Sheaf representation of monoidal categories

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Categories should be nice and easy

Category **Vect** of vector spaces is monoidal. So is **Vect** × **Vect**. Clearly **Vect** is easier: does not decompose as product.

Any monoidal category embeds into a nice one, and any nice monoidal category is dependent product of easy ones.
Nice and easy

\[ \prod_{i \in \{0,1\}} \textbf{Vect} \text{ is decomposable since } \{0,1\} \text{ is disjoint union} \]

Can reconstruct opens of \{0,1\} as \textit{subunits} of \textbf{Vect} \times \textbf{Vect}
Nice and easy

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Can reconstruct opens of \( \{0, 1\} \) as subunits of \( \text{Vect} \times \text{Vect} \)

Category is nice if subunits form frame respected by tensor product:

- stiff: subunits form semilattice
- universal joins of subunits: subunits form complete lattice
Nice and easy

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Category is easy if subunits are like singletons:
- (sub)local: any (finite) cover contains the open that is covered every net converges to a single focal point
Sheaves are continuously parametrised objects

Write \( \mathcal{O}(X) \) for open sets of space \( X \).

Presheaf on \( X \) is functor \( F : \mathcal{O}(X)^{\text{op}} \rightarrow \textbf{Set} \)
Elements of \( F(U) \) are called \textit{local sections}.
Elements of \( F(X) \) are called \textit{global sections}.
Map \( F(U \subseteq V) : F(V) \rightarrow F(U) \) is called \textit{restriction}.
Sheaf condition

**Sheaf** is continuous presheaf: \( F(\text{colim } U_i) = \lim F(U_i) \)

- Elements of \( F(U) \) are *global sections* over \( U = \text{colim } U_i = \bigcup U_i \)
- Elements of \( \lim F(U_i) \) are *compatible local sections*:

\[
\lim F(U_i) = \left\{ (s_i) \mid F(U_i \cap U_j \subseteq U_i)(s_i) = F(U_i \cap U_j \subseteq U_j)(s_j) \right\}
\]

Compatible local sections must glue together to unique global section

Example: \( F(U) = \{ \text{continuous functions } U \rightarrow \mathbb{R} \} \)
Sheaves of categories

What if $F$ takes values not in $\textbf{Set}$ but in $\textbf{V}$?

Then sheaf condition becomes equaliser in $\textbf{V}$:

$$F(\bigcup_i U_i) \xrightarrow{\langle F(U_i \subseteq \bigcup U_i) \rangle_i} \prod_i F(U_i) \xrightarrow{\langle F(U_i \cap U_j \subseteq U_i) \circ \pi_{i,j} \rangle_{i,j}} \prod_{i,j} F(U_i \cap U_j)$$
Stalk

of sheaf $F$ at point $x$ is $\text{colim}\{F(U) \mid x \in U\}$

Say $F$ is a “sheaf of …” when its stalks are “…”
E.g. sheaves of local rings
Sheaf representation

Literature:
- Boolean algebra is global sections of sheaf of spaces \( \{0, 1\} \)
- ring is ring of global sections of sheaf of local rings
- topos is category of global sections of sheaf of local toposes
- restricted monoids?

Will generalise all three into:
- monoidal category with universal join of subunits is category of global sections of sheaf of local monoidal categories

Corollary:
- stiff monoidal category embeds into category of global sections of sheaf of local monoidal categories
Subunits

How to recover $\mathcal{O}(X)$ from $\mathcal{Sh}(X)$?

Look at subobjects of terminal object $s: S \to 1$.

What if we want sheaves with values not in $\textbf{Set}$?

A subunit in a monoidal category $\mathcal{C}$ is a subobject $s: S \to 1$ such that $S \otimes s: S \otimes S \to S \otimes 1$ is invertible. They form set $\text{ISub}(\mathcal{C})$.

- $\text{ISub}(\mathcal{Sh}(X)) = \mathcal{O}(X)$
- $\text{ISub}(L) = L$ for semilattice $L$
- $\text{ISub}(\textbf{Mod}_R) = \{ I \subseteq R \text{ ideal} \mid I^2 = I \}$ for commutative ring $R$
- $\text{ISub}(\text{Hilb}_C(X)) = \mathcal{O}(X)$
Nice subunits

Draw subunit as $\circ_s$, and draw $\circ_s^\perp\circ_s^\perp$ for inverse of $\circ_s\circ_s^\perp = \circ_s\circ_s^\perp$

\[
\text{ISub}(\mathcal{C}) \text{ semilattice} \iff \mathcal{C} \text{ is stiff} \iff
\]

\[
S \otimes T \otimes A \xrightarrow{\perp} T \otimes A
\]

\[
S \otimes A \xrightarrow{\perp} A
\]

\[
\circ_s \circ_T \circ_A = \circ_s \circ_T \circ_A
\]
Nicer subunits

$s \leq t$ if there is unique $m : S \to T$ with $s = t \circ m$:

\[
\begin{array}{c}
S \quad T \\
\downarrow \quad \downarrow \\
S \quad T
\end{array}
\]

