A unified treatment of concrete sheaf models for higher-order recursion

Cristina Matache (joint work with Sean Moss and Sam Staton)

University of Oxford

Modelling higher-order programs with recursion

Model

- Cartesian closed category (CCC)
- Partiality monad, L
- Interpretation: Type $\longleftrightarrow$ Object
  Program $\longleftrightarrow$ Partial morphism

Examples:

(1) **Probabilistic programming**: partial maps that are measurable
[Heunen et al.'17, Vákár et al.'19]

(2) **Automatic differentiation**: partial maps that are smooth
[Huot et al.'20, Vákár'20]

(3) Piecewise differentiable programs [Lew et al.'21]

(4) **Full abstraction** for a sequential language: definable partial maps
[O'Hearn & Riecke'95], [Matache, Moss, Staton, FSCD'21]

Goal of this talk

Main Theorem [Matache, Moss, Staton, in preparation]

The examples

(1) **Probabilistic programming**
(2) **Automatic differentiation**
(3) Piecewise differentiation
(4) Full abstraction

} all model higher-order recursion using the same recipe

- using **concrete sheaves**
- using ideas from synthetic domain theory for **recursion**

In each case more domain specific work needs to be done.

Examples of concrete sheaves: subsequential spaces [Johnstone'79], C-spaces [Escardó & Xu'16]

Examples of concrete presheaves: [Rosolini & Streicher'99], finiteness spaces [Ehrhard'07]

Goal of this talk (continued)

Main Theorem [Matache, Moss, Staton, in preparation]

The examples

(1) **Probabilistic programming**
(2) **Automatic differentiation**
(3) Piecewise differentiation
(4) Full abstraction

} all model higher-order recursion using the same recipe

- using **concrete sheaves**
- using ideas from synthetic domain theory for **recursion**

Corollary: conservativity result for (1), (2), (3)

E.g.(2): Programs **real** $\to$ **real** are still interpreted as smooth maps even if they use higher-order recursion.

PCF_v: A call-by-value language

Call-by-value λ-calculus with:

- base types e.g. nat, real
- function types
- product and sum types
- recursive functions.

Model

- Cartesian closed category (CCC)
- Partiality monad, L
- Interpretation:
  Type $\longleftrightarrow$ Object
  Program $\longleftrightarrow$ Partial morphism

An interpretation looks like:

$$[\![\mathsf{nat}]\!] = 1 + 1 + \ldots \qquad [\![\tau_1 + \tau_2]\!] = [\![\tau_1]\!] + [\![\tau_2]\!] \qquad [\![\tau_1 \times \tau_2]\!] = [\![\tau_1]\!] \times [\![\tau_2]\!]$$

$$[\![\tau \to \tau']\!] = [\![\tau]\!] \Rightarrow L[\![\tau']\!] \qquad [\![\Gamma \vdash t : \tau]\!] : [\![\Gamma]\!] \to L[\![\tau]\!]$$

Outline

Why use categories of concrete sheaves?

Example: **first-order probabilistic computation** can be modelled in Sbs.
Sbs is NOT cartesian closed.

The category of **presheaves** on Sbs is cartesian closed.

Yoneda embedding

$$y : \mathsf{Sbs} \hookrightarrow \mathsf{PSh}(\mathsf{Sbs})$$

Full, faithful, preserves limits.
Does not preserve colimits.

Restricting to **sheaves** on a **site** (Sbs, J) preserves some colimits from Sbs.

Concrete sheaves $=$ sets with structure $+$ structure-preserving functions.

$$\mathsf{ConcSh}(\mathsf{Sbs}, J) \hookrightarrow \mathsf{Sh}(\mathsf{Sbs}, J) \hookrightarrow \mathsf{PSh}(\mathsf{Sbs})$$

Well-pointed categories and concrete sites

A category $\mathbb{C}$ is *well-pointed* if

- it has a terminal object $\star$
- $\mathbb{C}(\star, -) : \mathbb{C} \to$ **Set** is faithful
  i.e. maps $h : d \to c$ are distinguished *functions* $|h| : |d| \to |c|$

where $|c| = \mathbb{C}(\star, c)$. So $\mathbb{C}$ is a category of sets and certain functions.

