Sheaf representation of monoidal categories

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Categories should be nice and easy

Category **Vect** of vector spaces is monoidal. So is **Vect** $\times$ **Vect**.
Clearly **Vect** is easier: does not decompose as product.

Any monoidal category embeds into a nice one, and
any nice monoidal category is dependent product of easy ones.

Nice and easy

$\prod_{i\in\{0,1\}}$ **Vect** is decomposable since $\{0,1\}$ is disjoint union

Can reconstruct opens of $\{0,1\}$ as *subunits* of **Vect** $\times$ **Vect**

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- stiff: subunits form semilattice
- universal joins of subunits: subunits form complete lattice

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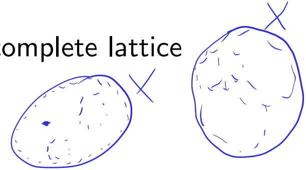
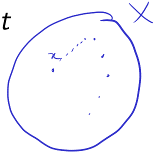


Category is easy if subunits are like singletons:

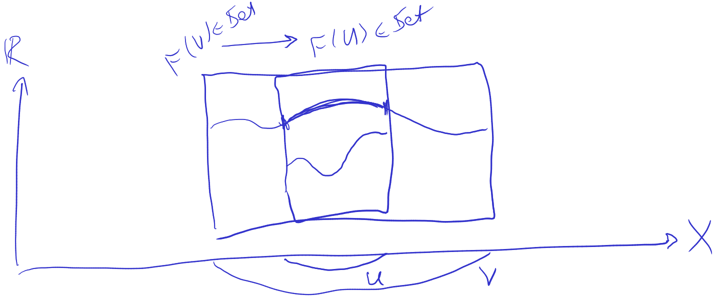
- (sub)local: any (finite) cover contains the open that is covered
  every net converges to a single *focal point*



Sheaves are continuously parametrised objects

Write $\mathcal{O}(X)$ for open sets of space X.

Presheaf on X is functor $F\colon \mathcal{O}(X)^{\mathrm{op}} \to \mathbf{Set}$
Elements of $F(U)$ are called *local sections*.
Elements of $F(X)$ are called *global sections*.
Map $F(U \subseteq V)\colon F(V) \to F(U)$ is called *restriction*.



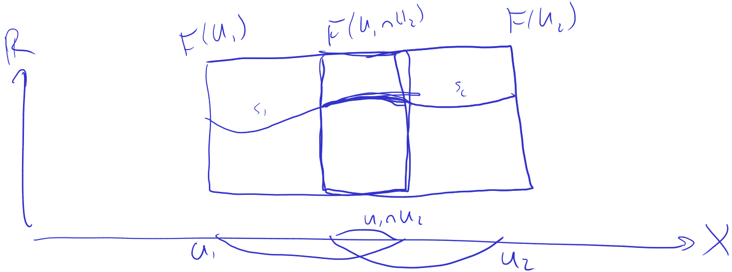
Sheaf condition

Sheaf is continuous presheaf: $F(\mathrm{colim}\, U_i) = \lim F(U_i)$

- Elements of $F(U)$ are *global sections* over $U = \mathrm{colim}\, U_i = \bigcup U_i$
- Elements of $\lim F(U_i)$ are *compatible local sections*:

$$\lim F(U_i) = \{(s_i) \mid F(U_i \cap U_j \subseteq U_i)(s_i) = F(U_i \cap U_j \subseteq U_j)(s_j)\}$$

Compatible local sections must glue together to unique global section



Example: F(U) = { continuous functions $U \to \mathbb{R}$ }

Sheaves of categories

What if F takes values not in **Set** but in **V**?