$I\text{Sub}(\mathbf{C})$ distributive lattice

$\iff \mathbf{C}$ has universal finite joins of subunits

$\iff I\text{Sub}(\mathbf{C})$ has finite joins, $0 \cong 0 \otimes A$ is initial, and

\[
\begin{array}{c}
S \otimes T \otimes A \\
\downarrow \\
S \otimes A
\end{array}
\xrightarrow{\ast} \quad
\begin{array}{c}
T \otimes A \\
\downarrow \\
(S \lor T) \otimes A
\end{array}
\]

\[
\begin{array}{c}
S \lor T \\
\downarrow \\
S
\end{array}
\quad
\begin{array}{c}
A \\
\downarrow \\
A
\end{array}
\quad =
\begin{array}{c}
S \lor T \\
\downarrow \\
S
\end{array}
\quad
\begin{array}{c}
T \\
\downarrow \\
T
\end{array}
\quad
\begin{array}{c}
A \\
\downarrow \\
A
\end{array}
\]
Embedding

**Stiff $\mathcal{C}$** embeds into category with universal finite joins of subunits

Embeds into category with universal joins of subunits

Universally, faithfully, preserving subunits and tensor products
Base space

\( \mathbf{C} \) has **universal** (finite) joins of subunits

\[ \implies \text{ISub}(\mathbf{C}) \text{ is a (distributive lattice) frame} \]

\[ \implies \text{Zariski spectrum } X = \text{Spec}(\text{ISub}(\mathbf{C})) \text{ is topological space} \]

points \( x \) are (completely) prime filters in \( \text{ISub}(\mathbf{C}) \)
Local sections $F(s) = \kappa_\ell (S \otimes -)$

- **Objects:** as in $\mathbf{C}$
- **Morphisms:** $A \otimes S \to B$ in $\mathbf{C}$
- **Composition:**

\[ f \circ g : B \otimes S \to C \]

- **Identity:**

\[ \text{id}_A : A \otimes S \to A \]

- **Tensor product:**

\[ T \subseteq \mathbf{C} \mathbf{C} \leftarrow \longleftarrow F(s) \]

\[ \mathcal{T} : \mathbf{ISub}(C) \to [C, C] \]
Sheaf condition

To specify a sheaf $F : \mathcal{O}(X)^{op} \to \text{MonCat}$, it’s enough to give a presheaf $F : \text{ISub}(C)^{op} \to \text{MonCat}$, such that $F(0)$ is terminal and the following is an equaliser:

$$F(s \lor t) \xrightarrow{\langle F(s \leq s \lor t), F(t \leq s \lor t) \rangle} F(s) \times F(t) \xrightarrow{F(s \land t \leq s) \circ \pi_1} F(s \land t)$$
Stalks $F(x)$ are (sub)local

- **Objects:** as in $\mathbf{C}$
- **Morphisms:** $A \otimes S \rightarrow B$ in $\mathbf{C}$ for $s \in x$, identified when

\[ f_{S,R}^A B = f_{S',R}^A B \quad \text{for some } r \in \text{Sub}(\mathbf{C}) \]

- **Composition of** $(s, f)$ and $(t, g)$ is

\[ I_{\text{Sub}(\mathbf{C})} \xrightarrow{c} I_{\text{Sub}(F(x))} \]
Theorem

Any small stiff category with universal (finite) joins of subunits is monoidally equivalent to category of global sections of sheaf of (sub)local categories.

Any small stiff category embeds into a category of global sections of a sheaf of local categories.
## Preservation

<table>
<thead>
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<th>( \mathcal{C} )</th>
<th>( F(s) \simeq \mathcal{C}/s )</th>
<th>( F(x) )</th>
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<tr>
<td>category</td>
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- \( I_{\text{Sub}}(\mathcal{C}) \)
- Boolean algebra
- \( = \{0,1\} \)
- Locally compactible
- ?
Conclusion

- Cleanly separate ‘spatial’ from ‘temporal’ directions
- Does for multiplicative linear logic what was known for intuitionistic logic
- Directly capture more examples
- Concrete proof

- Completeness theorem?
- Coherence theorem?
- Restriction categories?
- Applications in computer science? Probability? Quantum theory?
References

▶ “Space in monoidal categories” [arXiv:1704.08086]
P. Enrique Moliner, C. Heunen, S. Tull

▶ “Tensor topology” [arXiv:1810.01383]
P. Enrique Moliner, C. Heunen, S. Tull

▶ “Sheaf representation for monoidal categories” [arXiv:soon]
R. Soares Barbosa, C. Heunen

C. Heunen, J. S. Pacaud Lemay
Restriction categories

Turn restriction category $\mathbf{C}$ into monoidal category $S[\mathbf{C}]$:

- Objects: as in $\mathbf{C}$
- Morphisms: $A \otimes S \rightarrow B$ in $\mathbf{C}$
- Composition:
- Identity: $A \otimes I \rightarrow A$
- Tensor product:
- Restriction: $\begin{pmatrix}
\begin{array}{c}
\mathcal{B} \\
\mathcal{A}
\end{array}
\end{pmatrix}_{\mathcal{A} \mathcal{S}} = \left| \right|_{\mathcal{A} \mathcal{S}}$
Tensor-restriction categories

point is \( d: I \to S \) with restriction inverse that is tensor-total

\[
\begin{align*}
f \otimes g &= f \otimes g \\
\text{any } e = \bar{e}: I \to I \text{ factors via subunit } s \text{ and point } d \\
\text{any subunit } s \text{ has point as restriction section} \\
\text{any } f = \bar{f}: X \to X \text{ equals } f = e \bullet X \text{ for unique } e = \bar{e}: I \to I \\
\text{any tensor-total } f \text{ equals } f = g \circ \bar{f} \text{ for a unique restriction-total } g; \\
\text{points left-lift against subunits} \\
\text{points are closed under tensor product} \\
\text{points are determined by codomain up to unique scalar}
\end{align*}
\]