Categorical relativistic quantum theory

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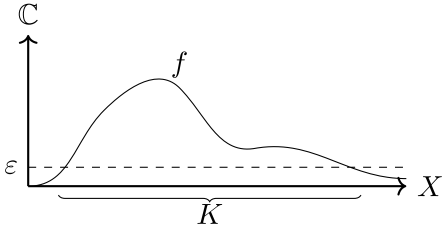
Idea

- Hilbert modules: naive quantum field theory
- Idempotent subunits: base space in any category
- Support: where morphisms live
- Causal structures: relativistic quantum information

Base space

Let X be locally compact Hausdorff space.
$C_0(X) = \{f\colon X \to \mathbb{C} \text{ cts} \mid \forall \varepsilon > 0\, \exists K \subseteq X \text{ cpt}\colon f(X \setminus K) < \varepsilon\}$



$C_b(X) = \{f\colon X \to \mathbb{C} \text{ cts} \mid \exists \|f\| < \infty\, \forall t \in X\colon |f(t)| \leq \|f\|\}$

Hilbert spaces

$\mathbb{C}$-module H with complete $\mathbb{C}$-valued inner product

tensor product over $\mathbb{C}$	monoidal category
tensor unit $\mathbb{C}$	tensor unit I
complex numbers $\mathbb{C}$	scalars $I \to I$
finite dimensional	dual objects
adjoints	dagger
orthonormal basis	commutative dagger Frobenius structure
fin-dim C*-algebra	dagger Frobenius structure

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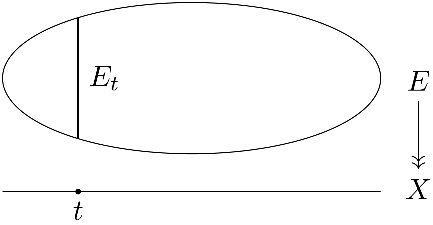
tensor unit $C_0(X)$	tensor unit I
complex numbers $C_b(X)$	scalars $I \to I$
finitely presented	dual objects
adjoints	dagger
finite coverings	commutative dagger Frobenius structure
unif fin-dim C*-bundles	dagger Frobenius structure

'Scalars are not numbers'

Bundles of Hilbert spaces

Bundle $E \twoheadrightarrow X$, each fibre Hilbert space, operations continuous



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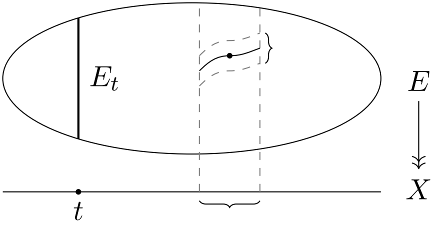

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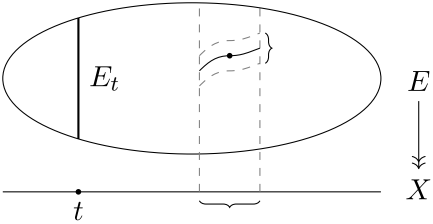

$$\begin{array}{rcl} \text{Hilbert } C_0(X)\text{-modules} & \simeq & \text{bundles of Hilbert spaces over } X \\ \text{sections vanishing at infinity} & \leftarrow\!\shortmid & E \twoheadrightarrow X \\ E & \mapsto & \text{localisation} \end{array}$$

Idempotent subunits

Definition: $\mathrm{ISub}(\mathbf{C}) = \{s \colon S \rightarrowtail I \mid \mathrm{id}_S \otimes s \colon S \otimes S \to S \otimes I \text{ iso}\}/\simeq$

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- Order theory: $\mathrm{ISub}(Q) = \{x \in Q \mid x^2 = x \leq 1\}$ for quantale Q:
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- Algebra: $\mathrm{ISub}(\mathbf{Mod}_R) = \{S \subseteq R \text{ ideal} \bigm| S = S^2\}$
  'idempotent subunits are idempotent ideals'

