combined with another by combining each alternative of the one separately with each of the other. Each combined alternative may then be combined with each alternative of a third plural term, and so on:

Thus (D+E) (B+C)=B (D+E) +C (D+E)
=BD+BE+CD+CE.

Law of unity. **69.** It is in the nature of thought and things that *same alternatives are together same in meaning, as any one taken singly.*

Thus, what is the same as A *or* A is the same as A, a self-evident truth.

A+A=A A+A+A=A A+A+B=A+B

This law is correlative to the Law of Simplicity, (§ 39), and is perhaps of equal importance and frequent use. It was not recognised by Professor Boole, when laying down the principles of his system.

Superfluous terms. **70.** In a plural term, any alternative may be removed, of which a part forms another alternative.

Thus the term *either B or BC* is the same in meaning with B alone, or B+BC=B. For it is a self-evident truth (§ 99) that B standing alone is either the same as BC, or as *B not-C*. Thus

B+BC=B *not*-C+BC+BC
=B *not*-C+BC=B.

Boole's algebra isn't Boolean algebra

Boole's Algebra Isn't Boolean Algebra

A description, using modern algebra, of what Boole really did create.

THEODORE HAILPERIN
Lehigh University
Bethlehem, PA 18015

To Boole and his mid-nineteenth century contemporaries, the title of this article would have been very puzzling. For Boole's first work in logic, *The Mathematical Analysis of Logic*, appeared in 1847 and, although the beginnings of modern abstract algebra can be traced back to the early part of the nineteenth century, the subject had not fully emerged until towards the end of the century. Only then could one clearly distinguish and compare algebras. (We use the term **algebra** here as standing for a formal system, not a structure which realizes, or is a model for, it—for instance, the algebra of integral domains as codified by a set of axioms *versus* a particular structure, e.g., the integers, which satisfies these axioms.) Granted, however, that this later full degree of understanding has been attained, and that one can conceptually distinguish algebras, is it not true that Boole's "algebra of logic" is Boolean algebra?

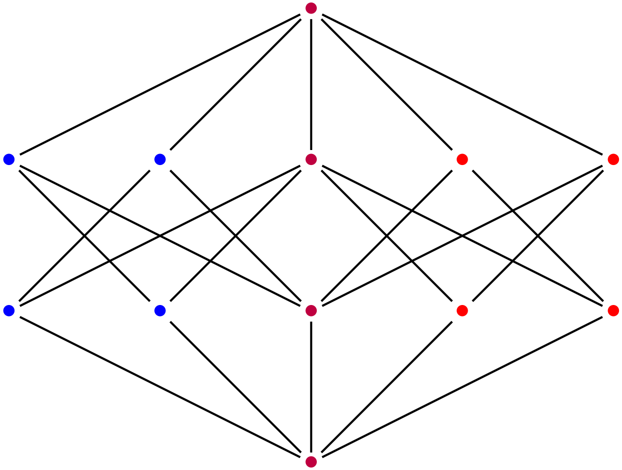
Piecewise Boolean algebra: definition

A piecewise Boolean algebra is a set B with:

- a reflexive symmetric binary relation $\odot \subseteq B^2$;
- a (partial) binary operation $\wedge\colon \odot \to B$;
- a (total) function $\neg\colon B \to B$;
- an element $1 \in B$ with $\{1\} \times B \subseteq \odot$;

such that every $S \subseteq B$ with $S^2 \subseteq \odot$ is contained in a $T \subseteq B$ with $T^2 \subseteq \odot$ where $(T, \wedge, \neg, 1)$ is a Boolean algebra.

