Sheaf representation of monoidal categories

Chris Heunen
<chris.heunen@ed.ac.uk>

based on arXiv:2106.08896
with Rui Soares Barbosa
“Categories should be nice and easy”

- Vect is monoidal, but so is Vect x Vect

[Diagram]

Any monoidal cat embeds into a nice one

Any nice monoidal cat is dependent prod of easy ones
\[ \text{Vect} \times \text{Vect} = \bigoplus_{i \in \{0,1\}} \text{Vect} \]

decomposable as \{0,1\} disjoint union

Can reconstruct
opens as
central idempotents

cat is nice if these behave:

- stiff: form semi-lattice respected by \( \otimes \)
- universal joins: complete lattice respected by \( \otimes \)

cat is easy if these are:

- (sub) local: any (fin) cover contains open that is covered
- topologically: any net converges to single focal pt
- logically: disjunction property if \( A \lor B \) then \( A \lor B \)

Overview
Sheaves: continuously parametrised objects

A presheaf on $X$ is a functor $F: \mathcal{O}(X)^\text{op} \to \text{Set}$, where $\mathcal{O}(X)$ is the frame of opens of $X$. The diagram illustrates the concepts of local sections and global sections.

- $F(V)$ represents the set associated to an open set $V$.
- $F(U)$ represents the set associated to a subset $U$ of $V$.
- $F(X)$ represents the set associated to the entire space $X$.

The diagram shows the restriction of a section from $F(V)$ to $F(U)$ and the concept of local sections and global sections.
Sheaf = continuous presheaf:

\[ F(\operatorname{colim} U_i) = \lim F(U_i) \]

global sections \{ (s_i) \mid F(U_i \cap U_j \subseteq U_i)(s_i) = F(U_i \cap U_j \subseteq U_j)(s_i) \}
compatible local sections
Q: What if values not in Set but in MonCat?
A: Sheaf condition becomes equalizer:

\[ \text{global sections} \quad \text{are} \quad \text{families of local sections} \quad \text{that are pairwise compatible} \]

\[ F(UU_i) \xrightarrow{\langle F(U_i \leq UU_i) \rangle} \prod_i F(U_i) \xrightarrow{\langle F(U_i \cap U_j \leq U_i) \cap U_j \rangle} \prod_{ij} F(U_i \cap U_j) \]
Stalk of sheaf $F$ at pt $x = \operatorname{colim}_{x \in U} F(U)$

"sheaf of $\mathbb{R}$" = "sheaf whose stalks are $\mathbb{R}$"  

e.g. "sheaf of local rings"
- Boolean algebra = global sections of spaces \{0,1\}
- ring = global sections of local rings
- topos = global sections of local toposes

\[ \text{mon}\text{'k cat w univ joins} \]
\[ = \]
\[ \text{global sections of local mon's cat} \]

\[ \text{UI} \]
\[ \text{stiff mon's cat} \]
opens of \( X \) = central idempotents of \( Sh(X) \)

- morphism \( U \xrightarrow{u} I \)
- half-braiding \( U \otimes - \Rightarrow - \otimes U \)
- s.t. \( U \otimes U \xrightarrow{u \otimes u = u \otimes u} U \)

satisfying:

\[
\begin{align*}
\text{id} & = 0 \\
u \circ \text{id} & = 0 \\
u \circ u & = u \circ u
\end{align*}
\]
<table>
<thead>
<tr>
<th>Category</th>
<th>Central Idempotents</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cartesian cat</td>
<td>Subterminal Obj</td>
</tr>
<tr>
<td>(Sh(X))</td>
<td>Opens of (X)</td>
</tr>
<tr>
<td>Semilattice</td>
<td>Everything</td>
</tr>
<tr>
<td>Quantale</td>
<td>Largest Subframe ({q^2 = q \leq 1})</td>
</tr>
<tr>
<td>(\text{Mod}_R)</td>
<td>Idempotent Ideals (I^2 = I \leq R(\ldots))</td>
</tr>
<tr>
<td>(\text{Mod}_A)</td>
<td>Central Idemp. Elts. of (A)</td>
</tr>
<tr>
<td>Hilbert modules</td>
<td>Opens of (X)</td>
</tr>
<tr>
<td>over (C(x))</td>
<td>Trivial</td>
</tr>
<tr>
<td>Endofunctors (C \rightarrow C)</td>
<td></td>
</tr>
</tbody>
</table>
central idempotents form semilattice: \( u \vee v = u \wedge v \)
\( 1 = I \)

\[ u \leq v \iff u = u \vee v \iff \begin{array}{c}
\begin{array}{c}
V \downarrow V \\
\downarrow I \\
V \uparrow V
\end{array}
\end{array} \iff \begin{array}{c}
\begin{array}{c}
V \downarrow V \\
\downarrow I \\
V \uparrow V
\end{array}
\end{array} \]

- \( C \) is shift \( \iff \begin{array}{c}
\begin{array}{c}
0 \downarrow 0 = \begin{array}{c}
0 \downarrow 0 \\
0 \downarrow 0 \\
0 \downarrow 0
\end{array}
\end{array}
\end{array} \) pullback