Concrete site $(\mathbb{C}, J)$

- A small well-pointed category $\mathbb{C}$
- For every $c \in \mathbb{C}$ a set $J(c)$ of **covering families** $\{f_i : c_i \to c\}_{i \in I}$ of c s.t.
  - (C) pullback stability
  - ($\star$) If $\{f_i : c_i \to c\}_{i \in I}$ covers c, then $\bigcup_{i \in I} \mathsf{Im}(|f_i|) = |c|$

Concrete sheaf on a concrete site $(\mathbb{C}, J)$

[*Concrete quasitopoi*, Dubuc'77]
[*Convenient categories of smooth spaces*, Baez & Hoffnung'11]

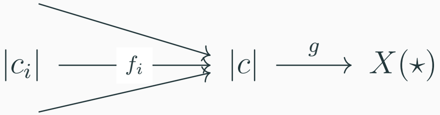
Well-pointed category $\mathbb{C}$	Concrete site $(\mathbb{C}, J)$
• has a terminal $\star$ • a map $h : d \to c$ is a function between sets $\lvert d\rvert = \mathbb{C}(\star, d)$ etc.	• small well-pointed $\mathbb{C}$ • For every $c \in \mathbb{C}$ a set $J(c)$ of **covering families** $\{f_i : c_i \to c\}_{i \in I}$ of c, with axioms (C) and $\star$.

A concrete sheaf $X : \mathbb{C}^{\mathrm{op}} \to$ Set is:

- a set $X(\star)$
- $X(c) \subseteq [\lvert c\rvert \to X(\star)]$

$X(h : d \to c)$ is precomposition by $\lvert h\rvert$.

Sheaf condition: for each function $g : \lvert c\rvert \to X(\star)$ and each covering family $\{f_i : c_i \to c\}_{i \in I} \in J(c)$, if each $g \circ \lvert f_i\rvert \in X(c_i)$, then $g : \lvert c\rvert \to X(\star) \in X(c)$.



A morphism $\alpha : X \to Y$ is a structure-preserving function $\alpha : X(\star) \to Y(\star)$.

Example: modelling probabilistic programming [Heunen et al.'17, Vákár et al.'19]

A functor $X : \mathbb{C}^{\mathrm{op}} \to$ **Set** is a **concrete sheaf** on a **concrete site** $(\mathbb{C}, J)$ if $X(c) \subseteq [|c| \to X(\star)]$ and X satisfies the **sheaf condition**.

Quasi-Borel spaces is the category of concrete sheaves on:

- **Sbs**: objects U are Borel subsets of $\mathbb{R}$
  morphisms are measurable functions between these sets.
- $J(U) =$ countable sets of measurable inclusions $\{U_i \hookrightarrow U\}_{i \in I}$ where $U = \bigcup_{i \in I} U_i$ and the U_i's are disjoint.

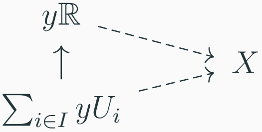
$X(\mathbb{R}) \subseteq [\mathbb{R} \to X(\star)]$ is the set of "random elements" of $X(\star)$.

Example: modelling probabilistic programming in $\mathsf{ConcSh}(\mathsf{Sbs}, J)$

A functor $X : \mathbb{C}^{\mathrm{op}} \to \mathsf{Set}$ is a **concrete sheaf** on a **concrete site** $(\mathbb{C}, J)$ if $X(c) \subseteq [|c| \to X(\star)]$ and X satisfies the **sheaf condition**.

Sbs = Borel subsets $U \subseteq \mathbb{R}$ + measurable functions
$J(U)$ = sets of inclusions $\{U_i \hookrightarrow U\}_{i \in I}$ where $U = \bigcup_{i \in I} U_i$ and the U_i's are disjoint.

