A Framework for Intuitionistic Modal Logics

(Extended Abstract)

Gordon Plotkin Colin Stirling
Department of Computer Science
Edinburgh University

Introduction

This abstract presents work on a Kripkean analysis of intuitionistic modal logic. As remarked in [BS] there ought to be such a subject, but in fact there is very little literature [B1,B2,F1,F2,V1,V2,V3]. One possible explanation is simply that it is hardly obvious what the applications would be. It seems to us however that there is a wide range of computational applications and indeed so many are the possibilities that it is worth beginning by sorting out the basic theory.

What we have in mind is to tie up Scott's theory of domains with the logic of programs where one wants to know whether or not

$$p \models A$$

holds. Here p is a computational entity (state, function, process, trace or whatever) taken from a type P of such entities and A is an assertion (of dynamic logic, epistemic logic, temporal logic or whatever). If one wanted to know whether A held of a program one would take p to be its denotation. Now suppose P is a Scott domain (say a complete partial order with least element $\bot$) and consider the assertion H, for "halts", or, "is defined". Then it seems natural that

$$\bot \models H \vee \neg H$$

does not hold as we would like the "truth value" of A above, viz $\{p \mid p \models A\}$, to be upper closed. Indeed one of us has proposed, see [AS], following Smyth's work [Sm] that for a liveness property (generalization of total correctness) one ought to take here as truth values all upper closed sets (rather than the G_δ sets proposed by Smyth for specifications - we want all intersections of opens and not just countable ones) and for safety properties (generalization of partial correctness) one would take the Scott closed sets. Hence one expects to look at intuitionistic logics.

As a concrete example we tried linear time temporal logic, taking P to be the set of total and partially defined execution sequences (or traces) from a given set S of states so that:

$$P = S^* \cup (S^* \times \{\bot\}) \cup S^\omega$$

with the ordering $w \sqsubseteq w'$ iff $w = w'$ or there are v, v' such that $w = v\bot$ and $w' = vv'$, using an obvious notation. However we soon became lost in the details and realized we first needed an understanding of intuitionistic modal logic before worrying about the more complex temporal connectives (or epistemic, or whatever).

Previous work is mainly concerned with relating intuitionistic modal logics either to fragments of first-order intuitionistic logic or to classical modal logics using translations. What is not presented is a general framework similar to the Kripkean framework for classical modal logic. There have been other proposals to apply intuitionistic logic to Computer Science [Ma,Mc] but these have concerned ideas centering around the realizability interpretation. The present proposal to look at the uses of the Kripkean (or more general such as topological) ones seems novel.

1. 'Minimal' Intuitionistic Modal Logic

The simplest version of a Kripkean intuitionistic modal frame consists of a set of worlds W and two relations on it, $\langle W, \sqsubseteq, R\rangle$: the relation $\sqsubseteq$ is the intuitionistic information partial ordering whereas R is the modal accessibility relation. Questions then arise as to the interrelationships between these two relations (compare the discussion of frames for combinations of different modalities [T]). Four possible conditions spring to mind:

1. if $w \sqsubseteq w'$ and wRv then $\exists v'.\ w'Rv'$ and $v \sqsubseteq v'$.
2. if $w \sqsubseteq w'$ and $w'Rv'$ then $\exists v.\ wRv$ and $v \sqsubseteq v'$
3. if $v \sqsubseteq v'$ and wRv then $\exists w'.\ w'Rv'$ and $w \sqsubseteq w'$
4. if $v \sqsubseteq v'$ and $w'Rv'$ then $\exists w.\ wRv$ and $w \sqsubseteq w'$

These can be represented as conditions for completing diagrams. For instance, 1 and 3 become:

Which of these conditions, if any, should be imposed on the frame depends to a large extent on the semantic clauses for the modal operators and their expected interrelation. Guided by the clauses for the intuitionistic quantifiers we might suggest the following pair:

$$w \models \Diamond A \quad \text{iff} \quad \exists u.\ wRu \ \text{and} \ u \models A$$

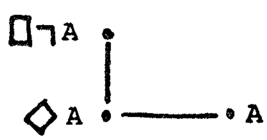
$$w \models \Box A \quad \text{iff} \quad \forall w' \sqsupseteq w\ \forall u.\ \text{if } w'Ru \ \text{then} \ u \models A$$

