Compact inverse categories

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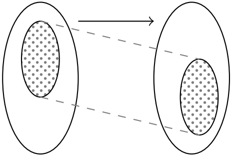
Inverse monoids

Every x has $x^\dagger$ with $x = xx^\dagger x$, and $x^\dagger x y^\dagger y = y^\dagger y x^\dagger x$

- any group
- any semilattice
- untyped reversible computation
- partial injections on fixed set



(Commutative) inverse monoids

Theorem (Ehresmann-Schein-Nambooripad):
$\{\text{inverse monoids}\} \simeq \{\text{inductive groupoids}\}$

(groupoid in category of posets,
étale for Alexandrov topology,
objects are semilattice)

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{inverse monoids} $\simeq$ {inductive groupoids}

(groupoid in category of posets,
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Theorem (Jarek):
{commutative inverse monoids} $\simeq$ {semilattices of abelian groups}

(functor from a semilattice to category of abelian groups)

Inverse categories

Every f has $f^\dagger$ with $f = ff^\dagger f$, and $f^\dagger f g^\dagger g = g^\dagger g f^\dagger f$

- fundamental groupoid of pointed topological space
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Theorem (DeWolf-Pronk):

{inverse categories} $\simeq$ {locally complete inductive groupoids}

(groupoid in category of posets,
étale for Alexandrov topology,
objects are coproduct of semilattices)

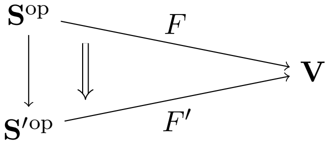
Structure theorems

objects	general case	commutative case
one	inductive groupoid	semilattice of abelian groups
many	locally inductive groupoid	semilattice of compact groupoids

Semilattices of categories

Semilattice is partial order with greatest lower bounds $s \wedge t$ and $\top$

Semilattice over a subcategory $\mathbf{V} \subseteq \mathbf{Cat}$ is functor $F\colon \mathbf{S}^{\mathrm{op}} \to \mathbf{V}$ where $\mathbf{S}$ is semilattice, all categories $F(s)$ have the same objects



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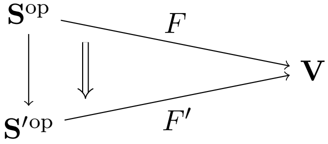

Theorem (Jarek):

$$\begin{array}{rcl} \mathbf{cInvMon} & \simeq & \mathbf{SLat}[\mathbf{Ab}] \\ M & \mapsto & S = \{s \in M \mid ss^\dagger = s\} \\ & & F(s) = \{x \in M \mid xx^\dagger = s\} \\ \coprod_s F(s) & \leftarrow & F \end{array}$$

The one-object case

$\{$commutative inverse monoids$\} \simeq \{$one-object compact inverse cats$\}$

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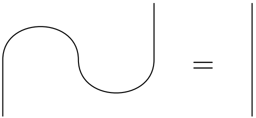
Symmetric monoidal, every object has dual
$\eta \colon I \to A^* \otimes A$ with $(\varepsilon \otimes 1) \circ (1 \otimes \eta) = 1$ for $\varepsilon = \sigma \circ \eta^\dagger$

- A and A^* adjoint in one-object 2-category
- any abelian group as discrete monoidal category
- fundamental groupoid of pointed topological space



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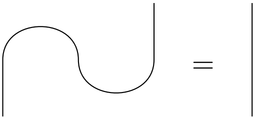


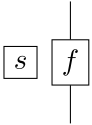
In any monoidal category:

- scalars $I \to I$ form commutative monoid
- I dual to itself

Compact categories

- scalar multiplication of $f\colon A \to B$ with $s\colon I \to I$

$$\begin{array}{ccc} A & \overset{s \bullet f}{\dashrightarrow} & B \\ \simeq \downarrow & & \uparrow \simeq \\ I \otimes A & \xrightarrow{\quad s \otimes f \quad} & I \otimes B \end{array}$$



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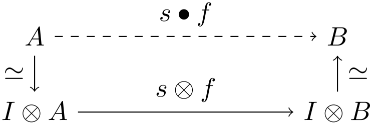
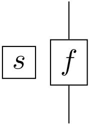
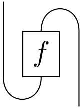




- dual morphism of $f\colon A \to B$

$$f^* = (1 \otimes \varepsilon) \circ (1 \otimes f \otimes 1) \circ (\eta \otimes 1)\colon B^* \to A^*$$



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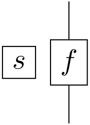

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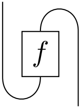
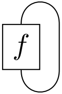


- trace of $f\colon A \to A$

$$\mathrm{Tr}(f) = \varepsilon \circ (f \otimes 1) \circ \eta\colon I \to I$$
$$\mathrm{tr}(f) = \mathrm{Tr}(f)^*$$



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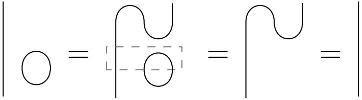
1. because $h = hh^\dagger h$: 

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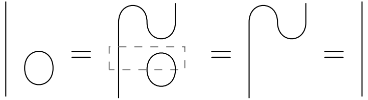
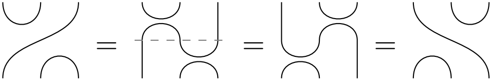
1. because $h = hh^\dagger h$: 

