Understanding Why Forward-Backward is Belief Propagation in Two Easy Steps

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Suppose that we run belief propagation on the HMM in Figure 1, with the following schedule: First all of the observed nodes o_t send a message $m_{o_t}(s_t)$ to their parent:

$$m_{o_t}(s_t) = p(o_t|s_t), \tag{1}$$

and then we send messages $m_{t-1,t}(j)$ forward, starting with s_0, and finally we send messages $m_{t+1,t}(i)$ backward, starting with s_T. These messages are depicted in Figure 1, and given by:

$$m_{t-1,t}(j) = \sum_i p(s_t = j|s_{t-1} = i) m_{t-2,t-1}(i) m_{o_{t-1}}(i) \tag{2}$$

$$m_{t+1,t}(i) = \sum_j p(s_{t+1} = j|s_t = i) m_{t+2,t+1}(j) m_{o_{t+1}}(j), \tag{3}$$

where the summations over i and j are over the possible labels of the HMM.

The purpose of this note is simply to say: forward-backward is equivalent to belief propagation using this schedule, because in Figure 1 each α is the product of the red messages and β is the blue message.

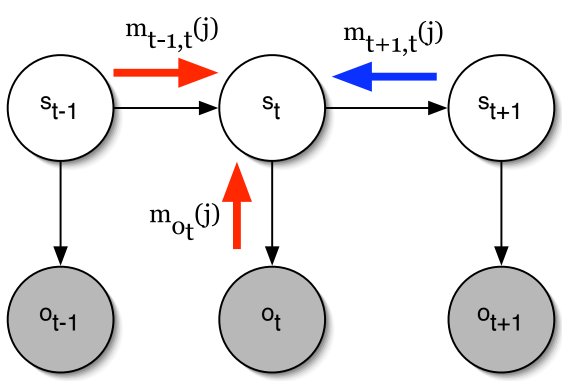

Figure 1: A sample HMM, marked with the messages sent by belief propagation. The forward-backward value for α_t is the product of the red messages, and β_t is the blue message.

This can be proven by induction. For $t = 0$, we allow by convention $m_{-1,0}(j) = p(s_0 = j)$, so that

$$\alpha_0(j) = p(s_0 = j)p(o_0|s_0 = j) = m_{s_{-1}}(s_0)m_{o_0}(j). \tag{4}$$

For $t > 0$, assume $\alpha_{t-1}(i) = m_{t-1,t}(i)m_{o_{t-1}}(i)$ for all i. Then

$$\alpha_t(j) = p(o_t|s_t = j)\sum_i p(s_t = j|s_{t-1} = i)\alpha_{t-1}(i) \tag{5}$$

$$= m_{o_t}(j)\sum_i p(s_t = j|s_{t-1} = i)m_{t-1,t}(i) \tag{6}$$

$$= m_{o_t}(j)m_{t-1,t}(j), \tag{7}$$

which completes the proof.

Similarly, it can be shown by induction that $\beta_t(i) = m_{t+1,t}(i)$. We assume that $m_{T+1,T}$ is uniformly 1, so that for all i

$$\beta_T(i) = p(o_T|s_T = i) = m_{o_T}(i)m_{T+1,T}(i). \tag{8}$$

Inductively, assume that $\beta_{t+1}(i) = m_{t+2,t+1}(i)$. Then

$$\beta_t(i) = \sum_j p(s_{t+1} = j|s_t = i)p(o_{t+1}|s_{t+1} = j)\beta_{t+1}(j) \tag{9}$$

$$= \sum_j p(s_{t+1} = j|s_t = i)m_{o_{t+1}}(j)m_{t+2,t+1}(j) \tag{10}$$

$$= m_{t+1,t}(i), \tag{11}$$

which completes the proof.