Tensor topology

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“Where things happen”

Wouldn’t it be great if

- control flow
- data flow
- provenance
- proof analysis
- causality

were all instances of a one theory?

Idempotent subunits

Categorify idempotents in ring

$$\mathrm{ISub}(\mathbf{C}) = \left\{ s\colon S \rightarrowtail I \mid \mathrm{id}_S \otimes s\colon S \otimes S \to S \otimes I \text{ iso } \right\}$$

Example: order theory

Frame: complete lattice, $\wedge$ distributes over $\bigvee$

e.g. open subsets of topological space

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Quantale: complete lattice, $\cdot$ distributes over $\bigvee$
e.g. $[0, \infty]$, $\mathrm{Pow}(M)$

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e.g. open subsets of topological space

Quantale: complete lattice, $\cdot$ distributes over $\bigvee$

e.g. $[0,\infty]$, $\mathrm{Pow}(M)$

$$\begin{tikzcd}
\mathbf{Frame} \arrow[rr, hook, shift left=2] \arrow[rr, phantom, "\perp"] & & \mathbf{Quantale} \arrow[ll, shift left=2, "\mathrm{ISub}"] \\
\{x \in Q \mid x^2 = x \leq 1\} & & Q \arrow[ll, mapsto]
\end{tikzcd}$$

'idempotent subunits are side-effect-free observations'

Example: logic

$$\begin{aligned}\mathrm{ISub}(\mathrm{Sh}(X)) &= \{S \rightarrowtail 1\} \\ &= \{S \subseteq X \mid S \text{ open}\} \in \textbf{Frame}\end{aligned}$$

'idempotent subunits are truth values'

Example: algebra

$$\mathrm{ISub}(\mathbf{Mod}_R) =$$

$$\left\{ S \subseteq R \text{ ideal} \;\middle|\; S = S^2 = \{x_1 y_1 + \cdots x_n y_n \mid x_i, y_i \in S\} \right\}$$

'idempotent subunits are idempotent ideals'

Example: analysis

Hilbert module is $C_0(X)$-module with $C_0(X)$-valued inner product
$C_0(X) = \{f\colon X \to \mathbb{C} \mid \forall \varepsilon > 0\ \exists K \subseteq X\colon |f(X \setminus K)| < \varepsilon\}$

$$\mathrm{ISub}(\mathbf{Hilb}_{C_0(X)}) = \{S \subseteq X \text{ open}\}$$

'idempotent subunits are open subsets of base space'

Example: geometry

Hilbert bundle is bundle $E \twoheadrightarrow X$ with Hilbert spaces for fibres

$$\mathrm{ISub}(\mathbf{Hilb}_X) = \{S \subseteq X \text{ open}\}$$

'idempotent subunits are open subsets of base space'

Semilattice

Proposition: $\mathrm{ISub}(\mathbf{C})$ is a semilattice, $\wedge = \otimes$, $1 = \mathrm{id}_I$

$$
\begin{tikzcd}
T \arrow[dr, tail] & \\
& I \\
S \arrow[ur, tail] \arrow[uu, dashed] &
\end{tikzcd}
$$

Caveat: $\mathbf{C}$ must be firm, i.e. $s \otimes \mathrm{id}_T$ monic, and size issue

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$$
\begin{tikzcd}
\mathbf{SemiLat} \arrow[loop left, "\mathrm{id}"] \arrow[r, dashed, shift left] \arrow[r, phantom, "\perp"] & \mathbf{FirmCat} \arrow[l, shift left, "\mathrm{ISub}"]
\end{tikzcd}
$$

Spatial categories

Call $\mathbf{C}$ spatial when $\mathrm{ISub}(\mathbf{C})$ is frame

$$\begin{tikzcd}
\mathbf{SemiLat} \arrow[rr, shift left=2] \arrow[rr, phantom, "\perp"] \arrow[d, shift right=2] \arrow[d, phantom, "\dashv"] & & \mathbf{Frame} \arrow[ll, hook', shift left=2] \arrow[d, shift right=2] \arrow[d, phantom, "\dashv"] \\
\mathbf{FirmCat} \arrow[rr, dashed, shift left=2, "?"] \arrow[rr, phantom, "\perp"] \arrow[u, shift right=2, "\mathrm{ISub}"'] & & \mathbf{SpatCat} \arrow[ll, hook', shift left=2] \arrow[u, shift right=2, "\mathrm{ISub}"']
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\begin{tikzcd}[column sep=huge, row sep=large]
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Idea: $\widehat{\mathbf{C}} = [\mathbf{C}^{\mathrm{op}}, \mathbf{Set}]$ is cocomplete

