Tensor topology

Pau Enrique Moliner Chris Heunen Sean Tull

THE UNIVERSITY of EDINBURGH
informatics

“Where things happen”

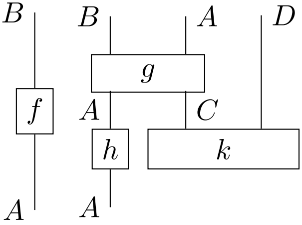
- Any monoidal category comes with built-in ‘space’
- Matches examples
- Universal notion of support
- Completion to actual space
- Embedding separates out spatial dimension
- Coproducts correspond to complements

See also
[Balmer, “Tensor triangular geometry”]
[Boyarchenko&Drinfeld, “Character sheaves of unipotent groups”]

Monoidal categories

- Objects $(A, B, C, \ldots)$ and morphisms $(f \colon A \to B, \ldots)$
- Two ways to compose: sequential $(\circ)$ and parallel $(\otimes)$
- Two ways to do nothing: $\mathrm{id}_A \colon A \to A$ and I

$$f \circ \mathrm{id} = \mathrm{id} = \mathrm{id} \circ f \qquad I \otimes A \simeq A \simeq A \otimes I$$



Morphisms $I \to I$ form commutative monoid of *scalars*

Many examples:

- Hilbert spaces
- Sets
- Lattices

Idempotent subunits

Categorify central idempotents in ring:

$$\mathrm{ISub}(\mathbf{C}) = \left\{ s \colon S \rightarrowtail I \;\middle|\; S \otimes s \colon S \otimes S \to S \otimes I \text{ iso} \right\}$$

Example: order theory

Frame: complete lattice, $\wedge$ distributes over $\bigvee$

e.g. open subsets of topological space

Example: order theory

Frame: complete lattice, $\wedge$ distributes over $\bigvee$
e.g. open subsets of topological space

Quantale: complete lattice, $\cdot$ distributes over $\bigvee$
e.g. $[0, \infty]$, $\mathrm{Pow}(M)$

Example: order theory

Frame: complete lattice, $\wedge$ distributes over $\bigvee$

e.g. open subsets of topological space

Quantale: complete lattice, $\cdot$ distributes over $\bigvee$

e.g. $[0, \infty]$, $\mathrm{Pow}(M)$

$$\mathbf{Frame} \underset{\mathrm{ISub}}{\overset{\hookrightarrow}{\underset{\longleftarrow}{\perp}}} \mathbf{Quantale}$$

$$\{x \in Q \mid x^2 = x \leq 1\} \longleftarrow\!\!\mid Q$$

Example: logic

$$\begin{aligned}\mathrm{ISub}(\mathrm{Sh}(X)) &= \{S \rightarrowtail 1\} \\ &= \{S \subseteq X \mid S \text{ open}\} \in \mathbf{Frame}\end{aligned}$$

Example: algebra

$$\mathrm{ISub}(\mathbf{Mod}_R) =$$

$$\big\{S \subseteq R \text{ ideal} \;\big|\; S = S^2 = \{x_1y_1 + \cdots + x_ny_n \mid x_i, y_i \in S\}\big\}$$

for nonunital bialgebra R in monoidal category

Example: analysis

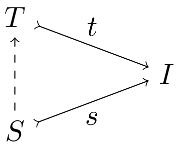
Hilbert module is $C_0(X)$-module with $C_0(X)$-valued inner product

$$C_0(X) = \{f \colon X \to \mathbb{C} \mid \forall \varepsilon > 0\ \exists K \subseteq X \colon |f(X \setminus K)| < \varepsilon\}$$

$$\mathrm{ISub}(\mathbf{Hilb}_{C_0(X)}) = \{S \subseteq X \text{ open}\}$$

Semilattice

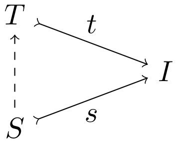
Proposition: ISub(**C**) is a semilattice, $\wedge = \otimes$, $1 = I$



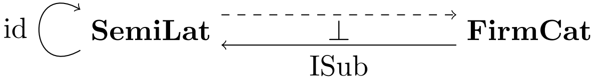
Caveat: **C** must be firm, i.e. $s \otimes T$ monic, and size issue

Semilattice

Proposition: $\mathrm{ISub}(\mathbf{C})$ is a semilattice, $\wedge = \otimes$, $1 = I$