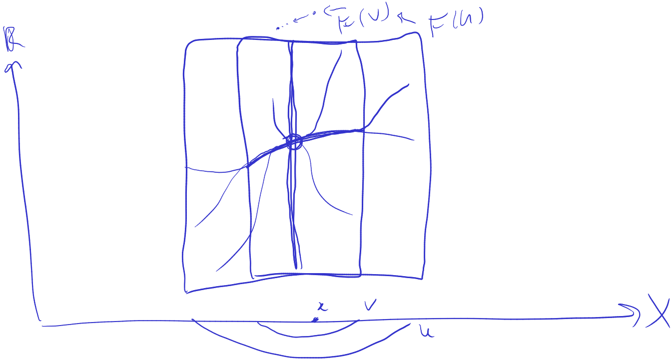
Then sheaf condition becomes equaliser in **V**:

$$
\begin{tikzcd}
F(\bigcup_i U_i) \arrow[r, "\langle F(U_i \subseteq \bigcup U_i)\rangle_i"] & \prod_i F(U_i) \arrow[r, shift left, "\langle F(U_i \cap U_j \subseteq U_i) \circ \pi_i\rangle_{i,j}"] \arrow[r, shift right, "\langle F(U_i \cap U_j \subseteq U_j) \circ \pi_j\rangle_{i,j}"'] & \prod_{i,j} F(U_i \cap U_j)
\end{tikzcd}
$$

global sections are families of local sections that are compatible on pairwise intersections

Stalk

of sheaf F at point x is $\operatorname{colim}\{F(U) \mid x \in U\}$



Say F is a "sheaf of ..." when its stalks are "..."
E.g. sheaves of local rings

Sheaf representation

Literature:

- Boolean algebra is global sections of sheaf of spaces $\{0, 1\}$
- ring is ring of global sections of sheaf of local rings
- topos is category of global sections of sheaf of local toposes
- restriction monoids?

Will generalise all three into:

- monoidal category with universal join of subunits is category of global sections of sheaf of local monoidal categories

$$s = \bigvee_i s_i \Rightarrow \exists i : s_i \simeq s$$

Corollary:

- stiff monoidal category embeds into category of global sections of sheaf of local monoidal categories

Subunits

How to recover $\mathcal{O}(X)$ from $\mathrm{Sh}(X)$?
Look at subobjects of terminal object $s\colon S \rightarrowtail 1$.

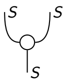
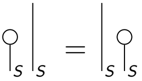
$$\begin{tikzcd} T \arrow[dr, tail, "t"] \arrow[dd, dotted, "\simeq"] & \\ & 1 \\ S \arrow[ur, tail, "s"'] & \end{tikzcd}$$

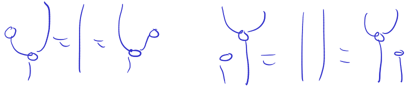
What if we want sheaves with values not in **Set**?
A subunit in a monoidal category **C** is a subobject $s\colon S \rightarrowtail I$
such that $S \otimes s\colon S \otimes S \to S \otimes I$ is invertible. They form set $\mathrm{ISub}(\mathbf{C})$.

- $\mathrm{ISub}(\mathrm{Sh}(X)) = \mathcal{O}(X)$
- $\mathrm{ISub}(L) = L$ for semilattice L

  $\{\sum_{i=1}^{n} x_i y_i \mid x_i, y_i \in I\}$ =
- $\mathrm{ISub}(\mathbf{Mod}_R) = \{I \subseteq R \text{ ideal} \mid I^2 = I\}$ for commutative ring R
- $\mathrm{ISub}(\mathbf{Hilb}_{C(X)}) = \mathcal{O}(X)$

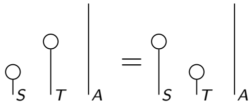
Nice subunits

Draw subunit as , and draw  for inverse of 



$\mathrm{ISub}(\mathbf{C})$ semilattice $\Longleftarrow$ $\mathbf{C}$ is stiff $\Longleftrightarrow$

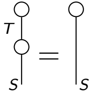
$$\begin{tikzcd}
S \otimes T \otimes A \arrow[r, tail] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & T \otimes A \arrow[d, tail] \\
S \otimes A \arrow[r, tail] & A
\end{tikzcd}$$



Nicer subunits

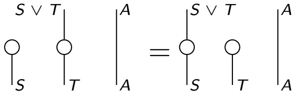
$s \leq t$ if there is unique $m\colon S \to T$ with $s = t \circ m$:

$$\begin{tikzcd} T \arrow[dr, tail, "t"] & \\ & I \\ S \arrow[uu, dotted, "m"] \arrow[ur, tail, "s"'] & \end{tikzcd}$$



