Dagger Category Theory

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informatics

Outline

- What are dagger categories?
- What are dagger monads?
- What are dagger limits?
- What are evils about daggers?

Dagger

A dagger is contravariant involutive identity-on-objects endofunctor

$$X \underset{f^{\dagger}}{\overset{f = f^{\dagger\dagger}}{\rightleftarrows}} Y$$

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Terminology: adjoints in Hilbert spaces $\langle f(x) \mid y \rangle_Y = \langle x \mid f^\dagger(y) \rangle_X$

If $S(X)$ is poset of closed subspaces, get $S(f) \colon S(X)^{\mathrm{op}} \to S(Y)$

Theorem [Palmquist 74]: $S(f)$ and $S(f^\dagger)$ adjoint, and
up to scalar any adjunction of this form

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- Cofree dagger category: same objects, pairs $X \leftrightarrows Y$
- Dagger functors and natural transformations
- Unitary representations and intertwiners

Way of the dagger

Category theory	Dagger category theory
isomorphism	unitary $f^{-1} = f^\dagger$
idempotent	projection $f = f^\dagger \circ f$
functor	dagger functor $F(f^\dagger) = F(f)^\dagger$
natural transform	natural transformation $(\alpha^\dagger)_X = (\alpha_X)^\dagger$
monoidal structure	monoidal dagger structure $(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger$

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	isn't this trivially trivial?

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- Dagger categories, dagger functors, and natural transformations: not just 2-category, but *dagger 2-category*

  2-cells have dagger, so should have unitary coherence laws
- Principle: if $P \implies Q$ for categories, then $P^\dagger + \text{laws} \implies Q^\dagger + \text{laws}$ for dagger categories

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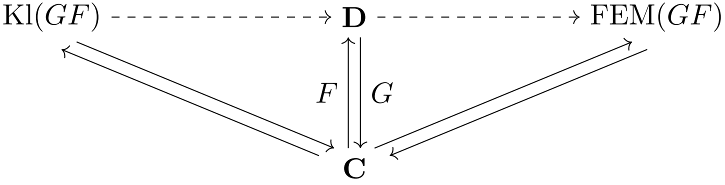

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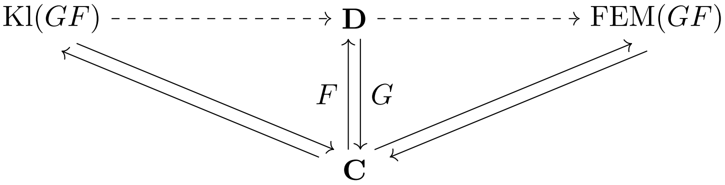
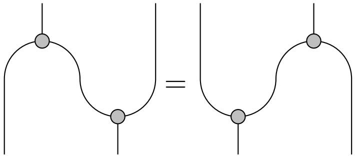


- *Dagger adjunction* is adjunction in **DagCat**: no left/right
- Dagger monad should at least be dagger functor: so comonad
- What interaction between monad and comonad?

Dagger monads

- A *dagger monad* is a monad that is a dagger functor satisfying

$$\mu T \circ T\mu^{\dagger} = T\mu \circ \mu^{\dagger} T$$



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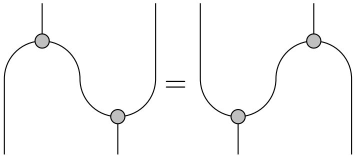

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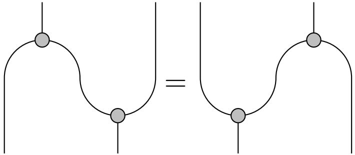

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- Dagger adjunctions induce dagger monads

Kleisli algebras

- If T is dagger monad on $\mathbf{C}$, then $\mathrm{Kl}(T)$ has dagger

$$\big(A \xrightarrow{f} T(B)\big) \;\mapsto\; \big(B \xrightarrow{\eta} T(B) \xrightarrow{\mu^\dagger} T^2(B) \xrightarrow{T(f^\dagger)} T(A)\big)$$

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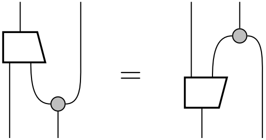
- Frobenius law for monoid M is Frobenius law for monad $- \otimes M$