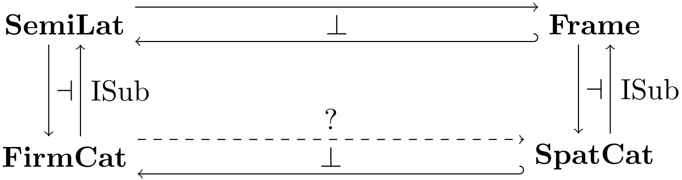


Caveat: $\mathbf{C}$ must be firm, i.e. $s \otimes T$ monic, and size issue



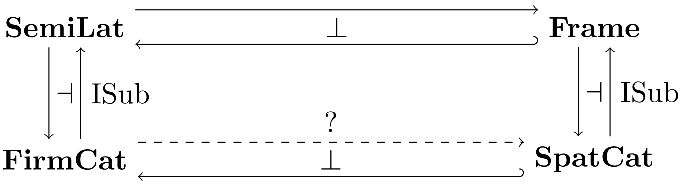
Spatial categories

Call **C** spatial when ISub(**C**) is frame



Spatial categories

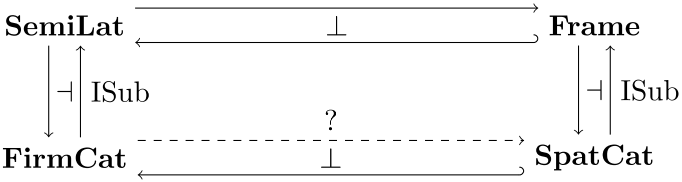
Call **C** spatial when ISub(**C**) is frame



Idea: $\widehat{\mathbf{C}} = [\mathbf{C}^{\mathrm{op}}, \mathbf{Set}]$ is cocomplete

Spatial categories

Call $\mathbf{C}$ spatial when $\mathrm{ISub}(\mathbf{C})$ is frame



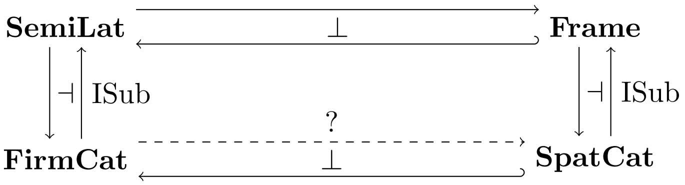
Idea: $\widehat{\mathbf{C}} = [\mathbf{C}^{\mathrm{op}}, \mathbf{Set}]$ is cocomplete

$$F \widehat{\otimes} G(A) = \int^{B,C} \mathbf{C}(A, B \otimes C) \times F(B) \times G(C)$$

Lemma: $\mathrm{ISub}(\widehat{\mathbf{C}}, \widehat{\otimes})$ is frame

Spatial categories

Call **C** spatial when ISub(**C**) is frame



Idea: $\widehat{\mathbf{C}} = [\mathbf{C}^{\mathrm{op}}, \mathbf{Set}]$ is cocomplete

$$F \mathbin{\widehat{\otimes}} G(A) = \int^{B,C} \mathbf{C}(A, B \otimes C) \times F(B) \times G(C)$$

Lemma: $\mathrm{ISub}(\widehat{\mathbf{C}}, \widehat{\otimes})$ is frame, but $\mathrm{ISub}(\widehat{\mathbf{C}}) \neq \widehat{\mathrm{ISub}(\mathbf{C})}$

Support

Say $s \in \mathrm{ISub}(\mathbf{C})$ supports $f\colon A \to B$ when

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & B \\ \downarrow & & \uparrow \simeq \\ B \otimes S & \xrightarrow{\quad B \otimes s \quad} & B \otimes I \end{array}$$

Support

Say $s \in \mathrm{ISub}(\mathbf{C})$ supports $f\colon A \to B$ when

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & B \\ \downarrow & & \uparrow \simeq \\ B \otimes S & \xrightarrow[B \otimes s]{} & B \otimes I \end{array}$$

$$\begin{array}{ccc} f & \longmapsto & \{s \mid s \text{ supports } f\} \\ \mathbf{C^2} & \xrightarrow{\quad \mathrm{supp} \quad} & \mathrm{Pow}(\mathrm{ISub}(\mathbf{C})) \end{array}$$

Support

Say $s \in \mathrm{ISub}(\mathbf{C})$ supports $f\colon A \to B$ when

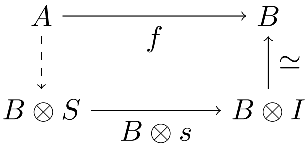
$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & B \\ \Big\downarrow & & \Big\uparrow \simeq \\ B \otimes S & \xrightarrow[\quad B \otimes s \quad]{} & B \otimes I \end{array}$$

