Let $\sigma\colon \Sigma \to \Sigma'$

Translating syntax

- *translation of variables*: $X \mapsto X'$, where $X'_{s'} = \biguplus_{\sigma(s)=s'} X_s$
- *translation of terms*: $\sigma\colon |T_\Sigma(X)|_s \to |T_{\Sigma'}(X')|_{\sigma(s)}$, for $s \in S$
- *translation of equations*: $\sigma(\forall X.t_1 = t_2)$ yields $\forall X'.\sigma(t_1) = \sigma(t_2)$

. . . and semantics

- *σ-reduct*: $\_|_\sigma\colon Alg(\Sigma') \to Alg(\Sigma)$, where for $A' \in Alg(\Sigma')$
  - $|A'|_\sigma|_s = |A'|_{\sigma(s)}$, for $s \in S$
  - $f_{A'|_\sigma} = \sigma(f)_{A'}$ for $f \in \Omega$

Note the contravariancy!

Satisfaction condition

Fact: *For all signature morphisms $\sigma\colon \Sigma \to \Sigma'$, Σ'-algebras A' and Σ-equations φ:*

$$\boxed{A'|_\sigma \models_\Sigma \varphi \iff A' \models_{\Sigma'} \sigma(\varphi)}$$

TRUTH is preserved (at least) under:

- *change of notation*
- *restriction/extension of irrelevant context*

Categories & functors around

- Σ-algebras and their homomorphisms form the category $\mathbf{Alg}(\Sigma)$
- algebraic signatures and their morphisms form the category $\mathbf{AlgSig}$
- sets and functions between them form the category $\mathbf{Set}$
- (sm)all categories and functors between them form the category $\mathbf{Cat}$
- σ-reduct extends to the functor $\_|_\sigma\colon \mathbf{Alg}(\Sigma') \to \mathbf{Alg}(\Sigma)$
- $\mathbf{Alg}\colon \mathbf{AlgSig}^{op} \to \mathbf{Cat}$ is a (contravariant) functor mapping signature Σ to the category $\mathbf{Alg}(\Sigma)$ and signature morphism $\sigma\colon \Sigma \to \Sigma'$ to the reduct functor $\_|_\sigma\colon \mathbf{Alg}(\Sigma') \to \mathbf{Alg}(\Sigma)$
- $\mathbf{Eq}\colon \mathbf{AlgSig} \to \mathbf{Set}$ is a (covariant) functor mapping signature Σ to the set $\mathbf{Eq}(\Sigma)$ of all Σ-equations and signature morphism $\sigma\colon \Sigma \to \Sigma'$ to the translation function $\sigma\colon \mathbf{Eq}(\Sigma) \to \mathbf{Eq}(\Sigma')$

Categories, functors

- *Category* $\mathbf{K}$ consists of:
  - "set" of *objects*: $|\mathbf{K}|$
  - sets of *morphisms*: $\mathbf{K}(A,B)$, for all $A, B \in |\mathbf{K}|$; $m\colon A \to B$ stands for $m \in \mathbf{K}(A,B)$
  - morphism *composition*: for $m\colon A \to B$ and $m'\colon B \to C$, we have $m;m'\colon A \to C$;
    the composition is associative and has identities.
- *Functor* $\mathbf{F}\colon \mathbf{K} \to \mathbf{K}'$ between two categories maps:
  - $\mathbf{K}$-objects to $\mathbf{K}'$-objects
  - $\mathbf{K}$-morphisms to $\mathbf{K}'$-morphisms, preserving their source and target, composition and identities

Diagrams, limits, colimits

- *Diagram* in $\mathbf{K}$ is a functor $D\colon J \to \mathbf{K}$ from (small) *shape* category J
- *Cocone* $\alpha\colon D \to X$ on diagram D with *vertex* $X \in |\mathbf{K}|$: consists of a family of morphisms $\alpha_n\colon D(n) \to X$, one for each *node* $n \in |J|$, such that $\alpha_n = D(e);\alpha_m$ for each *edge* $e\colon n \to m$ in J
- *Cone* $\beta\colon X \to D$: ... a family of morphisms $\beta_n\colon X \to D(n)$ ... *dually*
- *Colimit* of D is a cocone $colimD\colon D \to |colimD|$ such that for every cocone $\alpha\colon D \to X$ there exists a *unique* $h\colon |colimD| \to X$ such that $(colimD)_n;h = \alpha_n$ for $n \in |J|$
- *Limit* of D is a cone $limD\colon |limD| \to D$ ... *dually*

Some limits

diagram	limit	in **Set**
(empty)	*terminal object*	$\{*\}$
$A \qquad B$	*product*	$A \times B$
$A \overset{f}{\underset{g}{\rightrightarrows}} B$	*equalizer*	$\{a \in A \mid f(a) = g(a)\} \hookrightarrow A$
$A \xrightarrow{f} C \xleftarrow{g} B$	*pullback*	$\{(a,b) \in A \times B \mid f(a) = g(b)\}$

Fact:

- *All finite limits may be built using terminal object and pullbacks.*
- *Pullbacks may be built using products and equalizers.*

... & colimits

diagram	colimit	in **Set**
(empty)	*initial object*	$\emptyset$
$A \qquad B$	*coproduct*	$A \uplus B$
$A \overset{f}{\underset{g}{\rightrightarrows}} B$	*coequalizer*	$B \longrightarrow B/{\equiv}$ where $f(a) \equiv g(a)$ for all $a \in A$
$A \xleftarrow{f} C \xrightarrow{g} B$	*pushout*	$(A \uplus B)/{\equiv}$ where $f(c) \equiv g(c)$ for all $c \in C$

Limits and colimits (when they exist) are defined up to isomorphism

Pushout

$$\begin{tikzcd}
& \{0,[1=1],[2=3],2,3\} & \\
\{0,1,2,3\} \arrow[ur, "{\{0\mapsto 0,1\mapsto[1=1],2\mapsto[2=3],3\mapsto 3\}}"] & & \{1,2,3\} \arrow[ul, "{\{1\mapsto[1=1],2\mapsto 2,3\mapsto[2,3]\}}"'] \\
& \{a,b\} \arrow[ul, "{\{a\mapsto 1,b\mapsto 2\}}"] \arrow[ur, "{\{a\mapsto 1,b\mapsto 3\}}"'] &
\end{tikzcd}$$

Diagrams list objects indicating how they share components

Colimits add objects up taking account of the indicated sharing