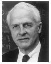
Piecewise Boolean algebra: example



Piecewise Boolean algebra $\lneq$ quantum logic

~~Subsets of a set~~

Subspaces of a Hilbert space




"The logic of quantum mechanics"
Annals of Mathematics 37:823–843, 1936

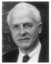
Piecewise Boolean algebra $\lneq$ quantum logic

~~Subsets of a set~~

Subspaces of a Hilbert space

An orthomodular lattice is:

- A partial order set $(B, \leq)$ with min 0 and max 1
- that has greatest lower bounds $x \wedge y$;
- an operation $\perp \colon B \to B$ such that
- $x^{\perp\perp} = x$, and $x \leq y$ implies $y^{\perp} \leq x^{\perp}$;
- $x \vee x^{\perp} = 1$;
- if $x \leq y$ then $y = x \vee (y \wedge x^{\perp})$

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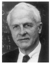
An orthomodular lattice is not distributive:

( or ) and 

$\neq$

( and ) or ( and )

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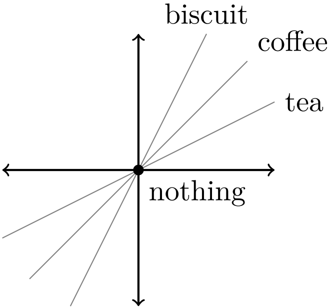

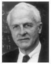

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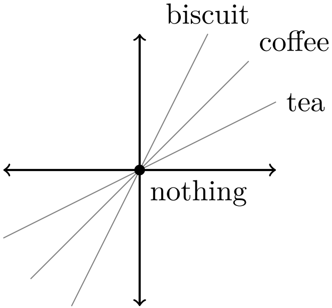
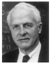


However: fine when within orthogonal basis (Boolean subalgebra)

"The logic of quantum mechanics"
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Boole's algebra $\neq$ Boolean algebra

Quantum measurement is probabilistic
(state $\alpha|0\rangle + \beta|1\rangle$ gives outcome 0 with probability $|\alpha|^2$)

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A hidden variable for a state is an assignment of a consistent outcome to any possible measurement
(homomorphism of piecewise Boolean algebras to $\{0, 1\}$)

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Quantum measurement is probabilistic
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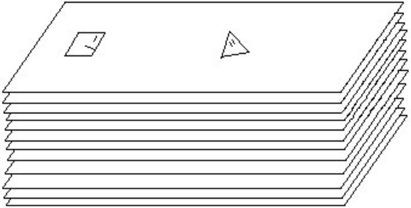
A hidden variable for a state is an assignment of a consistent outcome to any possible measurement
(homomorphism of piecewise Boolean algebras to $\{0,1\}$)

Theorem: hidden variables cannot exist
(if dimension $n \geq 3$, there is no homomorphism $\mathrm{Sub}(\mathbb{C}^n) \to \{0,1\}$ of piecewise Boolean algebras.)


"The problem of hidden variables in quantum mechanics"
Journal of Mathematics and Mechanics 17:59–87, 1967

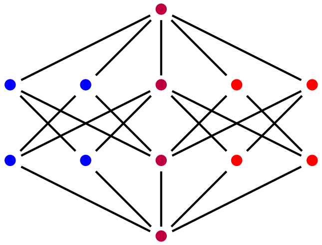
Piecewise Boolean domains: definition

Given a piecewise Boolean algebra B,
its piecewise Boolean domain $\mathrm{Sub}(B)$
is the collection of its Boolean subalgebras,
partially ordered by inclusion.

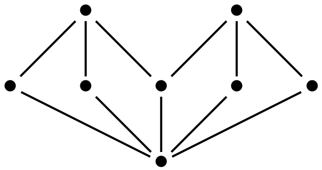


Piecewise Boolean domains: example

Example: if B is



then $\mathrm{Sub}(B)$ is



Piecewise Boolean domains: theorems

Can reconstruct B from $\mathrm{Sub}(B)$
($B \cong \mathrm{colim}\ \mathrm{Sub}(B)$)
(the parts determine the whole)


“Noncommutativity as a colimit”
Applied Categorical Structures 20(4):393–414, 2012

Piecewise Boolean domains: theorems

Can reconstruct B from $\mathrm{Sub}(B)$
($B \cong \mathrm{colim}\ \mathrm{Sub}(B)$)
(the parts determine the whole)