Semilattice

Proposition: $\mathrm{ISub}(\mathbf{C})$ is a semilattice, $\wedge = \otimes$, $1 = \mathrm{id}_I$

$$\begin{tikzcd}
T \arrow[dr, tail] & \\
& I \\
S \arrow[uu, dashed] \arrow[ur, tail] &
\end{tikzcd}$$

Caveat: $\mathbf{C}$ must be firm, i.e. $s \otimes \mathrm{id}_T$ monic, and size issue

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$$\begin{tikzcd} \mathbf{SemiLat} \arrow[loop left, "\mathrm{id}"] \arrow[r, dashed, shift left] \arrow[r, phantom, "\perp"] & \mathbf{FirmCat} \arrow[l, shift left, "\mathrm{ISub}"] \end{tikzcd}$$

Spatial categories

Call $\mathbf{C}$ spatial when $\mathrm{ISub}(\mathbf{C})$ is frame

$$\begin{tikzcd}
\mathbf{SemiLat} \arrow[rrr, shift left=2] \arrow[rrr, phantom, "\perp"] \arrow[d, shift right=2] \arrow[d, phantom, "\dashv"] & & & \mathbf{Frame} \arrow[lll, hook', shift left=2] \arrow[d, shift right=2] \arrow[d, phantom, "\dashv"] \\
\mathbf{FirmCat} \arrow[rrr, shift left=2] \arrow[rrr, phantom, "\perp"] \arrow[u, shift right=2, "\mathrm{ISub}"'] & & & \mathbf{SpatCat} \arrow[lll, hook', shift left=2] \arrow[u, shift right=2, "\mathrm{ISub}"']
\end{tikzcd}$$

$$(\mathbf{C}, \otimes) \longmapsto ([\mathbf{C}^{\mathrm{op}}, \mathbf{Set}]_{\mathrm{supp}}, \otimes_{\mathrm{Day}})$$

Support

Say $s \in \mathrm{ISub}(\mathbf{C})$ supports $f \colon A \to B$ when

$$\begin{tikzcd} A \arrow[rr, "f"'] \arrow[d, dashed] & & B \\ B \otimes S \arrow[rr, "\mathrm{id} \otimes s"'] & & B \otimes I \arrow[u, "\simeq"'] \end{tikzcd}$$

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$$\begin{array}{ccc}
f & \longmapsto & \{s \mid s \text{ supports } f\} \\
\mathbf{C}^{\mathbf{2}} & \xrightarrow{\mathrm{supp}} & \mathrm{Pow}(\mathrm{ISub}(\mathbf{C}))
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universal with $F(f) = \bigvee \{F(s) \mid s \in \mathrm{ISub}(\mathbf{C}) \text{ supports } f\}$

Restriction

Full subcategory $\mathbf{C}\big|_s$ of A with $\mathrm{id}_A \otimes s$ invertible:

- monoidal with tensor unit S
- coreflective: $\begin{tikzcd} \mathbf{C}\big|_s \arrow[r, hook, shift left] \arrow[r, phantom, "\perp"] & \mathbf{C} \arrow[l, dashed, shift left] \end{tikzcd}$
- tensor ideal: if $A \in \mathbf{C}$ and $B \in \mathbf{C}\big|_s$, then $A \otimes B \in \mathbf{C}\big|_s$
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Proposition: $\mathrm{ISub}(\mathbf{C}) \simeq \{$monocoreflective tensor ideals in $\mathbf{C}\}$

Localisation

A graded monad is a monoidal functor $\mathbf{E} \to [\mathbf{C}, \mathbf{C}]$

$(\eta\colon A \to T(1),\ \mu\colon T(t) \circ T(s) \to T(s \otimes t))$

Lemma: $s \mapsto \mathbf{C}\big|_s$ is an $\mathrm{ISub}(\mathbf{C})$-graded monad

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universal property of localisation for $\Sigma = \{\mathrm{id}_E \otimes s \mid E \in \mathbf{C}\}$

$$\begin{tikzcd}
\mathbf{C} \arrow[rr, "(-) \otimes S"] \arrow[drr, "F \text{ inverting } \Sigma"'] & & \mathbf{C}\big|_s = \mathbf{C}[\Sigma^{-1}] \arrow[d, dashed, "\simeq"'] \\
& & \mathbf{D}
\end{tikzcd}$$

Spacetime

What if X is more than just space?