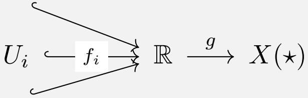
In $\mathsf{PSh}(\mathsf{Sbs})$, take X concrete. In Sbs, take $\mathbb{R} = \bigcup_{i \in I} U_i$ and U_i's disjoint:



by Yoneda lemma

$$(g : \mathbb{R} \to X(\star)) \in X(\mathbb{R})$$
$$\Updownarrow$$
$$\{(f_i : U_i \to X(\star)) \in X(U_i)\}_{i \in I}$$

Sheaf condition at $\mathbb{R}$: for each function $g : \mathbb{R} \to X(\star)$ and each covering family $\{f_i : U_i \to \mathbb{R}\}_{i \in I} \in J(c)$, if each $g \circ f_i \in X(U_i)$, then $g : \mathbb{R} \to X(\star) \in X(\mathbb{R})$.



Example: modelling differentiable programs [Huot et al.'20, Vákár'20]

A functor $X : \mathbb{C}^{\mathrm{op}} \to \mathsf{Set}$ is a **concrete sheaf** on a **concrete site** $(\mathbb{C}, J)$ if $X(c) \subseteq [|c| \to X(\star)]$ and X satisfies the **sheaf condition**.

Diffeological spaces is the category of concrete sheaves on:

- Site: objects are open subsets $U \subseteq \mathbb{R}^n$ for any n
  morphisms are smooth maps.
- $J(U) =$ countable sets of open inclusions $\{U_i \hookrightarrow U\}_{i \in I}$ where $U = \bigcup_{i \in I} U_i$.

$X(U) \subseteq [U \to X(\star)]$ is the set of "plots" of $X(\star)$.

Outline

Partial maps

In any category, a **partial map** $X \rightharpoonup Y$ is a pair (m, f):

$$\begin{array}{ccc} X' & \xrightarrow{f} & Y \\ {\scriptstyle m \in \mathcal{N}} \big\downarrow & & \\ X & & \end{array}$$

where $\mathcal{N}$ is **stable class of monos**:

- contains all isomorphisms
- closed under composition
- stable under pullback (with arbitrary maps)

(1) **Quasi-Borel spaces**: partial maps that are measurable, with Borel domain

(2) **Diffeological spaces**: partial maps that are smooth, with open domain

From partial maps to a lifting monad

How do we get a monad L with the following property?

For every $\begin{array}{ccc} X' & \xrightarrow{f} & Y \\ {\scriptstyle m\in\mathcal{N}}\big\downarrow & & \\ X & & \end{array}$, where $\mathcal{N}$ is a stable class of monos

there is exactly one corresponding total map $X \to LY$ such that

$$\begin{array}{ccc} X' & \xrightarrow{f} & Y \\ {\scriptstyle m\in\mathcal{N}}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle \eta_Y} \\ X & \xrightarrow[\exists!]{} & LY \end{array}$$

and conversely.

L might not exist in general.

From partial maps to a lifting monad (continued)

In a sheaf category $\mathsf{Sh}(\mathbb{C}, J)$:

Theorem

$\mathcal{N}$ has an associated lifting monad L if the class of monos $\mathcal{N}$ "comes from" a **class of pre-admissible monos $\mathcal{M}$ in $\mathbb{C}$.**

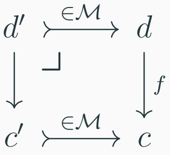
$\mathcal{M}$ is a class of pre-admissible monos in $\mathbb{C}$ if:

- stable class
- $\Delta_{\mathcal{M}} : \mathbb{C}^{\mathrm{op}} \to \mathsf{Set}$ is a J-**sheaf**, where:
  $\Delta_{\mathcal{M}}(c) =$ iso. classes of $c' \rightarrowtail c \in \mathcal{M}$
  $\Delta_{\mathcal{M}}(f : d \to c) =$ pullback along f

$\mathcal{N}$ stable class of monos:

- contains all isomorphisms
- closed under composition
- stable under pullback

Lifting monad:
for every $(m, f) : X \rightharpoonup Y$ with $m \in \mathcal{N}$, there is exactly one total map $X \to LY$.