$\mathrm{Sub}(B)$ determines B
($B \cong B' \iff \mathrm{Sub}(B) \cong \mathrm{Sub}(B')$)
(*shape* of parts determines whole)

"Noncommutativity as a colimit"
Applied Categorical Structures 20(4):393–414, 2012

"Subalgebras of orthomodular lattices"
Order 28:549–563, 2011

Piecewise Boolean domains: as complex as graphs

State space = Hilbert space
Sharp measurements = subspaces (projections)
Jointly measurable = overlapping or orthogonal (commute)

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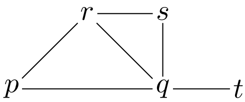
(In)compatibilities form graph:



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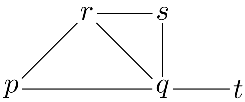

Theorem: Any graph can be realised as sharp measurements on some Hilbert space.

  

"Quantum theory realises all joint measurability graphs"
Physical Review A 89(3):032121, 2014

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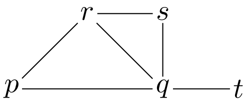

Theorem: Any graph can be realised as sharp measurements on some Hilbert space.

Corollary: Any piecewise Boolean algebra can be realised on some Hilbert space.

  

"Quantum theory realises all joint measurability graphs"
Physical Review A 89(3):032121, 2014

"Quantum probability – quantum logic"
Springer Lecture Notes in Physics 321, 1989

Piecewise Boolean domains: as complex as hypergraphs

State space = Hilbert space

Unsharp measurements = positive operator-valued measures

Jointly measurable = marginals of larger POVM

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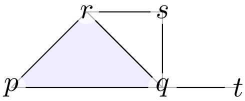
(In)compatibilities now form hypergraph:



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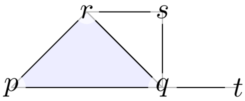
(In)compatibilities now form abstract simplicial complex:



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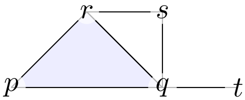

Theorem: Any abstract simplicial complex can be realised as POVMs on a Hilbert space.

"All joint measurability structures are quantum realizable"
Physical Review A 89(5):052126, 2014

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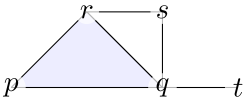

Theorem: Any abstract simplicial complex can be realised as POVMs on a Hilbert space.

Corollary: Any interval effect algebra can be realised on some Hilbert space.



“All joint measurability structures are quantum realizable”
Physical Review A 89(5):052126, 2014


“Hilbert space effect-representations of effect algebras”
Reports on Mathematical Physics 70(3):283–290, 2012

Piecewise Boolean domains: partition lattices

What does $\mathrm{Sub}(B)$ look like when B is an honest Boolean algebra?

Piecewise Boolean domains: partition lattices

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Boolean algebras are dually equivalent to Stone spaces

"The theory of representations of Boolean algebras"
Transactions of the American Mathematical Society 40:37–111, 1936

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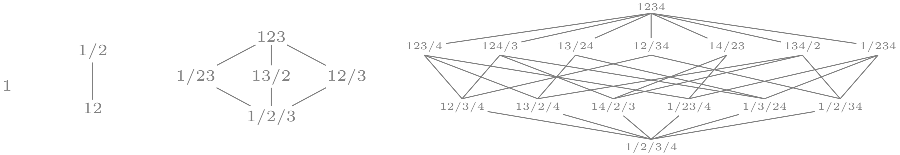
$\mathrm{Sub}(B)$ becomes a partition lattice




“The theory of representations of Boolean algebras”
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“On the lattice of subalgebras of a Boolean algebra”
Proceedings of the American Mathematical Society 36: 87–92, 1972

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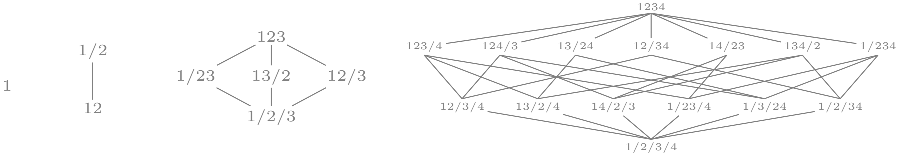
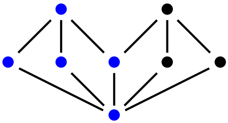


Idea: every downset in $\mathrm{Sub}(B)$ is a partition lattice (upside-down)!