Monoidal functor: $\mathrm{supp}(f) \wedge \mathrm{supp}(g) \leq \mathrm{supp}(f \otimes g)$

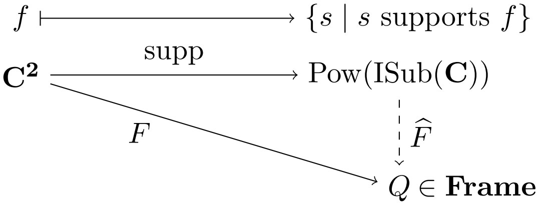
$$\begin{array}{ccc} f & \longmapsto & \{s \mid s \text{ supports } f\} \\ \mathbf{C^2} & \xrightarrow{\qquad \mathrm{supp} \qquad} & \mathrm{Pow}(\mathrm{ISub}(\mathbf{C})) \end{array}$$

Support

Say $s \in \mathrm{ISub}(\mathbf{C})$ supports $f\colon A \to B$ when



Monoidal functor: $\mathrm{supp}(f) \wedge \mathrm{supp}(g) \leq \mathrm{supp}(f \otimes g)$



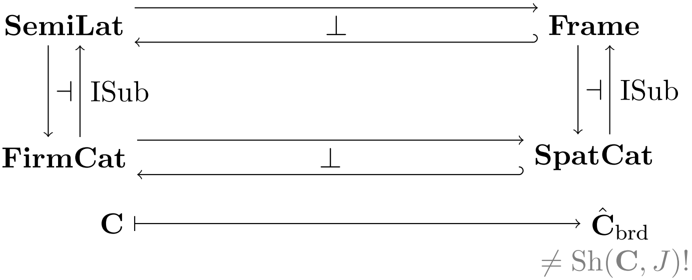
universal with $F(f) = \bigvee\{F(s) \mid s \in \mathrm{ISub}(\mathbf{C}) \text{ supports } f\}$

Spatial completion

Call $F\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ broad when

$$F(A) \simeq \left\{ (f, s)\colon A \to B \;\middle|\; s \in \mathrm{supp}(f) \cap U \right\}$$

for some $B \in \mathbf{C}$ and $U \subseteq \mathrm{ISub}(\mathbf{C})$.



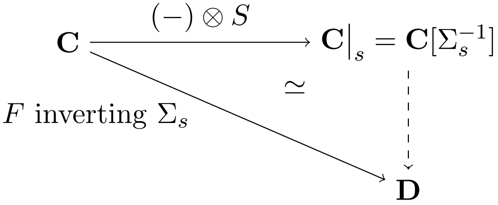
Simple categories

Say $\mathbf{C}$ is simple when $\mathrm{ISub}(\mathbf{C}) = \{\mathrm{id}_I\}$

Simple categories

Say $\mathbf{C}$ is simple when $\mathrm{ISub}(\mathbf{C}) = \{\mathrm{id}_I\}$

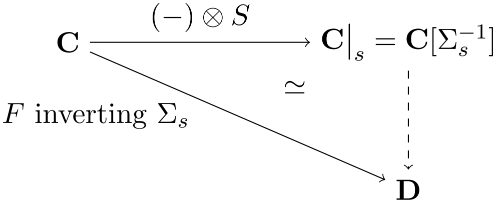
universal property of localisation for $\Sigma_s = \{A \otimes s \mid A \in \mathbf{C}\}$



Simple categories

Say $\mathbf{C}$ is simple when $\text{ISub}(\mathbf{C}) = \{\text{id}_I\}$

universal property of localisation for $\Sigma_s = \{A \otimes s \mid A \in \mathbf{C}\}$



Lemma: $\Sigma = \{A \otimes s \mid A \in \mathbf{C}, s \in \text{ISub}(\mathbf{C}\}$ calculus of right fractions gives functor $\mathbf{C} \to \text{Loc}(\mathbf{C}) = \mathbf{C}[\Sigma^{-1}]$ into simple category

Slim categories

Say **C** is slim when any object is (domain of) idempotent subunit
(Note: S determines s)

Definition: support structure is functor $\zeta\colon \mathbf{C} \to \mathbf{C}$ with morphisms

- $\beta_A\colon \zeta(A) \rightarrowtail I$;
- $\gamma_A\colon A \to \zeta(A) \otimes A$;
- $\delta_A\colon \zeta(\zeta(A)) \to \zeta(A)$;

satisfying five coherence conditions

Example: supported quantales

Proposition: δ_A is iso, β_A is idempotent, $\zeta\colon \mathbf{C} \to \mathrm{ISub}(\mathbf{C})$