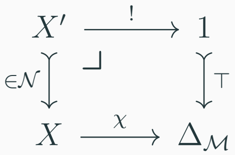
From partial maps to a lifting monad (continued)

In a sheaf category $\mathsf{Sh}(\mathbb{C}, J)$:

Theorem

$\mathcal{N}$ has an associated lifting monad L if the class of monos $\mathcal{N}$ "comes from" a **class of pre-admissible monos $\mathcal{M}$ in $\mathbb{C}$.**

$\mathcal{N}$ "comes from" $\mathcal{M}$ if
$\mathcal{N} =$ all pullbacks of $\top : 1 \to \Delta_{\mathcal{M}}$,
where $\top_c = [\mathrm{id}_c]$



$\mathcal{N}$ stable class of monos:
- contains all isomorphisms
- closed under composition
- stable under pullback

Lifting monad:
for every $(m, f) : X \rightharpoonup Y$ with $m \in \mathcal{N}$, there is exactly one total map $X \to LY$.

$\mathcal{M}$ stable class
$\Delta_{\mathcal{M}} : \mathbb{C}^{\mathrm{op}} \to$ **Set** a **J-sheaf**
$\Delta_{\mathcal{M}}(c) =$ iso. classes of $c' \rightarrowtail c \in \mathcal{M}$
$\Delta_{\mathcal{M}}(f : d \to c) =$ pullback along f

see e.g [Rosolini'86] for dominance, [Mulry'94], [Fiore&Plotkin'97] for constructing a lifting monad

Examples: classes of pre-admissible monos

Quasi-Borel spaces:

Site: objects U are Borel subsets of $\mathbb{R}$, morphisms are measurable functions.

$\mathcal{M} =$ for every U, the measurable monos with codomain U

Diffeological spaces:

Site: objects are open subsets $U \subseteq \mathbb{R}^n$ for any n, morphisms are smooth maps.

$\mathcal{M} =$ for every U, the open inclusion maps into U

For a concrete sheaf X, the lifting monad:

$$LX(\star) = X(\star) \uplus \{\bot\}$$

$$LX(U) = \left\{g : U \to X(\star) \uplus \{\bot\} \;\middle|\; \exists U' \rightarrowtail U \in \mathcal{M} \text{ s.t. } \mathsf{dom}(g) = U' \text{ and } g|_{U'} \in X(U')\right\}$$

In general, having the lifting monad is not enough to model recursion.

Outline

The ωcpo model of PCF_v

Types = partially ordered sets with least upper bounds of ω-chains

Terms = continuous functions

To model recursive functions:

- a lifting monad on $\omega\mathbf{cpo}$
- Tarski's fixed point theorem

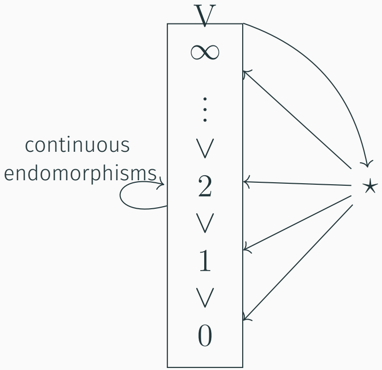
We want to recover this model as presheaves with a class of admissible monos in the site.

$\mathrm{V} = \{0 < 1 < 2 < \ldots < \infty\}$ = poset of vertical nat. numbers

See the category $\mathcal{H}$ from [Fiore & Rosolini'97, '01].

$\mathbb{V}$ = two-object category

vSet $= [\mathbb{V}^{\mathrm{op}}, \mathbf{Set}]$ = presheaves on $\mathbb{V}$



Concrete presheaf on $\mathbb{V}$

- a set $X(\star)$
- a set of functions $X(\mathrm{V}) \subseteq [\mathrm{V} \to X(\star)]$

$x \in X(\mathrm{V})$ is a **completed chain** of elements in $X(\star)$.

Map $X \to Y$ = function $X(\star) \to Y(\star)$ that **preserves chains**.

ω**cpo** is a full subcategory of **vSet**:

$D \mapsto \big(|D|, \omega\mathbf{cpo}(\mathrm{V}, D)\big)$

Lifting in vSet

$\mathbb{V}$ = vertical naturals as a two-object category
vSet = presheaves on $\mathbb{V}$

Theorem:
A class of pre-admissible monos $\mathcal{M}$ **in** $\mathbb{C}$ induces a lifting monad L on the sheaf category $\mathsf{Sh}(\mathbb{C}, J)$.