	$\mathrm{ISub}(\mathbf{C})$ distributive lattice
$\Longleftarrow$	$\mathbf{C}$ has universal $\cancel{\text{finite}}$ joins of subunits
$\Longleftrightarrow$	$\mathrm{ISub}(\mathbf{C})$ has $\cancel{\text{finite}}$ joins, $0 \simeq 0 \otimes A$ is initial, and

$$\begin{tikzcd} S \otimes T \otimes A \arrow[r, tail] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & T \otimes A \arrow[d, tail] \\ S \otimes A \arrow[r, tail] \arrow[ur, phantom, "\ulcorner", very near end] & (S \vee T) \otimes A \end{tikzcd}$$



Embedding

Stiff **C** embeds into category with universal finite joins of subunits
embeds into category with universal joins of subunits

stiff

$$\begin{tikzcd}
\mathbb{C} \arrow[r, tail, "\text{monoidal}", "\text{preserves subunits}"'] \arrow[dr] & \hat{\mathbb{C}}_{bnd} \arrow[d, dashed] \\
& \mathbb{D}
\end{tikzcd}$$

universal (finite) joins

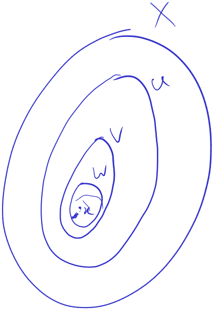
univ (fin) joins

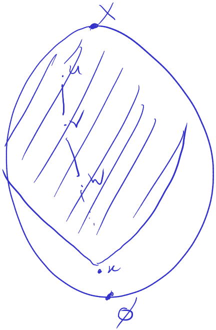
$\hat{U} \hat{\otimes} \hat{A}$

Universally, faithfully, preserving subunits and tensor products

Base space

C has universal (finite) joins of subunits
$\implies$ $\mathrm{ISub}(\mathbf{C})$ is a (distributive lattice) frame
$\implies$ *Zariski spectrum* $X = \mathrm{Spec}(\mathrm{ISub}(\mathbf{C}))$ is topological space

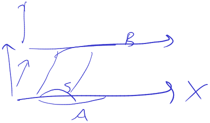




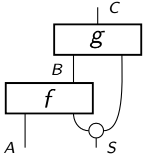
points x are (completely) prime filters in $\mathrm{ISub}(\mathbf{C})$

Local sections $F(s)$ $= kl(S \otimes -)$

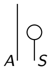
- Objects: as in $\mathbf{C}$



- Morphisms: $A \otimes S \to B$ in $\mathbf{C}$
- Composition:

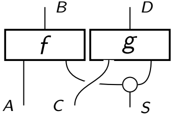


- Identity:



$= \mathrm{id}_A \otimes s : A \otimes S \to A \otimes I \simeq A$

- Tensor product:



$T_S \circlearrowright \mathbb{C} \underset{\perp}{\rightleftarrows} F(S)$

$T : \mathrm{ISub}(\mathbb{C}) \to [\mathbb{C}, \mathbb{C}]$

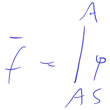
Sheaf condition

To specify a sheaf $F\colon \mathcal{O}(X)^{\mathrm{op}} \to \mathbf{MonCat}$,
it's enough to give a presheaf $F\colon \mathsf{ISub}(\mathbf{C})^{\mathrm{op}} \to \mathbf{MonCat}$,
such that $F(0)$ is terminal and the following is an equaliser:

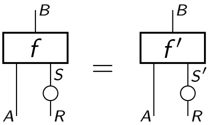
$$F(s \vee t) \xrightarrow{\langle F(s \leq s \vee t),\, F(t \leq s \vee t)\rangle} F(s) \times F(t) \underset{F(s \wedge t \leq t) \circ \pi_2}{\overset{F(s \wedge t \leq s) \circ \pi_1}{\rightrightarrows}} F(s \wedge t)$$

pairs of "local sections" that overlap

Stalks $F(x)$ are (sub)local

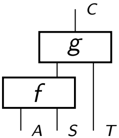


- Objects: as in $\mathbf{C}$
- Morphisms: $A \otimes S \xrightarrow{f} B$ in $\mathbf{C}$ for $s \in x$, identified when



for some $r \in \mathrm{ISub}(\mathbf{C})$

- Composition of (s, f) and (t, g) is



$$\begin{array}{ccc} \mathrm{ISub}(\mathbb{C}) & \longrightarrow & \mathrm{ISub}(F(x)) \\ s & \longmapsto & [s]_x \end{array}$$