Slim categories

Say **C** is slim when any object is (domain of) idempotent subunit
(Note: S determines s)

Definition: support structure is functor $\zeta\colon \mathbf{C} \to \mathbf{C}$ with morphisms

- $\beta_A\colon \zeta(A) \rightarrowtail I$;
- $\gamma_A\colon A \to \zeta(A) \otimes A$;
- $\delta_A\colon \zeta(\zeta(A)) \to \zeta(A)$;

satisfying five coherence conditions

Example: supported quantales

Proposition: δ_A is iso, β_A is idempotent, $\zeta\colon \mathbf{C} \to \mathrm{ISub}(\mathbf{C})$

Theorem: Any supported monoidal category embeds into product of simple and slim one: $\mathbf{C} \rightarrowtail \mathrm{Loc}(\mathbf{C}) \times \mathrm{ISub}(\mathbf{C})$

Complements

Subunit is split when $\mathrm{id} \circlearrowright S \underset{\twoheadleftarrow}{\overset{s}{\rightarrowtail}} I$

SISub($\mathbf{C}$) is a sub-semilattice of ISub($\mathbf{C}$)

(don't need firmness)

Complements

Subunit is split when $\mathrm{id} \circlearrowright S \overset{s}{\underset{\twoheadleftarrow}{\rightarrowtail}} I$

SISub($\mathbf{C}$) is a sub-semilattice of ISub($\mathbf{C}$)

(don't need firmness)

If $\mathbf{C}$ has zero object, ISub($\mathbf{C}$) has least element 0

$s, s^{\perp}$ are complements if $s \wedge s^{\perp} = 0$ and $s \vee s^{\perp} = 1$

Complements

Subunit is split when $\mathrm{id} \circlearrowright S \underset{\twoheadleftarrow\text{- - -}}{\overset{s}{\rightarrowtail}} I$

SISub(**C**) is a sub-semilattice of ISub(**C**)

(don't need firmness)

If **C** has zero object, ISub(**C**) has least element 0

$s, s^{\perp}$ are complements if $s \wedge s^{\perp} = 0$ and $s \vee s^{\perp} = 1$

Proposition: when **C** has finite biproducts,
then $s, s^{\perp} \in \mathrm{SISub}(\mathbf{C})$ are complements
if and only if they are biproduct injections

Corollary: if $\oplus$ distributes over $\otimes$,
then SISub(**C**) is a Boolean algebra
(universal property?)

Conclusion

- Any monoidal category comes with built-in ‘space’
- Matches examples
- Universal notion of support
- Completion to actual space
- Embedding separates out spatial dimension
- Coproducts correspond to complements

Further goals:

- Canonical status for support structure
- Dauns-Hofmann-like theorem
- Graphical calculus
- Applications: causality, concurrency

Restriction

The full subcategory $\mathbf{C}\big|_s$ of A with $A \otimes s$ invertible is:

- monoidal with tensor unit S
- coreflective: $\mathbf{C}\big|_s \underset{\perp}{\overset{\hookrightarrow}{\dashleftarrow}} \mathbf{C}$
- tensor ideal: if $A \in \mathbf{C}$ and $B \in \mathbf{C}\big|_s$, then $A \otimes B \in \mathbf{C}\big|_s$
- monocoreflective: counit ε_I monic (and $A \otimes \varepsilon_I$ iso for $A \in \mathbf{C}\big|_s$)

Restriction

The full subcategory $\mathbf{C}\big|_s$ of A with $A \otimes s$ invertible is:

- monoidal with tensor unit S
- coreflective: $\mathbf{C}\big|_s \underset{\dashleftarrow}{\overset{\hookrightarrow}{\perp}} \mathbf{C}$
- tensor ideal: if $A \in \mathbf{C}$ and $B \in \mathbf{C}\big|_s$, then $A \otimes B \in \mathbf{C}\big|_s$
- monocoreflective: counit ε_I monic (and $A \otimes \varepsilon_I$ iso for $A \in \mathbf{C}\big|_s$)

Proposition: $\mathrm{ISub}(\mathbf{C}) \simeq \{$monocoreflective tensor ideals in $\mathbf{C}\}$

Restriction

The full subcategory $\mathbf{C}\big|_s$ of A with $A \otimes s$ invertible is:

- monoidal with tensor unit S
- coreflective: $\mathbf{C}\big|_s \underset{\longleftarrow}{\overset{\hookrightarrow}{\perp}} \mathbf{C}$
- tensor ideal: if $A \in \mathbf{C}$ and $B \in \mathbf{C}\big|_s$, then $A \otimes B \in \mathbf{C}\big|_s$
- monocoreflective: counit ε_I monic (and $A \otimes \varepsilon_I$ iso for $A \in \mathbf{C}\big|_s$)

