Sheaf representation of monoidal categories

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Categories should be nice and easy

Category \textbf{Vect} of vector spaces is monoidal. So is \textbf{Vect} \times \textbf{Vect}. Clearly \textbf{Vect} is easier: does not decompose as product.

Any monoidal category embeds into a \textbf{nice} one, and any \textbf{nice} monoidal category is dependent product of \textbf{easy} ones.
Nice and easy

\[ \prod_{i \in \{0,1\}} \textbf{Vect} \text{ is decomposable since } \{0,1\} \text{ is disjoint union} \]

Can reconstruct opens of \( \{0,1\} \) as subunits of \( \textbf{Vect} \times \textbf{Vect} \)
Nice and easy

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Category is nice if subunits form frame respected by tensor product:

- **stiff**: subunits form semilattice
- **universal joins of subunits**: subunits form complete lattice
Nice and easy

\[ \prod_{i \in \{0, 1\}} \text{Vect} \] is decomposable since \( \{0, 1\} \) is disjoint union

Can reconstruct opens of \( \{0, 1\} \) as *subunits* of \( \text{Vect} \times \text{Vect} \)

Category is nice if subunits form frame respected by tensor product:
  - stiff: subunits form semilattice
  - universal joins of subunits: subunits form complete lattice

Category is easy if subunits are like singletons:
  - (sub)local: any (finite) cover contains the open that is covered every net converges to a single *focal point*
Sheaves are continuously parametrised objects

Write $\mathcal{O}(X)$ for open sets of space $X$.

**Presheaf** on $X$ is functor $F : \mathcal{O}(X)^{\text{op}} \to \text{Set}$

Elements of $F(U)$ are called *local sections*.

Elements of $F(X)$ are called *global sections*.

Map $F(U \subseteq V) : F(V) \to F(U)$ is called *restriction*. 
Sheaf condition

Sheaf is continuous presheaf: \( F(\text{colim } U_i) = \lim F(U_i) \)

- Elements of \( F(U) \) are **global sections** over \( U = \text{colim } U_i = \bigcup U_i \)
- Elements of \( \lim F(U_i) \) are **compatible local sections**:

\[
\lim F(U_i) = \{ (s_i) | F(U_i \cap U_j \subseteq U_i)(s_i) = F(U_i \cap U_j \subseteq U_j)(s_j) \}
\]

Compatible local sections must glue together to unique global section

Example: \( F(U) = \{ \text{continuous functions } U \to \mathbb{R} \} \)
Sheaves of categories

What if $F$ takes values not in $\textbf{Set}$ but in $\mathbf{V}$?

Then sheaf condition becomes equaliser in $\mathbf{V}$:

$$F(\bigcup U_i) \xrightarrow{\langle F(U_i \subseteq U_i) \rangle_i} \prod_i F(U_i) \xrightarrow{\langle F(U_i \cap U_j \subseteq U_i \circ \pi_i) \rangle_{i,j}} \prod_{i,j} F(U_i \cap U_j)$$
Stalk

of sheaf $F$ at point $x$ is $\text{colim}\{F(U) \mid x \in U\}$

Say $F$ is a “sheaf of ...” when its stalks are “...”
E.g. sheaves of local rings
Sheaf representation

Literature:

- Boolean algebra is global sections of sheaf of spaces \(\{0, 1\}\)
- ring is ring of global sections of sheaf of local rings
- topos is category of global sections of sheaf of local toposes

Will generalise all three into:

- monoidal category with universal join of subunits is category of global sections of sheaf of local monoidal categories

Corollary:

- stiff monoidal category embeds into category of global sections of sheaf of local monoidal categories
Subunits

How to recover $\mathcal{O}(X)$ from $\text{Sh}(X)$?
Look at subobjects of terminal object $s: S \rightarrow 1$.

What if we want sheaves with values not in $\textbf{Set}$?
A subunit in a monoidal category $\mathbf{C}$ is a subobject $s: S \rightarrow I$
such that $S \otimes s: S \otimes S \rightarrow S \otimes I$ is invertible. They form set $\text{ISub}(\mathbf{C})$.

$\text{ISub}(\text{Sh}(X)) = \mathcal{O}(X)$
$\text{ISub}(L) = L$ for semilattice $L$
$\text{ISub}(\text{Mod}_R) = \{ I \subseteq R \text{ ideal} \mid I^2 = I \}$ for commutative ring $R$
$\text{ISub}(\text{Hilb}_C(X)) = \mathcal{O}(X)$
Nice subunits

Draw subunit as $\bullet_s$, and draw $S$ for inverse of $\bullet_s \circ \bullet_s = \bullet_s \circ \bullet_s$

\[
\text{ISub}(C) \text{ semilattice} \iff C \text{ is stiff} \iff
\]

\[
\begin{array}{ccc}
S \otimes T \otimes A & \Rightarrow & T \otimes A \\
\downarrow & & \downarrow \\
S \otimes A & \Rightarrow & A
\end{array}
\]

\[
\text{S, T, A} = \text{S, T, A}
\]
Nicer subunits

\[
\begin{align*}
    s \leq t \text{ if there is unique } m: S \to T \text{ with } s = t \circ m:
\end{align*}
\]

\[
\begin{tikzpicture}
    \node (s) at (0,0) {S};
    \node (t) at (2,0) {T};
    \draw[->] (s) -- (t);
    \node (s) at (0,-1) {S};
    \node (t) at (2,-1) {T};
    \draw[->] (s) -- (t);
    \node (s) at (0,-2) {S};
    \node (t) at (2,-2) {T};
    \draw[->] (s) -- (t);
    \node (s) at (0,-3) {S};
    \node (t) at (2,-3) {T};
    \draw[->] (s) -- (t);
\end{tikzpicture}
\]