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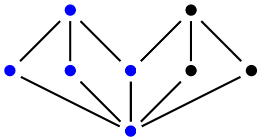
Piecewise Boolean domains: characterisation

Lemma: Piecewise Boolean domain D gives functor $F\colon D \to \mathbf{Bool}$ that preserves subobjects; "F is a piecewise Boolean diagram".
($\mathrm{Sub}(F(x)) \cong \downarrow x$, and $B = \mathrm{colim}\ F$)



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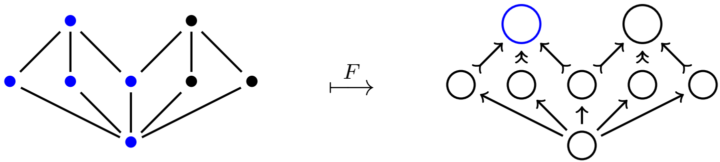

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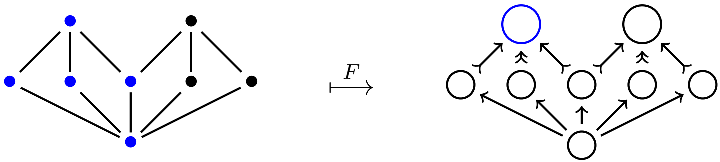


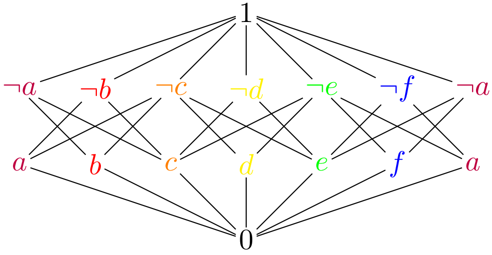
Theorem: A partial order is a piecewise Boolean domain iff:

- it has directed suprema;
- it has nonempty infima;
- each element is a supremum of compact ones;
- each downset is cogeometric with a modular atom;
- each element of height $n \leq 3$ covers $\binom{n+1}{2}$ elements.
- a set of atoms has a sup iff each finite subset does

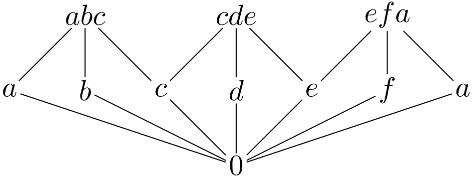
"Piecewise Boolean algebras and their domains"
ICALP Proceedings, Lecture Notes in Computer Science 8573:208–219, 2014

Orthoalgebras

This is almost a piecewise Boolean domain D:



That is of the form $D = \mathrm{Sub}(B)$ for this B:



But B is not a piecewise Boolean algebra: $\{a, c, e\}$ not in one block

Piecewise Boolean domains: higher order

Scott topology turns directed suprema into topological convergence
(closed sets = downsets closed under directed suprema)
Lawson topology refines it from dcpos to continuous lattices
(basic open sets = Scott open minus upset of finite set)

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If B_0 is piecewise Boolean algebra, $\mathrm{Sub}(B_0)$ is algebraic dcpo and complete semilattice,

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If B_0 is piecewise Boolean algebra, $\mathrm{Sub}(B_0)$ is algebraic dcpo and complete semilattice, hence a Stone space under Lawson topology!






"Continuous lattices and domains"
Cambridge University Press, 2003

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Lawson topology refines it from dcpos to continuous lattices
(basic open sets = Scott open minus upset of finite set)

If B_0 is piecewise Boolean algebra, $\mathrm{Sub}(B_0)$ is algebraic dcpo and complete semilattice, hence a Stone space under Lawson topology!