Proposition: $\mathrm{ISub}(\mathbf{C}) \simeq \{\text{monocoreflective tensor ideals in } \mathbf{C}\}$

Examples: $(\mathbf{Mod}_R)\big|_I = \mathbf{Mod}_I$, $\mathrm{Sh}(X)\big|_U = \mathrm{Sh}(U)$

Localisation

A graded monad is a monoidal functor $\mathbf{E} \to [\mathbf{C}, \mathbf{C}]$

$(\eta\colon A \to T(1),\ \mu\colon T(t) \circ T(s) \to T(s \otimes t))$

Lemma: $s \mapsto \mathbf{C}\big|_s$ is an $\mathrm{ISub}(\mathbf{C})$-graded monad

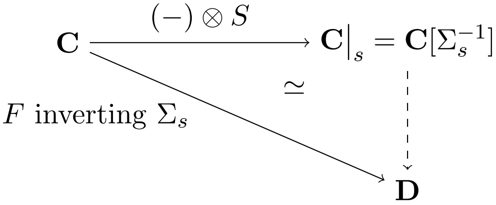
Localisation

A graded monad is a monoidal functor $\mathbf{E} \to [\mathbf{C}, \mathbf{C}]$

$(\eta \colon A \to T(1),\ \mu \colon T(t) \circ T(s) \to T(s \otimes t))$

Lemma: $s \mapsto \mathbf{C}\big|_s$ is an $\mathrm{ISub}(\mathbf{C})$-graded monad

universal property of localisation for $\Sigma_s = \{A \otimes s \mid A \in \mathbf{C}\}$



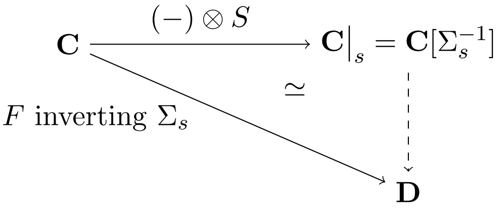
Localisation

A graded monad is a monoidal functor $\mathbf{E} \to [\mathbf{C}, \mathbf{C}]$

$(\eta \colon A \to T(1),\ \mu \colon T(t) \circ T(s) \to T(s \otimes t))$

Lemma: $s \mapsto \mathbf{C}\big|_s$ is an $\mathrm{ISub}(\mathbf{C})$-graded monad

universal property of localisation for $\Sigma_s = \{A \otimes s \mid A \in \mathbf{C}\}$



Lemma: $\Sigma = \{A \otimes s \mid A \in \mathbf{C}, s \in \mathrm{ISub}(\mathbf{C}\}$ calculus of right fractions gives functor $\mathbf{C} \to \mathrm{Loc}(\mathbf{C}) = \mathbf{C}[\Sigma^{-1}]$ into simple category

Coherence

$$
\begin{array}{ccc}
\zeta^2 A & \xrightarrow{\delta} & \zeta A \\
{\scriptstyle \zeta\beta}\downarrow & \searrow^{\beta} & \downarrow{\scriptstyle \beta} \\
\zeta I & \xrightarrow[\beta]{} & I
\end{array}
$$

$$
\begin{array}{ccccc}
 & & \zeta A \otimes A & & \\
 & {}^{\gamma}\nearrow & & \searrow^{\beta \otimes A} & \\
A & & \longrightarrow & & I \otimes A
\end{array}
$$

$$
\begin{array}{ccccc}
 & & I & & \\
 & {}^{\beta}\nearrow & & \searrow^{\gamma} & \\
\zeta I & & \longrightarrow & & \zeta I \otimes I
\end{array}
$$

$$
\begin{array}{ccccc}
 & & I \otimes \zeta^2 A \otimes \zeta A & & \\
 & {}^{\beta \otimes \gamma}\nearrow & & \searrow^{I \otimes \delta \otimes \zeta A} & \\
\zeta A \otimes \zeta A & & \longrightarrow & & I \otimes \zeta \otimes \zeta A
\end{array}
$$

$$
\begin{array}{ccc}
\zeta A & \xrightarrow{\gamma} & \zeta^2 A \otimes \zeta A \\
{\scriptstyle \beta}\downarrow & & \downarrow{\scriptstyle \beta \otimes \beta} \\
I & \longrightarrow & I \otimes I
\end{array}
$$