\(\text{ISub}(\mathcal{C})\) distributive lattice

\[\iff \quad \mathcal{C} \text{ has universal finite joins of subunits}\]

\[\iff \quad \text{ISub}(\mathcal{C}) \text{ has finite joins, } 0 \cong 0 \otimes A \text{ is initial, and}\]

\[
\begin{align*}
    S \otimes T \otimes A &\Rightarrow T \otimes A \\
    S \otimes A &\Rightarrow (S \lor T) \otimes A \\
    S \vee T &\Rightarrow A \\
    S &\Rightarrow S \\
    T &\Rightarrow T \\
    A &\Rightarrow A
\end{align*}
\]
Embedding

\textbf{Stiff} \(C\) embeds into category with \textit{universal finite joins} of subunits
embeds into category with \textit{universal joins} of subunits

Universally, faithfully, preserving subunits and tensor products
**Base space**

\( \textbf{C} \text{ has universal (finite) joins of subunits} \)

\( \implies \text{ ISub(} \textbf{C} \text{)} \text{ is a (distributive lattice) frame} \)

\( \implies \text{ Zariski spectrum } X = \text{ Spec(ISub(} \textbf{C} \text{))} \text{ is topological space} \)

points \( x \) are (completely) prime filters in ISub(\textbf{C})
Local sections $F(s)$

- **Objects**: as in $\mathbf{C}$
- **Morphisms**: $A \otimes S \rightarrow B$ in $\mathbf{C}$

\[ \begin{array}{c}
\text{Composition:} \\
\begin{array}{c}
\text{Identity:} \\
\text{Tensor product:}
\end{array}
\end{array} \]

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Sheaf condition

To specify a sheaf $F : \mathcal{O}(X)^{\text{op}} \to \text{MonCat}$, it’s enough to give a presheaf $F : \text{ISub}(C)^{\text{op}} \to \text{MonCat}$, such that $F(0)$ is terminal and the following is an equaliser:

$$F(s \vee t) \xrightarrow{\langle F(s \leq s \vee t), F(t \leq s \vee t) \rangle} F(s) \times F(t) \xrightarrow{\begin{array}{c} F(s \wedge t \leq s) \circ \pi_1 \\ F(s \wedge t \leq t) \circ \pi_2 \end{array}} F(s \wedge t)$$
Stalks $F(x)$ are (sub)local

- **Objects:** as in $\mathbf{C}$
- **Morphisms:** $A \otimes S \to B$ in $\mathbf{C}$ for $s \in x$, identified when $f_{SR} = f'_{SR}$

Composition of $(s, f)$ and $(t, g)$ is
Theorem

Any small stiff category with universal (finite) joins of subunits is monoidally equivalent to category of global sections of sheaf of (sub)local categories.

Any small stiff category embeds into a category of global sections of a sheaf of local categories.
### Preservation

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<th>stalks</th>
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Conclusion

- Cleanly separate ‘spatial’ from ‘temporal’ directions
- Does for multiplicative linear logic what was known for intuitionistic logic
- Directly capture more examples
- Concrete proof

- Completeness theorem?
- Coherence theorem?
- Restriction categories?
- Applications in computer science? Probability? Quantum theory?
References

- “Space in monoidal categories” [arXiv:1704.08086]
  P. Enrique Moliner, C. Heunen, S. Tull

- “Tensor topology” [arXiv:1810.01383]
  P. Enrique Moliner, C. Heunen, S. Tull

- “Sheaf representation for monoidal categories” [arXiv:soon]
  R. Soares Barbosa, C. Heunen

  C. Heunen, J. S. Pacaud Lemay
Restriction categories

Turn monoidal category $\mathbf{C}$ into restriction category $S[\mathbf{C}]$:

- Objects: as in $\mathbf{C}$
- Morphisms: $A \otimes S \to B$ in $\mathbf{C}$
- Composition:
- Identity: $A \otimes I \to A$
- Tensor product:

Restriction: $\begin{pmatrix} B \\ f \\ A \otimes S \end{pmatrix} = \begin{pmatrix} A \otimes S \\ f \end{pmatrix}$
Tensor-restriction categories

point is \( d: I \rightarrow S \) with restriction inverse that is tensor-total

\[
\begin{align*}
I \xrightarrow{e = \bar{e}} I \\
I \xleftarrow{d} S \xrightarrow{s} I
\end{align*}
\]

\[
\begin{align*}
I \xrightarrow{d} S \xleftarrow{s} I \\
I \xrightarrow{\bar{d}} X
\end{align*}
\]

\[
\begin{align*}
I \xrightarrow{d} X \\
S \xleftarrow{s} I
\end{align*}
\]

- \( f \otimes g = \bar{f} \otimes \bar{g} \)
- any \( e = \bar{e}: I \rightarrow I \) factors via subunit \( s \) and point \( d \)
- any subunit \( s \) has point as restriction section
- any \( f = \bar{f}: X \rightarrow X \) equals \( f = e \bullet X \) for unique \( e = \bar{e}: I \rightarrow I \)
- any tensor-total \( f \) equals \( f = g \circ \bar{f} \) for a unique restriction-total \( g \);
- points left-lift against subunits
- points are closed under tensor product
- points are determined by codomain up to unique scalar