2. $gg^\dagger$ and $hh^\dagger$ commute: 

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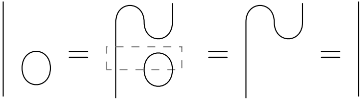

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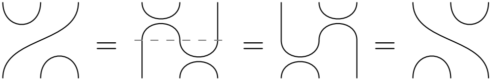

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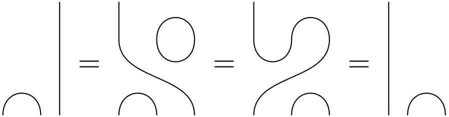

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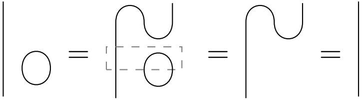

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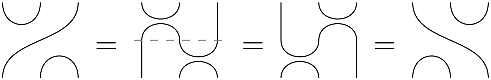

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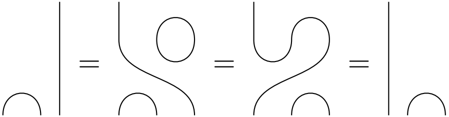
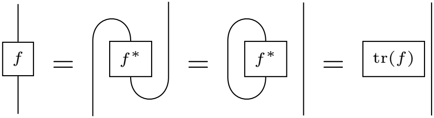


4. therefore:



Arbitrary morphisms

Corollary: compact dagger category is compact inverse category

$$\iff$$

every morphism f satisfies $f = \mathrm{tr}(ff^\dagger) \bullet f$

Proof: $\Longrightarrow$: $ff^\dagger = \mathrm{tr}(ff^\dagger ff^\dagger) \bullet 1 = \mathrm{tr}(ff^\dagger) \bullet 1$

$\Longleftarrow$: restriction category with $\overline{f} = \mathrm{tr}(ff^\dagger) \bullet 1$
every map is restriction isomorphism

Semilattices of groupoids

Theorem: If **C** is compact inverse category

- $S = \{s \colon I \to I \mid ss^\dagger = s\}$ is semilattice
- $s \in S$ induces compact groupoid $F(s)$ with same objects, and morphisms $F(s)(A, B) = \{f \colon A \to B \mid \mathrm{tr}(ff^\dagger) = s\}$
- semilattice $F \colon S^{\mathrm{op}} \to \mathbf{CptGpd}$ of compact groupoids

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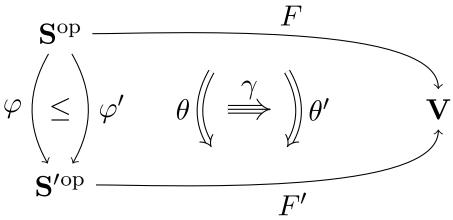
Equivalence $\mathbf{CptInvCat} \simeq \mathbf{SLat}[\mathbf{CptGpd}]$

2-categories

Redefinition of $\mathbf{SLat}[\mathbf{V}]$ as 2-category:



Write $\mathbf{SLat}_=[\mathbf{V}]$ for full subcategory where all $F(s)$ same objects

2-categories

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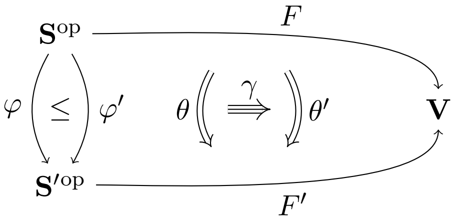

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Lemma: $\mathbf{SLat}[\mathbf{CptGpd}] \simeq \mathbf{SLat}_=[\mathbf{CptGpd}]$
(Compare inductive groupoids)

Compact groupoids

Proposition [Baez-Lauda]: compact groupoids $\mathbf{C}$ are, up to $\simeq$:

- abelian group G of isomorphism classes of $\mathbf{C}$ under $\otimes$, I, A^*
- abelian group H of scalars $\mathbf{C}(I, I)$ under $\circ$, 1, $f^\dagger$
- conjugation action $G \times H \to H$ given by $(A, s) \mapsto \mathrm{tr}(A \otimes s)$
- 3-cocycle $G \times G \times G \to H$ given by $(A, B, C) \mapsto \mathrm{Tr}(\alpha_{A,B,C})$

Proof: make $\mathbf{C}$ skeletal, strictify everything but associators

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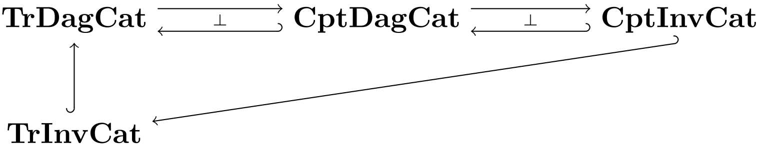
Theorem: $\mathbf{CptInvCat} \simeq \mathbf{SLat}[\mathbf{Cocycle}]$

Traced inverse categories

What do traced inverse categories look like?



Open ends

- **SLat**[**V**] as completion procedure?
- Bratelli diagrams?
- description internal to **Rel**?