It then gives rise to a new Boolean algebra B_1. Repeat: $B_2, B_3, \ldots$
(Can handle domains of Boolean algebras with Boolean algebra!)






"Continuous lattices and domains"
Cambridge University Press, 2003


"Domains of commutative C*-subalgebras"
Logic in Computer Science, ACM/IEEE Proceedings 450–461, 2015

Piecewise Boolean diagrams: topos

- Consider "contextual sets" over piecewise Boolean algebra B
  assignment of set $S(C)$ to each $C \in \mathrm{Sub}(B)$
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  $\underline{B}(C) = C$

- $\mathcal{T}(B)$ believes that $\underline{B}$ is an honest Boolean algebra!

  

"A topos for algebraic quantum theory"
Communications in Mathematical Physics 291:63–110, 2009

Operator algebra

C*-algebras: main examples of piecewise Boolean algebras.

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Example: $C(X) = \{f\colon X \to \mathbb{C} \text{ continuous}\}$
Theorem: Every commutative -algebra is of this form.


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piecewise Boolean algebras $\xleftarrow{\text{projections}}$ -algebras

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$$\begin{tikzcd} \text{piecewise Boolean algebras} \arrow[r, bend left] \arrow[r, phantom, "\perp"] & \text{-algebras} \arrow[l, bend left, "\text{projections}"] \end{tikzcd}$$

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Journal of Mathematical Analysis and Applications 416:289–313, 2014

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If $\mathrm{Sub}(A) \cong \mathrm{Sub}(B)$ preserves $U(A) \times \mathrm{Sub}(A) \to \mathrm{Sub}(A)$,
then $A \cong B$ as  -algebras.
Needs orientation!


"Characterizations of categories of commutative C*-subalgebras"
Communications in Mathematical Physics 331(1):215–238, 2014


"Isomorphisms of ordered structures of abelian C*-subalgebras of C*-algebras"
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"Active lattices determine AW*-algebras"
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Scatteredness

A space is scattered if every nonempty subset has an isolated point.

Precisely when each continuous $f\colon X \to \mathbb{R}$ has countable image.

Example: $\{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\}$.

"Inductive Limits of Finite Dimensional C*-algebras"
Transactions of the American Mathematical Society 171:195–235, 1972

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Nonexample: $C(\text{Cantor})$ is approximately finite-dimensional
Nonexample: $C([0,1])$ is not even approximately finite-dimensional

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Transactions of the American Mathematical Society 171:195–235, 1972

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Scatteredness

Theorem: the following are equivalent for a  -algebra A:

- Sub(A) is algebraic
- Sub(A) is continuous
- Sub(A) is meet-continuous
- Sub(A) is quasi-algebraic
- Sub(A) is quasi-continuous
- Sub(A) is atomistic
- A is scattered

"A characterization of scattered C*-algebras and application to crossed products"
Journal of Operator Theory 63(2):417–424, 2010

 

"Domains of commutative C*-subalgebras"
Logic in Computer Science, ACM/IEEE Proceedings 450–461, 2015

Back to quantum logic

For  -algebra $C(X)$, projections are clopen subsets of X.
Can characterize in order-theoretic terms: (if $|X| \geq 3$)

closed subsets of X = ideals of $C(X)$ = elements of $\mathrm{Sub}(C(X))$
clopen subsets of X = 'good' pairs of elements of $\mathrm{Sub}(C(X))$

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Georgia Institute of Technology, 1996

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Each projection of -algebra A is in some maximal $C \in \mathrm{Sub}(A)$.
Can recover poset of projections from $\mathrm{Sub}(A)$! (if $\dim(Z(A)) \geq 3$)

“Compactifications and functions spaces”
Georgia Institute of Technology, 1996

“$\mathcal{C}(A)$”
Radboud University Nijmegen, 2015

Back to piecewise Boolean domains

$\mathrm{Sub}(B)$ determines B

$(B \cong B' \iff \mathrm{Sub}(B) \cong \mathrm{Sub}(B'))$

(*shape* of parts determines whole)

Caveat: not 1-1 correspondence!


