On a Question of H. Friedman

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Abstract

In this paper we answer a question of Friedman, providing an ω-separable model $\mathcal{M}$ of the $\lambda\beta\eta$-calculus. There therefore exists an α-separable model for any $\alpha \geq 0$. The model $\mathcal{M}$ permits no non-trivial enrichment as a partial order; neither does it permit an enrichment as a category with an initial object. The open term model embeds in $\mathcal{M}$: by way of contrast we provide a model which cannot embed in any non-trivial model separating all pairs of distinct elements.

1 Introduction

Separability is a recurring topic in the λ-calculus. It is usually defined syntactically, but there is also an interesting model-theoretic definition. Say that a subset A of an applicative structure $(X, \cdot)$ is *separable* if any function $f : A \to X$ is realised by some $\hat{f}$ in X, by which is meant, that for all a in A, $f(a) = \hat{f} \cdot a$. This idea first appears in work of Flagg and Myhill [FM].

They termed the concept "discreteness," employing a topological analogy; we prefer to extend the usual λ-calculus terminology. Harvey Friedman asked if there is an ω-separable model of the $\lambda\beta\eta$-calculus. (An applicative structure is α-*separable* iff every subset of cardinality strictly less than α is separable.) Flagg and

Myhill noticed that there is a close connection between separability and order: if two distinct elements of an applicative structure are separable, then they cannot be ordered in any partial ordering of the structure for which application is monotone. So a model in which any two distinct elements are separable admits no non-trivial such partial order. It can therefore not be constructed using the order-theoretic methods pioneered by Scott.

In Section 2 we answer Friedman's question positively, giving a term model construction of an ω-separable model of the $\lambda\beta\eta$-calculus. Friedman's question arose when he noticed [FM] that —provided ω-separable models exist— one could use the model-theoretic notion of saturation to provide models which are even α-*separable* for strongly inaccessible α. We repeat his argument below, using it to show the existence of an α^+-separable model of power 2^α, for every $\alpha \geq \omega$.

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Lehmann considered an extension of Scott's methods, using categories rather than partial orders ([Leh] —see also [Abr]). Therefore, in Section 3 we extend our consideration of enrichment by partial orders to enrichment by categorical structure. Finally, in Section 4 we consider a technical question. The construction of the model (essentially) proceeds by adding new elements to the open term model of the $\lambda\beta\eta$-calculus. We show that not every model can be enlarged in this way. Indeed we provide a model of the $\lambda\beta\eta$-calculus with two elements which are *absolutely inseparable*, in the sense that there is no homomorphism of combinatory algebras from the model to a combinatory algebra such that the images of the two elements are separable.

It can be argued that the construction does not provide a model sufficiently distinguished from those provided by Scott's methods: it is countable and Scott's methods (directly) provide uncountable models. This objection can be met, as it is easy to adapt the construction to provide uncountable models —just add enough constants; alternatively take countable submodels of Scott's models, for example those of all the recursively enumerable elements (e.g., see [Smy]). Even so, it seems not quite reasonable to ask for the model to be provided, up to isomorphism, by Scott's methods. It is conceivable that, although the model itself admits no non-trivial partial order, it can be embedded in a model which does. We conjecture there is a model of the $\lambda\beta\eta$-calculus which (considered as a combinatory algebra) cannot be embedded in any combinatory algebra admitting a non-trivial partial order. It may also be interesting to consider which combinatory algebras admit non-trivial partial orders; for example, it is not even known if the open term model of the $\lambda\beta\eta$-calculus does. We follow [Bar] for standard λ-calculus definitions and notation.

2 Construction of the Model

It is known to be consistent to add to the $\lambda\beta$-calculus a constant, Church's $\boldsymbol{\delta}$, which separates distinct closed β-normal forms (and also some terms containing $\boldsymbol{\delta}$ itself). One can adapt this to obtain a $\boldsymbol{\delta}$ separating non-convertible closed terms not containing $\boldsymbol{\delta}$. However, for the present purposes, the associated term model is deficient in two respects. The first is that it is not possible to separate non-convertible closed terms containing $\boldsymbol{\delta}$; the second is that the resulting model is not extensional. The solution to the first difficulty is to iterate the construction infinitely many times, adding a sequence of $\boldsymbol{\delta}$s; the solution to the second is to work with the $\lambda\beta\eta$-calculus, and to add a second sequence of constants, serving as "generic" arguments.

Let us now turn to the technical details. Given a set of constants C we write $\Lambda(C)$ for the set of λ-terms built up from variables, the elements of C and using application and abstraction in the usual way. We feel free to use the usual syntactic concepts and notation for these terms. Let $\boldsymbol{\delta}_i$ and a_i be two sequences of distinct constants, with no $\boldsymbol{\delta}_i$ equal to any $a_{i'}$. For $i \geq 0$, set $\Lambda_i = \Lambda(\{\boldsymbol{\delta}_j, a_j \mid j < i\})$ and let Λ_ω be the union of the Λ_i.

