

# Tensor topology

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# “Where things happen”

Wouldn't it be great if

- ▶ control flow
- ▶ data flow
- ▶ provenance
- ▶ proof analysis
- ▶ causality

were all instances of a one theory?

# Idempotent subunits

Categorify idempotents in ring

$$\mathbf{ISub}(\mathbf{C}) = \{ s: S \rightarrow I \mid \text{id}_S \otimes s: S \otimes S \rightarrow S \otimes I \text{ iso} \}$$

## Example: order theory

**Frame:** complete lattice,  $\wedge$  distributes over  $\vee$   
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$$\begin{array}{ccc} \mathbf{Frame} & \begin{array}{c} \xrightarrow{\quad} \\ \perp \\ \xleftarrow{\quad} \\ \text{ISub} \end{array} & \mathbf{Quantale} \\ \{x \in Q \mid x^2 = x \leq 1\} & \xleftarrow{\quad} & Q \end{array}$$

*'idempotent subunits are side-effect-free observations'*

## Example: logic

$$\begin{aligned} \text{ISub}(\text{Sh}(X)) &= \{S \multimap 1\} \\ &= \{S \subseteq X \mid S \text{ open}\} \in \mathbf{Frame} \end{aligned}$$

*‘idempotent subunits are truth values’*

## Example: algebra

$$\text{ISub}(\mathbf{Mod}_R) = \{S \subseteq R \text{ ideal} \mid S = S^2 = \{x_1y_1 + \cdots + x_ny_n \mid x_i, y_i \in S\}\}$$

*'idempotent subunits are idempotent ideals'*



## Example: analysis

**Hilbert module** is  $C_0(X)$ -module with  $C_0(X)$ -valued inner product

$$C_0(X) = \{f: X \rightarrow \mathbb{C} \mid \forall \varepsilon > 0 \exists K \subseteq X: |f(X \setminus K)| < \varepsilon\}$$

$$\text{ISub}(\mathbf{Hilb}_{C_0(X)}) = \{S \subseteq X \text{ open}\}$$

*‘idempotent subunits are open subsets of base space’*

## Example: geometry

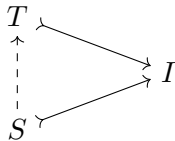
Hilbert bundle is bundle  $E \rightarrow X$  with Hilbert spaces for fibres

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## Semilattice

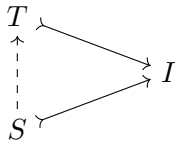
**Proposition:**  $\text{ISub}(\mathbf{C})$  is a semilattice,  $\wedge = \otimes$ ,  $1 = \text{id}_I$



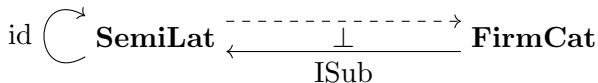
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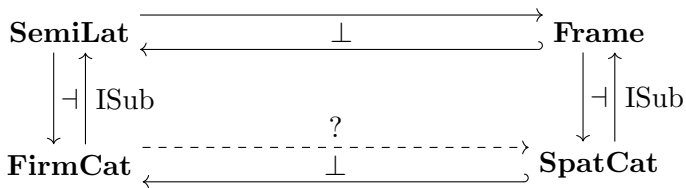


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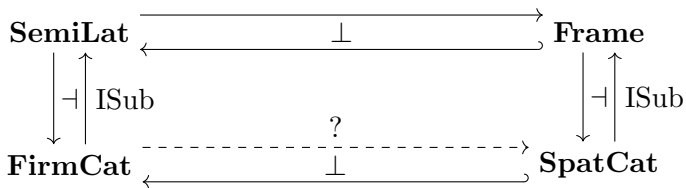
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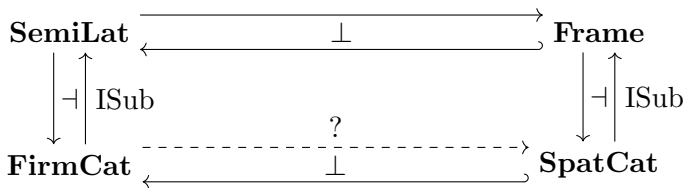
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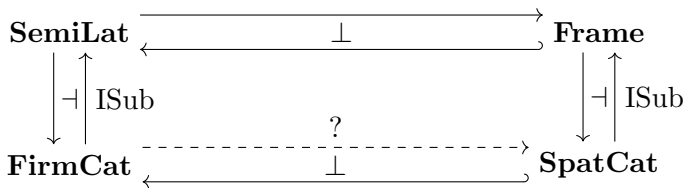
Idea:  $\widehat{\mathbf{C}} = [\mathbf{C}^{\text{op}}, \mathbf{Set}]$  is cocomplete

$$F \widehat{\otimes} G(A) = \int^{B, C} \mathbf{C}(A, B \otimes C) \times F(B) \times G(C)$$

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**Lemma:**  $\text{ISub}(\widehat{\mathbf{C}}, \widehat{\otimes})$  is frame, but  $\text{ISub}(\widehat{\mathbf{C}}) \neq \widehat{\text{ISub}(\mathbf{C})}$