“Subalgebras of orthomodular lattices”
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If B Boolean algebra, then Sub(B) partition lattice

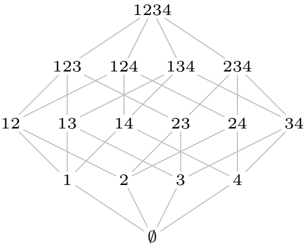
Caveat: not constructive, not categorical


"Subalgebras of orthomodular lattices"
Order 28:549–563, 2011



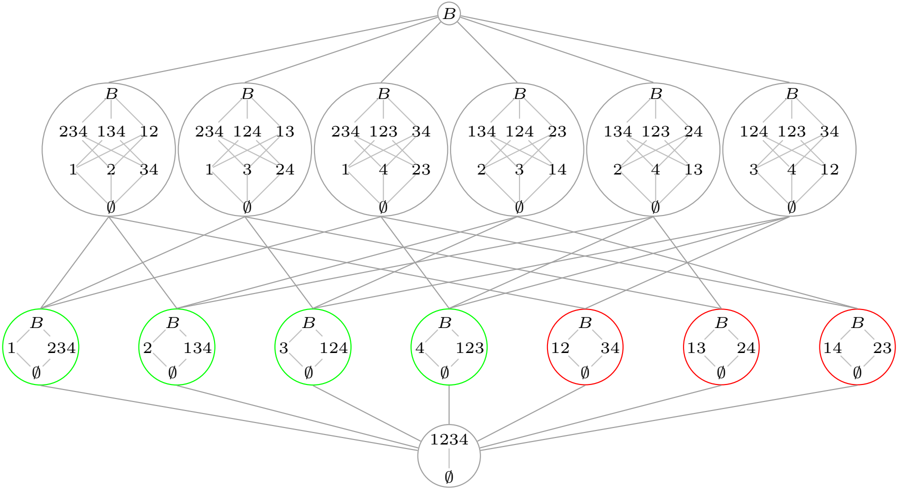
"On the lattice of subalgebras of a Boolean algebra"
Proceedings of the American Mathematical Society 36: 87–92, 1972

Different kinds of atoms

If $B =$ 

, then $\mathrm{Sub}(B) = \cdots$

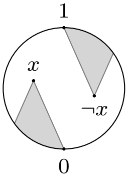
Different kinds of atoms



Principal pairs

Reconstruct pairs $(x, \neg x)$ of B:

- *principal ideal subalgebra* of B is of the form



- they are the elements p of $\mathrm{Sub}(B)$ that are
  dual modular and $(p \vee m) \wedge n = p \vee (m \wedge n)$ for $n \geq p$
  atom or *relative complement* $a \wedge m = a$, $a \vee m = B$ for atom a

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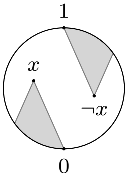

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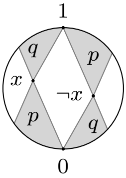

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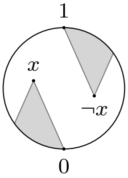

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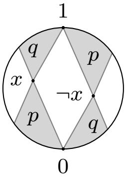

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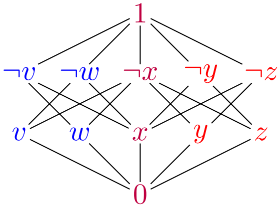
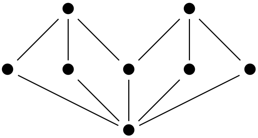
Theorem: $B \simeq \mathrm{Pp}(\mathrm{Sub}(B))$ for Boolean algebra B of size ≥ 4
$D \simeq \mathrm{Sub}(\mathrm{Pp}(D))$ for Boolean domain D of size ≥ 2