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\begin{tikzcd}[column sep=huge, row sep=large]
\mathbf{SemiLat} \arrow[r, shift left=1.5ex] \arrow[r, phantom, "\perp"] \arrow[d, shift right=1ex] \arrow[d, phantom, "\dashv"] & \mathbf{Frame} \arrow[l, hook', shift left=1.5ex] \arrow[d, shift right=1ex] \arrow[d, phantom, "\dashv"] \\
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Idea: $\widehat{\mathbf{C}} = [\mathbf{C}^{\mathrm{op}}, \mathbf{Set}]$ is cocomplete

$$F \widehat{\otimes} G(A) = \int^{B,C} \mathbf{C}(A, B \otimes C) \times F(B) \times G(C)$$

Lemma: $\mathrm{ISub}(\widehat{\mathbf{C}}, \widehat{\otimes})$ is frame

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$$F \widehat{\otimes} G(A) = \int^{B,C} \mathbf{C}(A, B \otimes C) \times F(B) \times G(C)$$

Lemma: $\mathrm{ISub}(\widehat{\mathbf{C}}, \widehat{\otimes})$ is frame, but $\mathrm{ISub}(\widehat{\mathbf{C}}) \neq \widehat{\mathrm{ISub}(\mathbf{C})}$

Support

Say $s \in \mathrm{ISub}(\mathbf{C})$ supports $f\colon A \to B$ when

$$\begin{tikzcd} A \arrow[r, "f"'] \arrow[d, dashed] & B \\ B \otimes S \arrow[r, "\mathrm{id} \otimes s"'] & B \otimes I \arrow[u, "\simeq"'] \end{tikzcd}$$

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$$\begin{array}{ccc} f & \longmapsto & \{s \mid s \text{ supports } f\} \\ \mathbf{C^2} & \xrightarrow{\quad\mathrm{supp}\quad} & \mathrm{Pow}(\mathrm{ISub}(\mathbf{C})) \end{array}$$

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Monoidal functor: $\mathrm{supp}(f) \wedge \mathrm{supp}(g) \leq \mathrm{supp}(f \otimes g)$

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$$\begin{tikzcd} f \arrow[r, mapsto] & \{s \mid s \text{ supports } f\} \\ \mathbf{C^2} \arrow[r, "\mathrm{supp}"] \arrow[dr, "F"'] & \mathrm{Pow}(\mathrm{ISub}(\mathbf{C})) \arrow[d, dashed, "\widehat{F}"] \\ & Q \in \mathbf{Frame} \end{tikzcd}$$

universal with $F(f) = \bigvee\{F(s) \mid s \in \mathrm{ISub}(\mathbf{C}) \text{ supports } f\}$

Spatial categories

Call $F\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ supported when

$$F(A) \simeq \{f\colon A \to B \mid \mathrm{supp}(f) \cap U \neq \emptyset\}$$

for some $B \in \mathbf{C}$ and $U \subseteq \mathrm{ISub}(\mathbf{C})$.

$$\begin{tikzcd}
\mathbf{SemiLat} \arrow[rr, shift left=2] \arrow[rr, phantom, "\perp"] \arrow[d, shift right=2] \arrow[d, phantom, "\dashv"] & & \mathbf{Frame} \arrow[ll, hook', shift left=2] \arrow[d, shift right=2] \arrow[d, phantom, "\dashv"] \\
\mathbf{FirmCat} \arrow[u, shift right=2, swap, "\mathrm{ISub}"] \arrow[rr, shift left=2] \arrow[rr, phantom, "\perp"] & & \mathbf{SpatCat} \arrow[u, shift right=2, swap, "\mathrm{ISub}"] \arrow[ll, hook', shift left=2] \\
\mathbf{C} \arrow[rr, mapsto] & & {[\mathbf{C}^{\mathrm{op}}, \mathbf{Set}]_{\mathrm{supp}}} \\
& & \neq \mathrm{Sh}(\mathbf{C}, J)!
\end{tikzcd}$$