# Support

Say  $s \in \text{ISub}(\mathbf{C})$  **supports**  $f: A \rightarrow B$  when

$$\begin{array}{ccc} A & \xrightarrow{\quad} & B \\ \vdots & & \uparrow \simeq \\ B \otimes S & \xrightarrow{\quad \text{id} \otimes s \quad} & B \otimes I \end{array}$$

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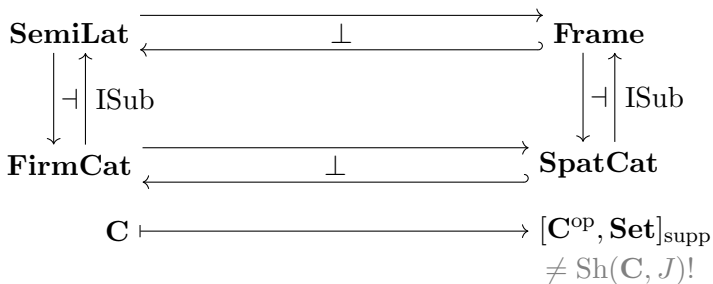
universal with  $F(f) = \bigvee \{F(s) \mid s \in \text{ISub}(\mathbf{C}) \text{ supports } f\}$

# Spatial categories

Call  $F: \mathbf{C}^{\text{op}} \rightarrow \mathbf{Set}$  **supported** when

$$F(A) \simeq \{f: A \rightarrow B \mid \text{supp}(f) \cap U \neq \emptyset\}$$

for some  $B \in \mathbf{C}$  and  $U \subseteq \text{ISub}(\mathbf{C})$ .



## Complements

Subunit is **split** when  $\text{id} \circlearrowleft S \overset{s}{\dashrightarrow} I$   
 $\text{SISub}(\mathbf{C})$  is a sub-semilattice of  $\text{ISub}(\mathbf{C})$   
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**Proposition:** when  $\mathbf{C}$  has finite biproducts,  
then  $s, s^\perp \in \text{SISub}(\mathbf{C})$  are complements  
if and only if they are biproduct injections

**Corollary:** if  $\oplus$  distributes over  $\otimes$ ,  
then SISub( $\mathbf{C}$ ) is a **Boolean** algebra  
(universal property?)



## Linear logic

if  $T: \mathbf{C} \rightarrow \mathbf{C}$  monoidal monad,  $\text{Kl}(T)$  is monoidal  
semilattice morphism

$\{\eta_I \circ s \mid s \in \text{ISub}(\mathbf{C}), T(s) \text{ is monic in } \mathbf{C}\} \rightarrow \text{ISub}(\text{Kl}(T))$   
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model for linear logic: \*-autonomous category  $\mathbf{C}$  with finite  
products, monoidal comonad  $!: (\mathbf{C}, \otimes) \rightarrow (\mathbf{C}, \times)$   
(then  $\text{Kl}(!)$  cartesian closed)

if  $\varepsilon$  epi, then  $\text{ISub}(\mathbf{C}, \times) \simeq \text{ISub}(\text{Kl}(!), \times)$   
(but hard to compare to  $\text{ISub}(\mathbf{C}, \otimes)$ )

## Further

Do you work with maps into a tensor unit?

- ▶ causality
- ▶ proof analysis
- ▶ control flow
- ▶ data flow
- ▶ concurrency
- ▶ graphical calculus

# Restriction

The full subcategory  $\mathbf{C}|_s$  of  $\mathbf{C}$  with  $\text{id}_A \otimes s$  invertible is:

- ▶ monoidal with tensor unit  $S$
- ▶ **coreflective**:  $\mathbf{C}|_s \begin{array}{c} \xrightarrow{\quad} \\ \dashleftarrow{\perp} \\ \xrightarrow{\quad} \end{array} \mathbf{C}$
- ▶ **tensor ideal**: if  $A \in \mathbf{C}$  and  $B \in \mathbf{C}|_s$ , then  $A \otimes B \in \mathbf{C}|_s$
- ▶ **monocoreflective**: counit  $\varepsilon_I$  monic (and  $\text{id}_A \otimes \varepsilon_I$  iso for  $A \in \mathbf{C}|_s$ )

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- ▶ monoreflective: counit  $\varepsilon_I$  monic (and  $\text{id}_A \otimes \varepsilon_I$  iso for  $A \in \mathbf{C}|_s$ )

**Proposition:**  $\text{ISub}(\mathbf{C}) \simeq \{\text{monoreflective tensor ideals in } \mathbf{C}\}$

## Localisation

A **graded monad** is a monoidal functor  $\mathbf{E} \rightarrow [\mathbf{C}, \mathbf{C}]$   
( $\eta: A \rightarrow T(1)$ ,  $\mu: T(t) \circ T(s) \rightarrow T(s \otimes t)$ )

**Lemma:**  $s \mapsto \mathbf{C}|_s$  is an  $\text{ISub}(\mathbf{C})$ -graded monad

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universal property of **localisation** for  $\Sigma = \{\text{id}_E \otimes s \mid E \in \mathbf{C}\}$

$$\begin{array}{ccc} \mathbf{C} & \xrightarrow{(-) \otimes S} & \mathbf{C}|_s = \mathbf{C}[\Sigma^{-1}] \\ & \searrow F \text{ inverting } \Sigma & \downarrow \text{dashed} \\ & & \mathbf{D} \end{array}$$

$\simeq$