Further, the following two schemata seem very natural:

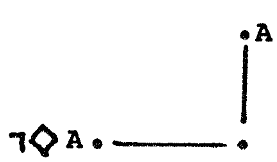
$$\Diamond A \rightarrow \neg\Box\neg A$$

$$\neg\Diamond A \rightarrow \Box\neg A$$

Frame condition 1 guarantees the first of these: for a counterexample would be (for some A);



More generally, frame condition 1 guarantees that if $w \models \Diamond A$ and $w \sqsubseteq w'$ then $w' \models \Diamond A$. In contrast it is frame condition 3 that guarantees the second schema: for a counterexample would be (for some A):



Therefore it is these two conditions which we here impose on the definition of an intuitionistic modal frame. We believe that condition 1 is natural whereas the other condition is less so. For example, when R is a monotonic function, the first condition holds, but the second, in general, will not.

The sentential modal language $\mathcal{L}$ is given by the set of formulas A, where q ranges over a set of atomic sentences Q and ff is the false sentence:

$$A ::= ff \mid q \mid A \wedge A \mid A \vee A \mid A \rightarrow A \mid \Diamond A \mid \Box A$$

Negation is defined as usual: $\neg A$ is $A \rightarrow ff$. An intuitionistic modal model for $\mathcal{L}$ is a pair $\langle \mathcal{F}, V \rangle$ where $\mathcal{F}$ is an intuitionistic modal frame and V is a mapping from W into subsets of Q with the property:

if $w \sqsubseteq w'$ then $V(w) \subset V(w')$

We define the satisfaction relation $w \models_{\mathcal{M}} A$. The index $\mathcal{M}$ is dropped:

$w \not\models ff$		
$w \models q$	iff	$q \in V(w)$
$w \models A \wedge B$	iff	$w \models A$ and $w \models B$
$w \models A \vee B$	iff	$w \models A$ or $w \models B$
$w \models A \rightarrow B$	iff	$\forall w' \sqsupseteq w$ if $w' \models A$ then $w' \models B$
$w \models \Diamond A$	iff	$\exists u.\ wRu$ and $u \models A$
$w \models \Box A$	iff	$\forall w' \sqsupseteq w\ \forall u.$ if $w'Ru$ then $u \models A$

The clauses for $\Diamond$ and $\Box$ are as discussed before and the others are standard.

The following lemma depends on the frame condition 1.

Lemma 1.1 If $w \sqsubseteq w'$ and $w \models A$ then $w' \models A$

The system below, IK, is an axiomatization of validity relative to arbitrary intuitionistic modal frames.

Axioms

1. Any intuitionistic sentential theorem instance
2. $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$
3. $\Box(A \rightarrow B) \rightarrow (\Diamond A \rightarrow \Diamond B)$
4. $\neg\Diamond ff$
5. $\Diamond(A \vee B) \rightarrow \Diamond A \vee \Diamond B$
6. $(\Diamond A \rightarrow \Box B) \rightarrow \Box(A \rightarrow B)$

Rules

MP If $\vdash A$ and $\vdash A \rightarrow B$ then $\vdash B$

Nec If $\vdash A$ then $\vdash \Box A$

Axiom A6 corresponds (see below) to the frame condition 2. Some derived theorems and rules are:

$\Box\neg A \rightarrow \neg\Diamond A$ $\qquad\qquad$ $\Diamond\neg A \rightarrow \neg\Box A$

$\Box(A \wedge B) \leftrightarrow \Box A \wedge \Box B$ $\qquad\qquad$ $\Box A \wedge \Diamond B \rightarrow \Diamond(A \wedge B)$

If $\vdash A \rightarrow B$ then $\vdash \Diamond A \rightarrow \Diamond B$

IK has the disjunction property and is also both sound and complete:

Theorem 1.2 i. If $\vdash A \vee B$ then $\vdash A$ or $\vdash B$

ii. $\vdash A$ iff $\models A$

If $A \vee \neg A$ is added as an axiom to IK then the resulting system is just minimal classical normal modal logic, the system K in [C].

2. Further Modal Logics

The variety of standard classical modal logics, extensions of K, arise by relating extra conditions on the modal accessibility relation to extra axioms. The result is correspondence theorems [VB]: for instance, the axiom $\Box A \rightarrow \Box\Box A$ (or, equivalently, $\Diamond\Diamond A \rightarrow \Diamond A$) corresponds to transitivity of the accessibility relation. The situation is more intricate for intuitionistic modal logics. There are two related features.