Eilenberg-Moore algebras

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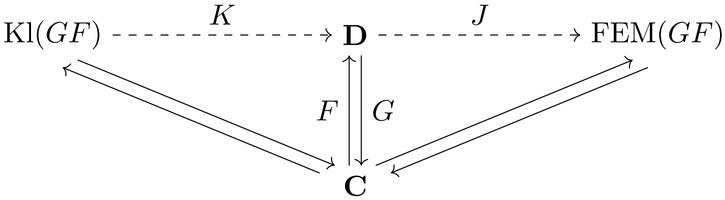
- There are EM-algebras that are not FEM

Dagger monads

Theorem

If F, G are dagger adjoint, there are unique dagger functors with



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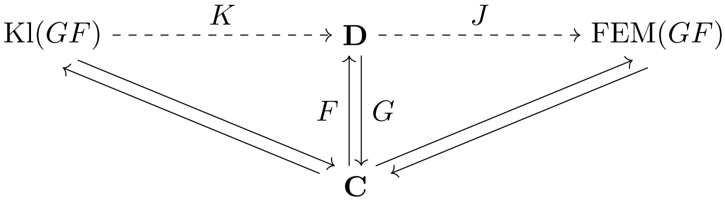

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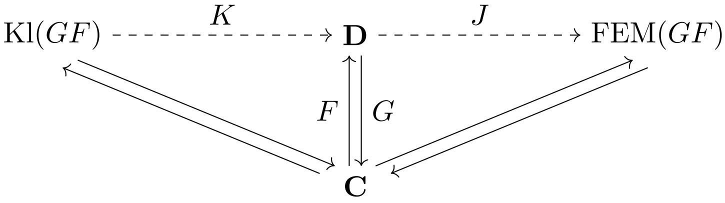

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Proof.

- EM-algebra (A, a) is FEM iff $a^\dagger$ is morphism $(A, a) \to (TA, \mu_A)$
- $(A, a) \in \mathrm{Im}(J)$ associative $\implies \big((TA, \mu_A) \xrightarrow{a} (A, a)\big) \in \mathrm{Im}(J)$
  $\implies a^\dagger \in \mathrm{Im}(J)$
  $\implies (A, a) \in \mathrm{FEM}(GF)$ □

Strength

- Monad T is *strong* when coherent natural $A \otimes T(B) \to T(A \otimes B)$
- monoids in $\mathbf{C}$ $\simeq$ monads on $\mathbf{C}$

$$\begin{array}{rcl} \text{monoids in } \mathbf{C} & \simeq & \text{monads on } \mathbf{C} \\ M & \mapsto & - \otimes M \\ T(I) & \leftarrowtail & T \end{array}$$

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- If T commutative, then $\mathrm{Kl}(T)$ dagger symmetric monoidal

Dagger limits

Should:

- be unique up to unique unitary
- be defined canonically (without e.g. enrichment)
- generalize dagger biproducts and dagger equalisers
- connect to dagger adjunctions and dagger Kan extensions

Unique up to unitary

- Two limits (L, l_A), (M, m_A) of same diagram are iso $L \xrightarrow{f} M$. Now f^{-1} is iso of limits $M \to L$. But $f^\dagger$ is iso of colimits.

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- Right notion of dagger limit means fixing maps $A \to L \to B$.

Dagger-shaped limits

Definition

The *dagger limit* of dagger functor $D\colon \mathbf{J} \to \mathbf{C}$ is a limit (L, l_J) with

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Proof.

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Constructing dagger-shaped limits

- *Dagger product*: product $J \xleftarrow{p_J} J \times K \xrightarrow{p_K} K$ with $p_K^\dagger p_J = \delta_{JK}$
- *Dagger equaliser*: equaliser $E \xrightarrow{e} J \rightrightarrows K$ with $e^\dagger e = \mathrm{id}$

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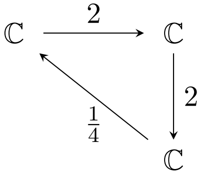
- $\mathbf{C}$ has dagger limits of dagger shapes with κ components $\iff$ $\mathbf{C}$ has dagger limits of
  - dagger products of size κ
  - dagger stabilisers
  - dagger projections

Non-dagger shapes?

What to do with loops?



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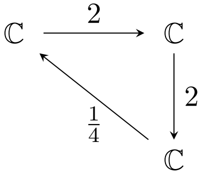

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- Theorem: If there is unitary $GFA \to A$ for each A, can replace F, G with isomorphic functors that lift to dagger equivalence.

Conclusion

- **DagCat** is not just a 2-category
  so dagger category theory nontrivial

- Dagger monads = monad + dagger functor + Frobenius law

- Dagger-shaped limits = limit + dagger + idempotent
  Dagger limits = ?

- Dagger categories can't be that evil