- \( C \) has univ fin joins \( \iff \begin{array}{c}
\begin{array}{c}
0 \downarrow 0 = \begin{array}{c}
0 \downarrow 0 \\
0 \downarrow 0 \\
0 \downarrow 0
\end{array}
\end{array}
\end{array} \) distrib. lattice pullback & pushout

\( A \otimes 0 = 0 \)

- \( C \) has univ joins \( \iff \begin{array}{c}
\begin{array}{c}
0 \downarrow 0 = \colim \begin{array}{c}
0 \\
0 \\
0
\end{array}
\end{array}
\end{array} \) frame
Stiff monoidal cat embeds into monoidal cat w univ fin joins
embeds into monoidal cat w univ joins

\[ F(A) \otimes F(U) \cong F(A \otimes U) \]
\[ F(I) \cong I \]
\[ F(\frac{\gamma}{\alpha}) = \frac{\gamma}{\alpha \cdot \beta} \]

nice(r) central idempotents
Base space \( X = \text{Zariski spectrum of } \text{central idempotents of } C \)

\[ = \{ \text{(completely) prime filters of } \text{central idempotents of } C \} \]
Local sections \( F(u) = C \downarrow u = \ker(\ominus \otimes u) \) \( \cong C / u \) (if \( \ominus = x \))

- **objs:** as in \( C \)

- **mor's:** \( A \otimes U \rightarrow B \)

- **comp.:**

- **identity:**

- **tensor:**
Sheaf condition
binary equaliser enough

\[ C_{uvw} \rightarrow C_{lu} \times C_{lv} \rightarrow C_{lwv} \]

pairs of local sections
that overlap
Stalks \( C|_x \) are (sub)local

- obj's: as in \( C \)

- mor's: \( A \otimes U \xrightarrow{f} B \) for \( u \in x \), identified when

- comp.: \([v, g] \circ [u, f] = \)
stiff mon'ld cat
\[ \leq \]
mon'ld cat w univ (fin) joins
\[
= 
\]
global sections of (sub)local mon'ld cats

\[ [\text{Lambek-Moerdijk-Avaodey}] \text{ toposes} \]
\[ \Rightarrow \text{not toposes} \]
\{ 
  \[ [\text{Stone}] \text{ Boolean alg} \]
  \[ [\text{Takahashi}] \text{ Hilbert modules} \]
  \[ [\text{Pierce}] \text{ modules over comm. ring} \]
\}
<table>
<thead>
<tr>
<th>Category $\mathcal{C}$</th>
<th>Local sections $\mathcal{E}^L_u$</th>
<th>Stalks $\mathcal{E}^L_x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>closed</td>
<td>closed</td>
<td>closed</td>
</tr>
<tr>
<td>traced</td>
<td>traced</td>
<td>traced</td>
</tr>
<tr>
<td>compact</td>
<td>compact</td>
<td>compact</td>
</tr>
<tr>
<td>$N=1$</td>
<td>Boolean</td>
<td>two-valued</td>
</tr>
<tr>
<td>$uv-u=1$</td>
<td>complete</td>
<td>complete</td>
</tr>
<tr>
<td>$u^u-u=0$</td>
<td>proj. cocomplete</td>
<td>fin. cocomplete</td>
</tr>
</tbody>
</table>
• obj's: \text{mon} \text{ cat} \ C
  w \text{ univ (fin) joins}

• mor's: \ F: C \to D \text{ lax mon} \text{'}
\[ F(A) \otimes F(U) \cong F(A \otimes U) \]
\[ F(I) \cong I \]
\[ F(\lambda) = \lambda \]

\[ \text{Mon Cat(\#)} \overset{\text{op}}{\to} \text{Mon Schema(\#)} \]

\[ \begin{align*}
\text{obj's:} & \quad P: G(X) \to \text{MonCat} \\
& \text{s.t. } P(x) \text{ (sub)local} \\
& \text{“flabby” } G\!P(u) \xrightarrow{1} P(v) \\
& \xleftarrow{P(u \leq v)} \\
\text{mor's:} & \quad \text{ct3, fn. } q: X \to Y \\
& \text{n.t. transf. } F_q: Q(v) \to P(q^{-1}(v)) \\
& \text{s.t. } F_q: Q(q(x)) \to P(x) \\
& \text{conservative on central idemp: } \\
& F_q(v) = 1 \implies v = 1
\end{align*} \]
✓ cleanly separate "spatial" from "temporal" directions
✓ capture more examples
✓ concrete proof

? completeness? coherence?
• linear logic?
• alg/geom?
... causality?
... concurrency? Petri nets?
... localisable monads?
[arXiv:2106.08896] “Sheaf representation of monoidal categories”
R. Soares Barbosa & C. Heunen

C. Heunen & J. S. Pascaud Lemay

P. Enrique Moliner & C. Heunen & S. Tull
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Elec. Proc. Th. Comp. Sci. 266, 2018