Ontological models as functors

Andru Gheorghiu Chris Heunen

arXiv:1905.09055

(Finite-dimensional) Quantum theory

state	unit vector in complex Hilbert space	$\lvert\psi\rangle \in H, \lVert\lvert\psi\rangle\rVert^2 = 1$
transformation	unitary operator	$uu^\dagger = u^\dagger u = 1$
composition	tensor product	$H_{AB} = H_A \otimes H_B$
observation	orthonormal basis	$\{\lvert i\rangle\}, \langle i \mid j\rangle = \delta_{ij}$

Ontological interpretation

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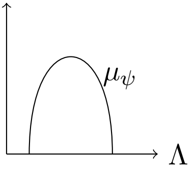
$\|\psi\rangle$	$\longmapsto$	

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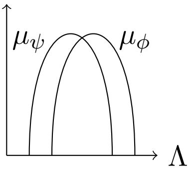
$$\int_\Lambda \xi_i(\lambda)\mathrm{d}\mu_\psi(\lambda)) = |\langle i \mid \psi\rangle|^2$$

$$\forall \lambda \in \Lambda \colon \; \sum_{i=1}^{\dim(H)} \xi_i = 1$$

Ontological interpretation

$$0 < |\langle\psi|\phi\rangle| < 1 \quad \longmapsto$$



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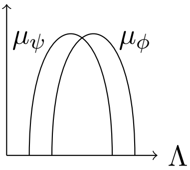

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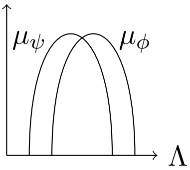

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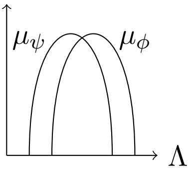

Epistemic model
(otherwise ontic model)

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[Leifer arXiv:1409.1570]

No-go results for epistemic models

- [Pusey-Barrett-Rudolph arXiv:1111.3328]
  Preparation independence:
  $\{|\psi\rangle \otimes |\phi\rangle\}_{\psi \in H_A, \phi \in H_B} \mapsto (\Lambda_A \times \Lambda_B, \Sigma_{\Lambda_A} \otimes \Sigma_{\Lambda_B})$
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- [Gheorghiu-Heunen arXiv:1905.09055]:
  one approach to rule them all

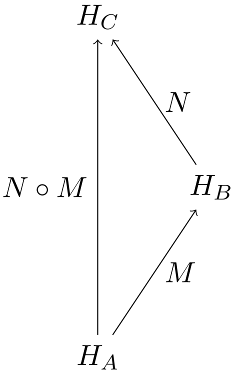
Category theory

Explicitly invented to translate structure between different areas:

- Algebraic topology: topology $\mapsto$ groups
- Algebraic geometry: varieties $\mapsto$ schemes
- Logic: theories $\mapsto$ models
- Computer compilers: high-level language $\mapsto$ assembly
- Complexity theory: algorithm $\mapsto$ function
- Semantics: computer programs $\mapsto$ mathematical model
- Physics: physical systems $\mapsto$ mathematical abstractions

Here: quantum physics $\mapsto$ statistical physics

Categorical approach



FHilb

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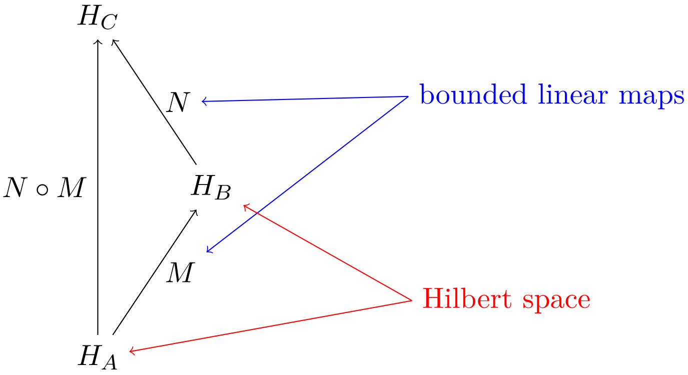