Complements

$$\text{Subunit is split when } \text{id} \circlearrowright S \underset{\twoheadleftarrow\text{-----}}{\overset{s}{\rightarrowtail}} I$$

$\mathrm{SISub}(\mathbf{C})$ is a sub-semilattice of $\mathrm{ISub}(\mathbf{C})$

(don't need firmness)

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SISub(**C**) is a sub-semilattice of ISub(**C**)

(don't need firmness)

If **C** has zero object, ISub(**C**) has least element 0

$s, s^{\perp}$ are complements if $s \wedge s^{\perp} = 0$ and $s \vee s^{\perp} = 1$

Complements

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SISub($\mathbf{C}$) is a sub-semilattice of ISub($\mathbf{C}$)

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Proposition: when $\mathbf{C}$ has finite biproducts,
then $s, s^\perp \in$ SISub($\mathbf{C}$) are complements
if and only if they are biproduct injections

Corollary: if $\oplus$ distributes over $\otimes$,
then SISub($\mathbf{C}$) is a Boolean algebra
(universal property?)

Linear logic

if $T\colon \mathbf{C} \to \mathbf{C}$ monoidal monad, $\mathrm{Kl}(T)$ is monoidal

semilattice morphism

$$\{\eta_I \circ s \mid s \in \mathrm{ISub}(\mathbf{C}),\ T(s) \text{ is monic in } \mathbf{C}\} \to \mathrm{ISub}(\mathrm{Kl}(T))$$

is not injective, nor surjective

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is not injective, nor surjective

model for linear logic: $*$-autonomous category $\mathbf{C}$ with finite products, monoidal comonad $!\colon (\mathbf{C}, \otimes) \to (\mathbf{C}, \times)$
(then $\mathrm{Kl}(!)$ cartesian closed)
if ε epi, then $\mathrm{ISub}(\mathbf{C}, \times) \simeq \mathrm{ISub}(\mathrm{Kl}(!), \times)$
(but hard to compare to $\mathrm{ISub}(\mathbf{C}, \otimes)$)

Further

Do you work with maps into a tensor unit?

- causality
- proof analysis
- control flow
- data flow
- concurrency
- graphical calculus

Restriction

The full subcategory $\mathbf{C}\big|_s$ of A with $\mathrm{id}_A \otimes s$ invertible is:

- monoidal with tensor unit S
- coreflective: $\begin{tikzcd} \mathbf{C}\big|_s \arrow[r, hook, shift left] \arrow[r, phantom, "\perp"] & \mathbf{C} \arrow[l, dashed, shift left] \end{tikzcd}$
- tensor ideal: if $A \in \mathbf{C}$ and $B \in \mathbf{C}\big|_s$, then $A \otimes B \in \mathbf{C}\big|_s$
- monocoreflective: counit ε_I monic (and $\mathrm{id}_A \otimes \varepsilon_I$ iso for $A \in \mathbf{C}\big|_s$)

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- monocoreflective: counit ε_I monic (and $\mathrm{id}_A \otimes \varepsilon_I$ iso for $A \in \mathbf{C}\big|_s$)

Proposition: $\mathrm{ISub}(\mathbf{C}) \simeq \{$monocoreflective tensor ideals in $\mathbf{C}\}$

Localisation

A graded monad is a monoidal functor $\mathbf{E} \to [\mathbf{C}, \mathbf{C}]$

$(\eta\colon A \to T(1),\ \mu\colon T(t) \circ T(s) \to T(s \otimes t))$

Lemma: $s \mapsto \mathbf{C}\big|_s$ is an $\mathrm{ISub}(\mathbf{C})$-graded monad

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universal property of localisation for $\Sigma = \{\mathrm{id}_E \otimes s \mid E \in \mathbf{C}\}$

$$\begin{tikzcd}
\mathbf{C} \arrow[rr, "(-) \otimes S"] \arrow[drr, "F \text{ inverting } \Sigma"'] & & \mathbf{C}\big|_s = \mathbf{C}[\Sigma^{-1}] \arrow[d, dashed, "\simeq"'] \\
& & \mathbf{D}
\end{tikzcd}$$