We wish to prove a Church-Rosser theorem, using the method of "parallel one-step reduction relations" introduced by Martin-Löf and Tait. For the $\lambda\beta\eta$-calculus, this reduction relation can be inductively defined by the following clauses:

1. $M \trianglerighteq M$

2. if $M \trianglerighteq M'$ then $\lambda x.M \trianglerighteq \lambda x.M'$

3. if $M \trianglerighteq M'$ and $N \trianglerighteq N'$ then $MN \trianglerighteq M'N'$

4. if $M \trianglerighteq M'$ and $N \trianglerighteq N'$ then $(\lambda x.M)N \trianglerighteq M'[x := N']$

5. if $M \trianglerighteq M'$ and x is not in $FV(M)$ then $\lambda x.Mx \trianglerighteq M'$

The two main properties of parallel one-step reduction relations are:

Substitution If $M \trianglerighteq M'$ and $N \trianglerighteq N'$ then $M[x := N] \trianglerighteq M'[x := N']$.
Diamond If $L \trianglerighteq M$ and $L \trianglerighteq N$ then for some K, $M \trianglerighteq K$ and $N \trianglerighteq K$.

The proof of Substitution is by induction on the structure of M, and cases on which inductive clause applies to show that $M \trianglerighteq M'$; the proof of Diamond is similar, and makes use of Substitution. Details are given for the $\lambda\beta$-calculus in [Bar] (the last clause of the inductive definition is then omitted); the extension to the $\lambda\beta\eta$-calculus is straightforward. This implies the Diamond property holds for $\lambda\beta\eta$-reduction, as that is the transitive closure of $\trianglerighteq$; the Church-Rosser theorem follows.

We now define parallel one-step reduction relations $\trianglerighteq_i$ for Λ_i, by induction on i. For $i = 0$ we take the above standard one for the $\lambda\beta\eta$-calculus. For $i > 0$ we take $\trianglerighteq_i$ to be the relation given inductively by the above clauses (but now applied to Λ_i), and:

6. for $j < i$, if M and N are closed terms in Λ_j and $M =_j N$ then $\boldsymbol{\delta}_j MN \trianglerighteq \boldsymbol{T}$

7. for $j < i$, if M and N are closed terms in Λ_j and $M \neq_j N$ then $\boldsymbol{\delta}_j MN \trianglerighteq \boldsymbol{F}$

(where is $=_i$ is the transitive, symmetric closure of $\trianglerighteq_j$).

It is easily seen that $\trianglerighteq_i$ $(i \geq 0)$ is an increasing sequence of relations, and the inclusions are conservative, in the sense that if $M \trianglerighteq_{i+1} M'$ for M in Λ_i, then $M \trianglerighteq_i M'$. Substitution holds for $\trianglerighteq_i$; one proves this by induction on i. The case $i = 0$ is already established; for $i > 0$ one proceeds as usual with two more cases for the two new inductive clauses, and these are easy as the term M is closed. With this one can prove the Diamond property. One again proceeds by induction on i; the ground case is already known and the inductive case proceeds as usual. The only new possibilities are that the inductive clauses showing that $L \trianglerighteq M$ and $L \trianglerighteq N$ are either both clause 6, both clause 7, or else one is either clause 6 or 7, and the other is clause 3 (applied twice); in the first two cases $M = N$, and in the third case one uses the conservativity of the inclusion of $\trianglerighteq_j$ in $\trianglerighteq_i$ (for $j < i$).

Now define relations on Λ_ω, taking $\trianglerighteq_\omega$ to be the union of the $\trianglerighteq_i$, and $\rightarrow_\omega$ to be its transitive closure, and $=_\omega$ to be its transitive, symmetric closure. Since the Diamond property holds for $\trianglerighteq_i$, it holds for $\trianglerighteq_\omega$ and $\rightarrow_\omega$, and we have Church-Rosser for $=_\omega$. Consequently, we have a non-trivial combinatory algebra $\mathcal{M} = (M, \cdot, k, s)$ whose elements are the equivalence classes of closed terms of Λ_ω, with respect to $=_\omega$, and with the evident definitions of application, k and s.

Lemma 1 *For M in Λ_i, if $M[x := a_i] \trianglerighteq_{i+1} N'$, then there is an N in Λ_i such that $M \trianglerighteq_i N$ and $N' = N[x := a_i]$.*

Proof The proof is a straightforward induction on the structure of M and by cases on which clause of the inductive definition of $\trianglerighteq_{i+1}$ applies to show $M[x := a_i] \trianglerighteq_{i+1} N'$. □

Theorem 1 *$\mathcal{M}$ is a non-trivial ω-separable extensional combinatory algebra.*

Proof We already know it is a non-trivial combinatory algebra. For extensionality, suppose that we have elements $[M]$ and $[N]$ such that $[M] \cdot u = [N] \cdot u$, for all u in $\mathcal{M}$. Then for some i, M and N are in Λ_i, and taking u to be $[a_i]$, we have $Ma_i =_{i+1} Na_i$. Applying the Church-Rosser property of $=_i$ and Lemma 1, we have $Mx =_i Nx$ for a variable x, and so $M =_i N$,(by η-conversion —use clause 5); so $[M] = [N]$, establishing extensionality. Finally, if we have k distinct elements $[M_j]$ ($j = 1, ..., k$), then for some i all the M_j are in Λ_i and are not related by $=_i$; one can then use $\boldsymbol{\delta}_i$ to separate them. □

As remarked above, Friedman pointed out [FM] a connection between separability and saturation: if an applicative structure $(X, \cdot)$ is ω-separable and α-saturated then it is α-separable. For if $f : A \to X$, where $A \subset X$, then $\hat{f}$ realises f iff it realises the type $\{x \cdot c_a = c_{f(a)} \mid a \in A\}$ (using the notation of [CK]).