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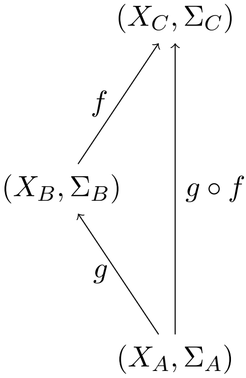
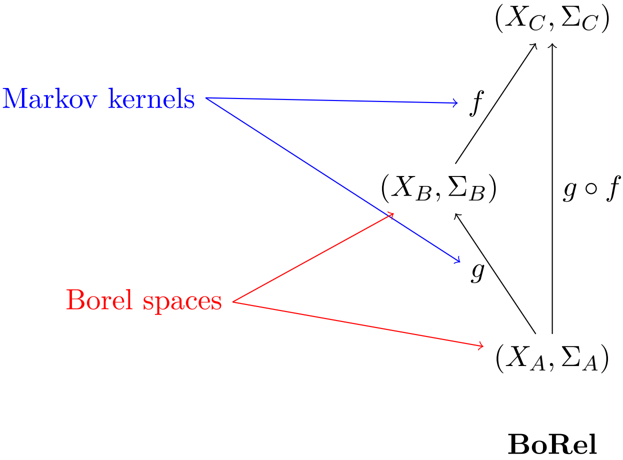


BoRel

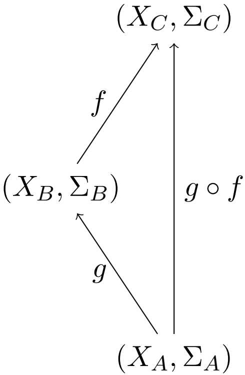
Categorical approach



Categorical approach

Borel space:
topological measurable space

Markov kernels:
$f\colon X_A \times \Sigma_B \to [0,1]$
$f(-,W)\colon X_A \to [0,1]$ bounded measurable
$f(x,-)\colon \Sigma_B \to [0,1]$ probability measure



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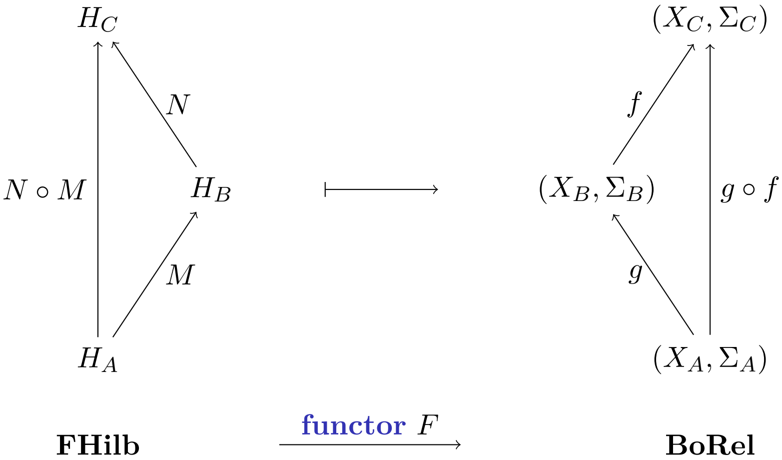

States

$$\mathbb{C} \xrightarrow{\;|\psi\rangle\;} H$$

States

$$
\begin{array}{ccc}
H & & (\Lambda, \Sigma_\Lambda) \\
\uparrow |\psi\rangle & \overset{F}{\longmapsto} & \mu_\psi \uparrow \\
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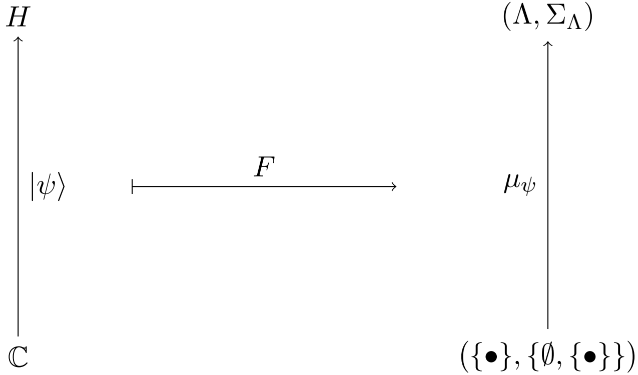

$$F(|\psi\rangle)(\bullet, -)\colon \Sigma_\Lambda \to [0,1]$$

probability measure

Effects

$$\begin{array}{c} H \\ \Big\downarrow \langle \psi | \\ \mathbb{C} \end{array}$$

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$$F(\langle\psi|)(-, \{\bullet\})\colon \Lambda \to [0,1]$$

response function

Operational category

- is monoidal (⊗,I)
- has distinguishing object 2
- has set Ω of elements called probabilities
- has evaluation $\langle - \rangle \colon \mathbf{C}(I, 2) \to \Omega$