$\mathbb{V}$ has a **class of pre-admissible monos**:

$$\mathcal{M}_{\mathbb{V}} = \{(\lambda x.x + n) \in \mathbb{V}(\mathrm{V}, \mathrm{V}) \mid n \in \mathbb{N}\} \cup \{\mathrm{id}_\star : \star \to \star\}$$

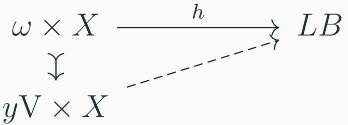
which induces a **lifting monad** L on vSet, where for a concrete presheaf X:

$$(LX)(\star) = X(\star) \uplus \{\bot\} \qquad (LX)(\mathrm{V}) = \{\bot\} \uplus \coprod_{n \in \mathbb{N}} (X(\mathrm{V}))_n$$

$(X(\mathrm{V}))_n \approx$ chains from $X(\mathrm{V})$ with n $\bot$'s added at the beginning.

Fixed point theorem in vSet

We can construct a fixed point of a map $(A \Rightarrow LB) \to (A \Rightarrow LB)$ if LB is "complete".



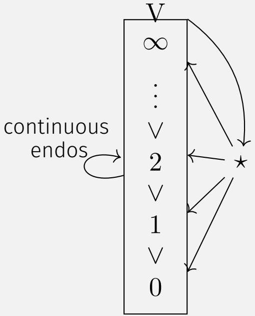
see also [Fiore & Plotkin'97]

ω = greatest subobject of $y\mathrm{V}$ without ∞

Theorem

vSet is an adequate model of $\mathsf{PCF_v}$ where types are concrete presheaves.

$\mathbb{V}$ = two-object category



vSet = $[\mathbb{V}^{\mathrm{op}}, \mathsf{Set}]$

The interpretation of $\mathsf{PCF_v}$ commutes with the inclusion ωcpo $\hookrightarrow$ vSet.

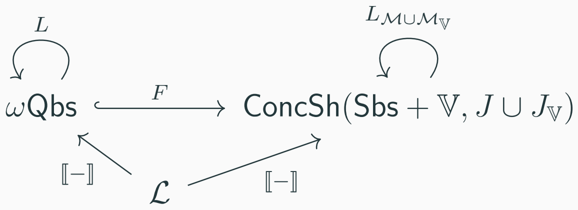
Outline

Modelling $\mathsf{PCF_v}$ in a category of concrete sheaves

Main Theorem [Matache, Moss, Staton, in preparation]

Given a **concrete site** with a **class of admissible monos** $(\mathbb{C}, J, \mathcal{M})$, "combine" it with the site for vSet, $(\mathbb{V}, J_{\mathbb{V}}, \mathcal{M}_{\mathbb{V}})$.
The category of **concrete sheaves** on the combined concrete site $(\mathbb{C} + \mathbb{V}, J \cup J_{\mathbb{V}}, \mathcal{M} \cup \mathcal{M}_{\mathbb{V}})$ is an adequate model of $\mathsf{PCF_v}$.

Example: we recover the ωQbs model



Concrete site for Qbs:
Sbs: objects U are Borel subsets of $\mathbb{R}$, morphisms are measurable functions.
$J(U)$ = countable sets of inclusions $\{U_i \hookrightarrow U\}_{i \in I}$ where $U = \bigcup_{i \in I} U_i$ and the U_i's are disjoint.
$\mathcal{M}$ = all monos.

Summary

Main Theorem [Matache, Moss, Staton, in preparation]

Given a **concrete site** with a **class of admissible monos** $(\mathbb{C}, J, \mathcal{M})$, "combine" it with the site for vSet, $(\mathbb{V}, J_{\mathbb{V}}, \mathcal{M}_{\mathbb{V}})$.
The category of **concrete sheaves** on the combined concrete site $(\mathbb{C} + \mathbb{V}, J \cup J_{\mathbb{V}}, \mathcal{M} \cup \mathcal{M}_{\mathbb{V}})$ is an adequate model of $\mathrm{PCF}_{\mathrm{v}}$.

Model higher-order recursion for:

(1) **Probabilistic programming**
(2) **Automatic differentiation**
(3) Piecewise differentiation
(4) Full abstraction

Using:

- sheaves on a concrete site
- class of admissible monos in the site
- presheaves on the vertical naturals