First, because of the breakdown in duality between $\Box$ and $\Diamond$, axioms like $\Box A \rightarrow A$ and $A \rightarrow \Diamond A$ are not equivalent (and are therefore unlikely to correspond to the same frame condition). This increases the variety of intuitionistic modal logics: for instance, there will be an 'S4' logic with axiom $\Diamond\Diamond A \rightarrow \Diamond A$ but without $\Box A \rightarrow \Box\Box A$. Secondly, the correspondence theorems do not involve just conditions on R but also include some interaction between it and $\sqsubseteq$. The general correspondence theorem below, the intuitionistic version of the $G^{k,l,m,n}$ schema [C], gives rise to a family of completeness results. Let R^n for $n \geq 0$ be defined as:

$$wR^{o}v \quad \text{iff} \quad w = v$$
$$wR^{n+1}v \quad \text{iff} \quad \exists u.\ wRu \text{ and } uR^{n}v$$

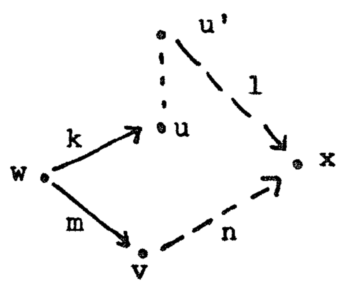
Let $G^{k,l,m,n}$ be the schema, for $k,l,m,n \geq 0$:

$$\Diamond^{k} \Box^{l} A \rightarrow \Box^{m} \Diamond^{n} A$$

Theorem 2.1 An intuitionistic modal frame validates $G^{k,l,m,n}$ iff the frame satisfies:

$$\text{if } wR^{k}u \text{ and } wR^{m}v \text{ then } \exists u' \sqsupseteq u\ \exists x(u'R^{l}x \text{ and } vR^{n}x)$$

Diagrammatically, the frame condition is:



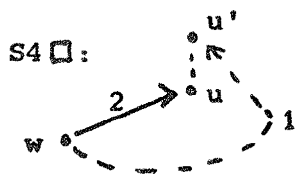
Note the presence of $\sqsubseteq$ between u and u'. Consider standard instances of the schema:

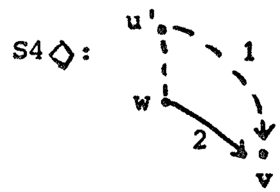
T$\Box$:	$\Box A \rightarrow A$	T$\Diamond$:	$A \rightarrow \Diamond A$
S4$\Box$:	$\Box A \rightarrow \Box\Box A$	S4$\Diamond$:	$\Diamond\Diamond A \rightarrow \Diamond A$
B$\Box$:	$A \rightarrow \Box\Diamond A$	B$\Diamond$:	$\Diamond\Box A \rightarrow A$

D : $\Box A \rightarrow \Diamond A$

S4.2 : $\Diamond\Box A \rightarrow \Box\Diamond A$

$T\Box(S4\Box, B\Box)$ corresponds to a different frame condition than $T\Diamond(S4\Diamond, B\Diamond)$. For instance, $S4\Box$ and $S4\Diamond$ are:





Addition of the axiom $A \vee \neg A$ (which corresponds to the frame condition: if $w \sqsubseteq w'$ then $w = w'$) means that $S4\Box$ and $S4\Diamond$ correspond to the same frame condition, transitivity of R (similarly, for T and B.) Thus, theorem 2.1 appears to capture the intuitionistic correlate of the classical correspondence theorem.