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FHilb is operational:

- $2 = \mathbb{C}^2$, $\Omega = [0, 1]$
- $\eta \colon \mathbb{C} \to \mathbb{C}^2$ $\langle \eta \rangle = |a|^2$ if $\eta(1) = (a, b)$, $|a|^2 + |b|^2 = 1$

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BoRel is operational:

- $2 = (\{0, 1\}, \{\emptyset, \{0\}, \{1\}, \{0, 1\}\})$, $\Omega = [0, 1]$
- $f \colon I \to 2$, $\langle f \rangle = f(\bullet, \{0\})$ if $f(\bullet, \{0\}) = 1 - f(\bullet, \{1\})$

Operational model

is functor $F\colon \mathbf{C} \to \mathbf{D}$ between operational categories satisfying:

$$F(I) = I$$
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For $\mathbf{C} = \mathbf{FHilb}$ and $\mathbf{D} = \mathbf{BoRel}$:

$$\int_\Lambda \xi_i(\lambda) \mathrm{d}\mu_\psi(\lambda) = |\langle i|\psi\rangle|^2$$
$$F(|\psi\rangle) = \mu_\psi$$
$$F(\langle i|) = \xi_i$$

Distinguishability

If $\mathbf{C}$ operational category with $\Omega = [0,1]$,
$\Psi \subseteq \mathbf{C}(I, A)$ collection of states
$\chi\colon A \to 2$ measurement,

χ distinguishes ψ from Ψ when

$$\langle \chi \circ \psi \rangle = 1$$
$$\sum_{\phi \in \Psi, \phi \neq \psi} \langle \chi \circ \phi = 0$$

Epistemic operational models

Operational model is epistemic when
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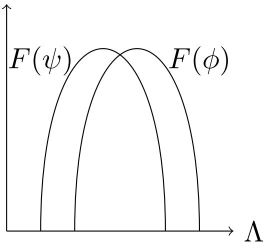
i.e. "distributions overlap":



Operational vs ontological

- operational model is more restrictive
- composition needs to be preserved
- trivial ontic models can be constructed
- not clear whether ontic operational models exist at all

No-go results: Pusey-Barrett-Rudolph

No epistemic ontological model when: preparation independence

$$\{|\psi\rangle \otimes |\phi\rangle\}_{\psi \in H_A, \phi \in H_B} \mapsto (\Lambda_A \times \Lambda_B, \Sigma_{\Lambda_A} \otimes \Sigma_{\Lambda_B})$$
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Monoidal operational model implies this

So cannot have monoidal epistemic operational model!

No-go results: Leifer-Maroney

No maximally epistemic ontological model

$$\forall |\psi\rangle, |\phi\rangle : |\langle\psi|\phi\rangle|^2 = \int_{\mathrm{supp}(\mu_\phi)} \mathrm{d}\mu_\psi(\lambda)$$

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So cannot have duality preserving operational model!

No-go results: Aaronson-Bouland-Chua-Lowther

No symmetric epistemic ontological model

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Implied by equivariance of operational model:

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So cannot have equivariant operational model!

What about a “go” result?

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signed Markov kernels:
$f\colon X_A \times \Sigma_B \to [-1,1]$
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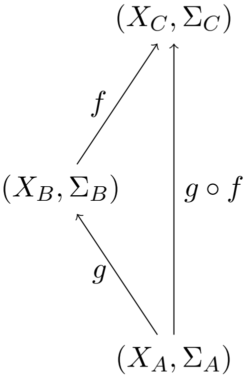

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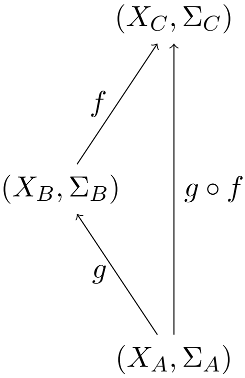

QBoRel

- Possible! In fact monoidal (in odd dimension)!
- Wigner functions
- quasi-probabilistic epistemic model [Ferrie arXiv:1010.2701]

Summary

- Unify ontological interpretations
- Many questions
- Can have operational model at all?
- What about target category of *quantum measures*?

$$\mu(U \cup V) \neq \mu(U) + \mu(V)$$
$$\mu(U \cup V \cup W) = \mu(U \cup V) + \mu(V \cup W) + \mu(W \cup U) - \mu(U) - \mu(V)$$