Directions

If B is



then $\mathrm{Sub}(B)$ is



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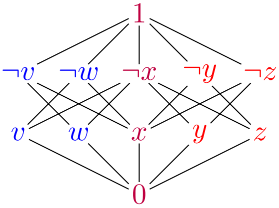
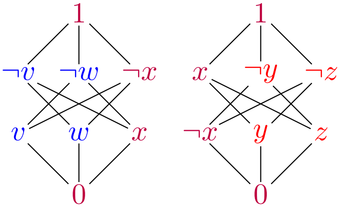
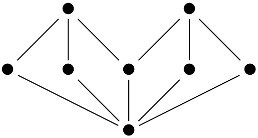
If B is 

or 

then Sub(B) is 

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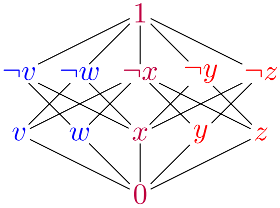

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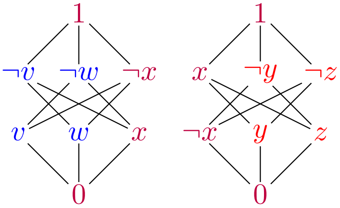

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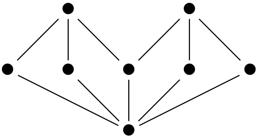

A direction for a Boolean domain is map $d\colon D \to D^2$ with

- $d(1) = (p, q)$ is a principal pair
- $d(m) = (p \wedge m, q \wedge m)$

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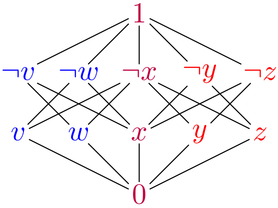

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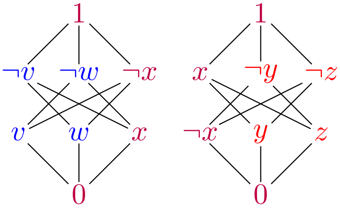

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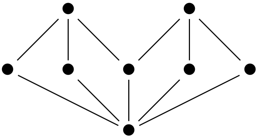

A direction for a piecewise Boolean domain is map $d\colon D \to D^2$ with

- if $a \leq m$ then $d(m)$ is a principal pair with meet a in m
- $d(m) = \bigvee\{(m, m) \wedge f(n) \mid a \leq n\}$
- if m, n cover a, $d(m) = (a, m)$, $d(n) = (n, a)$, then $m \vee n$ exists

Orthoalgebras

Almost theorem:

- $B \simeq \mathrm{Dir}(\mathrm{Sub}(B))$ for orthoalgebra B of size ≥ 4
- $D \simeq \mathrm{Sub}(\mathrm{Dir}(D))$ for piecewise orthodomain D of size ≥ 2

Problems:

- subalgebras of a Boolean orthoalgebra need not be Boolean
- intersection of two Boolean subalgebras need not be Boolean
- two Boolean subalgebras might have no meet
- two Boolean subalgebras might have upper bound but no join

"Boolean subalgebras of orthoalgebras"
Ongoing work

Conclusion

- Should consider piecewise Boolean algebras
- Give rise to domain of honest Boolean subalgebras
- Complicated structure, but can characterize
- Shape of parts enough to determine whole
- Same trick works for scattered operator algebras
- Direction needed for almost categorical equivalence

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Question

Theorem: any Boolean algebra is isomorphic to
the global sections of a sheaf on its Stone space

Question: is any *piecewise* Boolean algebra isomorphic to
the global sections of a *sheaf* on its *Stone space*?

Would give logic of contextuality

"The theory of representations of Boolean algebras"
Transactions of the American Mathematical Society 40:37–111, 1936

"Representations of algebras by continuous sections"
Bulletin of the American Mathematical Society 78(3):291–373, 1972


"The sheaf-theoretic structure of nonlocality and contextuality"
New Journal of Physics 13:113036, 2011

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