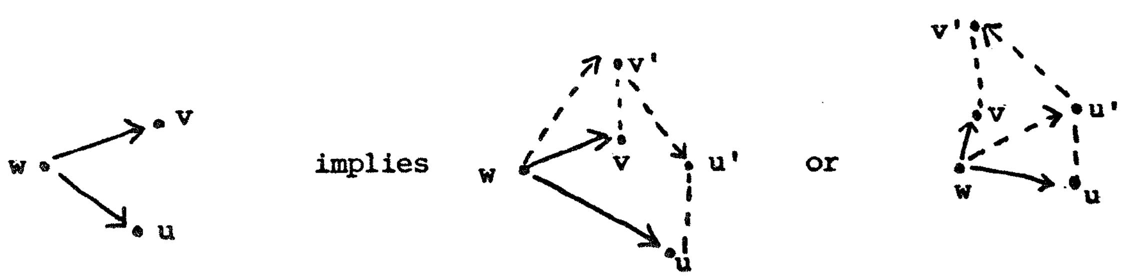
Of particular interest, given the computational motivation, are intuitionistic versions of the Diodorean modal systems [HC], S4.3 and S4.3.1. Classically, these correspond to linear time modal frames. Recall a standard classical connectedness axiom H (which when added to the modal system S4 results in S4.3):

$$\Diamond A \wedge \Diamond B \rightarrow \Diamond(A \wedge \Diamond B) \vee \Diamond(B \wedge \Diamond A)$$

<u>Theorem 2.2</u> An intuitionistic modal frame validates H iff the frame satisfies:

if wRv and wRu then $\exists v' \sqsupseteq v\ \exists u' \sqsupseteq u$ (wRv'Ru' or wRu'Rv')

Diagrammatically:



Again, note that addition of $A \vee \neg A$ collapses the frame condition to connectedness. Work is still in progress to find correspondence theorems for the variety of classically equivalent version of H and for the discreteness axiom (and its classical equivalents) of S4.3.1.

Work is also in progress generalizing theorem 2.1 in a way analogous to [Sa]. Also we are working on finite model properties (and complexity): standard filtration techniques do not easily work because of the existential nature of the frame conditions. Finally, we would like to obtain results without imposing the conditions on frames, especially the second.

References

[AS] B. Alpern and F. Schneider 'Defining liveness'. To appear in Information Processing Letters.

[B1] R. Bull 'A modal extension of intuitionistic logic' Notre Dame Journal of Formal Logic pp. 142-146 (1965).

[B2] R. Bull 'MIPC as the formalization of an intuitionist concept of modality' Journal of Symbolic Logic, pp. 609-616 (1966).

[B5] R. Bull and K. Segerberg 'Basic modal logic' in [GG] pp.1-88 (1984).

[C] B. Chellas 'Modal Logic' Cambridge University Press (1980).

[F1] G. Fischer Servi 'On modal logic with an intuitionist base' Studia Logica , pp.141-149 (1977).

[F2] G. Fischer Servi 'Semantics for a class of intuitionist modal calculi' in ed. M. Chiara 'Italian Studies in the Philosophy of Science' Reidel, Dordrecht, pp 59-72 (1981).

[GG] ed. D. Gabbay and F. Guenthner 'Handbook of Philosophical Logic Vol II' Reidel, Dordrecht, (1984).

[HC] G. Hughes and M. Cresswell 'An Introduction to Modal Logic' Methuen (1968).

[Ma] P. Martin-Löf 'Constructive mathematics and computer programming', in 'Logic, Methodology and Philosophy of Science VI' North Holland (1982).

[Mc] D. McCarty 'Realizability and recursive mathematics' Report CMU-CS-84-131, Dept. of Computer Science, Carnegie-Mellon (1984).

[Sa] H. Sahlqvist 'Completeness and correspondence in the first and second order semantics for modal logic' in ed. S. Kanger 'Procs. of 3rd Scandinavian Logic Symposium' pp 110-143, North-Holland (1975).

[Sm] M. Smyth 'Powerdomains and predicate transformers: a topological view' Research Report, Dept. of Computer Science, Edinburgh University SR-126-83 (1983).

[T] R. Thomason 'Combinations of tense and modality' in [GG] pp 135-166 (1984).

[V1] D.I. Vakarelov 'Simple Examples of Incomplete Logics' Comptes rendus de l'Academie Bulgare des Sciences, Tome 33, No. 5 (1980).

[V2] D.I. Vakarelov 'Intuitionistic Modal Logics Incompatible With the Law of the Excluded Middle' Studia Logica XL, 2 pp 103, 111, (1981).

[V3] D.I. Vakarelov 'An Application of Rieger-Nishimura Formulas to the Intuitionistic Modal Logics' Studia Logica, XLIV, 7 pp 79, 85 (1985).

[VB] J. Van Bentham 'Correspondence theory' in [GG] pp 167-248 (1984).