Lorentzian manifold with time orientation:

$s \ll t$: there is future-directed timelike curve $s \to t$

$s \prec t$: there is future-directed non-spacelike curve $s \to t$

	chronological	causal
future	$I^+(t) = \{s \in X \mid t \ll s\}$	$J^+(t) = \{s \in X \mid t \prec s\}$
past	$I^-(t) = \{s \in X \mid s \ll t\}$	$J^-(t) = \{s \in X \mid s \prec t\}$

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If $S \subseteq X$ open, then $I^+(S) = \bigcup_{s \in S} I^+(s) = \bigcup_{s \in S} J^+(s) = J^+(S)$
I^+ and I^- give 'future' and 'past' operators

Causal structure

Closure operator on partially ordered set P is function $C\colon P \to P$:

- if $s \leq t$, then $C(s) \leq C(t)$;
- $s \leq C(s)$;
- $C(C(s)) \leq C(s)$.

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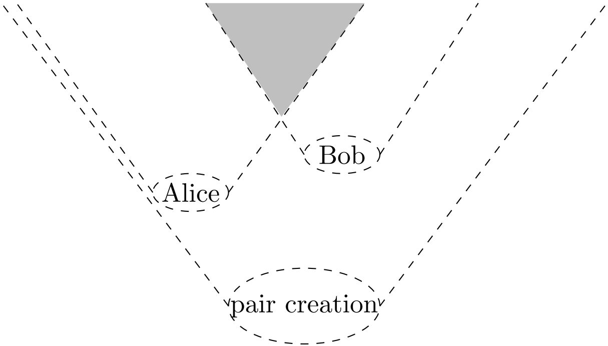
Proposition: if $r \in \mathrm{ISub}(\mathbf{C})$ and C is closure operator on $\mathbf{C}$,
then $D(s) = C(s) \wedge r$ is closure operator on $\mathbf{C}\big|_r$
'Causal structure restricts'

Teleportation

'Restriction = propagation'



compact category + support + causal structure
=
teleportation only successful on intersection of future sets

Further

- relativistic quantum information protocols
- causality
- proof analysis
- control flow
- data flow
- concurrency
- graphical calculus

Complements

$$\text{Subunit is split when } \quad \mathrm{id} \circlearrowright S \underset{\longleftarrow\!\!\!\!\leftarrow}{\overset{s}{\rightarrowtail}} I$$

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Proposition: when $\mathbf{C}$ has finite biproducts,
then $s, s^{\perp} \in$ SISub($\mathbf{C}$) are complements
if and only if they are biproduct injections

Corollary: if $\oplus$ distributes over $\otimes$,
then SISub($\mathbf{C}$) is a Boolean algebra
(universal property?)

Linear logic

if $T\colon \mathbf{C} \to \mathbf{C}$ monoidal monad, $\mathrm{Kl}(T)$ is monoidal

semilattice morphism

$$\{\eta_I \circ s \mid s \in \mathrm{ISub}(\mathbf{C}),\ T(s) \text{ is monic in } \mathbf{C}\} \to \mathrm{ISub}(\mathrm{Kl}(T))$$

is not injective, nor surjective

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model for linear logic: $*$-autonomous category $\mathbf{C}$ with finite products, monoidal comonad $!\colon (\mathbf{C}, \otimes) \to (\mathbf{C}, \times)$
(then $\mathrm{Kl}(!)$ cartesian closed)
if ε epi, then $\mathrm{ISub}(\mathbf{C}, \times) \simeq \mathrm{ISub}(\mathrm{Kl}(!), \times)$
(but hard to compare to $\mathrm{ISub}(\mathbf{C}, \otimes)$)