Theorem

Any small stiff category with universal (finite) joins of subunits
is monoidally equivalent to category of global sections of sheaf
of (sub)local categories.

Any small stiff category
embeds into a category of global sections of a sheaf
of local categories.

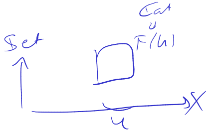
Preservation

	if $\mathbb{C}$ cartesian	
$\mathbb{C}$	$F(s) \simeq \mathbb{C}/s$	$F(x)$
category	local sections	stalks
stiff	monoidal	stiff
closed	closed	closed
traced	traced	traced
compact	compact	compact
Boolean		two-valued
limits	limits	limits
projective colimits	colimits	colimits
weakly distributive	—	— ?

$\mathrm{ISub}(\mathbb{C})$ Boolean algebra

$\mathrm{ISub}(F(x)) = \{0,1\}$

Conclusion



- Cleanly separate 'spatial' from 'temporal' directions
- Does for multiplicative linear logic
  what was known for intuitionistic logic
- Directly capture more examples
- Concrete proof

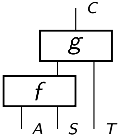
- Completeness theorem?
- Coherence theorem?
- Restriction categories?
- Applications in computer science? Probability? Quantum theory?

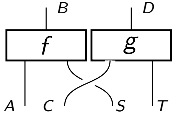
References

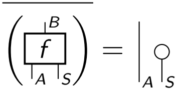
- "*Space in monoidal categories*" [arXiv:1704.08086]
  P. Enrique Moliner, C. Heunen, S.Tull
- "*Tensor topology*" [arXiv:1810.01383]
  P. Enrique Moliner, C. Heunen, S. Tull
- "*Sheaf representation for monoidal categories*" [arXiv:soon]
  R. Soares Barbosa, C. Heunen
- "*Tensor-restriction categories*" [arXiv:2009.12432]
  C. Heunen, J. S. Pacaud Lemay

Restriction categories

Turn ~~restriction~~ mon category **C** into ~~monoidal~~ restr category $S[\mathbf{C}]$:

- Objects: as in **C**
- Morphisms: $A \otimes S \to B$ in **C**
- Composition: 

- Identity: $A \otimes I \to A$
- Tensor product: 

- Restriction: 

Tensor-restriction categories

point is $d\colon I \to S$ with restriction inverse that is tensor-total

$$\begin{tikzcd} I \arrow[rr, "e = \overline{e}"] \arrow[dr, dashed, "d"'] & & I \\ & S \arrow[ur, dashed, tail, "s"'] & \end{tikzcd} \qquad \begin{tikzcd} & S \arrow[dr, tail, "s"] & \\ I \arrow[ur, dashed, "d"] \arrow[rr, "\overline{d}"'] & & I \end{tikzcd} \qquad \begin{tikzcd} I \arrow[r, "d"] \arrow[d, "f"'] & X \arrow[d, "g"] \arrow[dl, dashed, "m"] \\ S \arrow[r, tail, "s"'] & I \end{tikzcd}$$

- $\overline{f \otimes g} = \overline{f} \otimes \overline{g}$
- any $e = \overline{e}\colon I \to I$ factors via subunit s and point d
- any subunit s has point as restriction section
- any $f = \overline{f}\colon X \to X$ equals $f = e \bullet X$ for unique $e = \overline{e}\colon I \to I$
- any tensor-total f equals $f = g \circ \overline{f}$ for a unique restriction-total g;
- points left-lift against subunits
- points are closed under tensor product
- points are determined by codomain up to unique scalar