Now we may apply Lemma 5.1.4 of [CK] to obtain, for any $\alpha \geq \omega$, an α^+-saturated elementary extension of the extensional combinatory algebra $\mathcal{M}$ of power 2^α. Applying Friedman's observation we find:

Corollary 1 *For any $\alpha \geq \omega$, there is an α^+-separable extensional combinatory algebra of power 2^α.*

3 Enrichment

We have already seen that the applicative structure $(M, \cdot)$ does not admit any non-trivial partial ordering with application monotone, that is, it cannot be non-trivially enriched as a partial order. Furthermore, it cannot even be enriched as a preorder (other than as one in which either no two distinct elements are related or else all elements are equivalent). Curiously, it can be non-trivially enriched as a category (meaning a category not equivalent to a set). Indeed:

Proposition 1 *Every non-trivial applicative structure can be non-trivially enriched as a category.*

Proof Let $(X, \cdot)$ be a non-trivial applicative structure. For the enrichment, we add endomorphisms: take $\mathcal{U}(x, y)$ to be empty if $x \neq y$, and otherwise to be the set of natural numbers. The identity is given by zero, and composition is given by addition, as is the action of application on morphisms. □

This proposition clearly holds much more generally, extending to single-sorted algebras over any signature. It would be interesting to know if any combinatory algebra can be so enriched (meaning as a combinatory algebra in the category of small categories).

The categories considered by Lehmann possess initial objects, although the functors are not required to preserve them. Taking the existence of an initial object as an extra requirement we find:

Proposition 2 *The applicative structure* $(M, \cdot)$ *cannot be non-trivially enriched as a category, if that category has an initial object.*

Proof Let $\perp$ be initial. Then for any element x there is an $f : \perp \to x$. Choose d so that $d \cdot \perp = x$ and $d \cdot x = \perp$. Then $(id_d \cdot f) : x \to \perp$ and $(id_d \cdot (id_d \cdot f)) : \perp \to x$. From the last, by initiality, $(id_d \cdot (id_d \cdot f)) = f$. But, by initiality again, $(id_d \cdot f) \circ f = id_\perp$. So we have: $f \circ (id_d \cdot f) = (id_d \cdot (id_d \cdot f)) \circ (id_d \cdot f) = (id_d \circ id_d) \cdot ((id_d \cdot f) \circ f) = id_x$. Therefore x, an arbitrary element, is isomorphic to $\perp$, the initial element, and so the enrichment is trivial. □

4 An Unembeddable Model

Suppose we have a non-trivial combinatory algebra $\mathcal{U}$ with distinct elements a,b and an element c such that the following three equations hold:

$$\begin{array}{rcll} c \cdot \boldsymbol{T} & = & \boldsymbol{T} & \\ c \cdot \boldsymbol{F} & = & \boldsymbol{F} & \\ c \cdot (u \cdot a) & = & c \cdot (u \cdot b) & \text{(for all u)} \end{array}$$

(where we do not distinguish between a combinator and its denotation). Then a and b cannot be separated. For suppose, for the sake of contradiction that, for some d, $d \cdot a = \boldsymbol{T}$ and $d \cdot b = \boldsymbol{F}$. Then: $\boldsymbol{T} = c \cdot \boldsymbol{T} = c \cdot (d \cdot a) = c \cdot (d \cdot b) = \boldsymbol{F}$.

Furthermore, if $\mathcal{U}$ is extensional, a and b are absolutely inseparable. This is because any homomorphism $\theta : \mathcal{U} \to \mathcal{V}$ preserves the equations. That is evident for the first two, where no free variables are involved; for the third we use the abstraction operator of combinatory logic. Let t be $\lambda^* yzx.y(xz)$. Then, $(t)^{\mathcal{U}} \cdot c \cdot a \cdot u = (t)^{\mathcal{U}} \cdot c \cdot b \cdot u$ (for all u). So, by extensionality, $(t)^{\mathcal{U}} \cdot c \cdot a = (t)^{\mathcal{U}} \cdot c \cdot b$. But then, for any v in $\mathcal{V}$:

$$\begin{array}{rcll} lcl\theta(c) \cdot (v \cdot \theta(a)) & = & (t)^{\mathcal{V}} \cdot \theta(c) \cdot \theta(a) \cdot v & (1) \\ & = & \theta((t)^{\mathcal{U}}) \cdot \theta(c) \cdot \theta(a) \cdot v & (2) \\ & = & \theta((t)^{\mathcal{U}} \cdot c \cdot a) \cdot v & (3) \\ & = & \theta((t)^{\mathcal{U}} \cdot c \cdot b) \cdot v & (4) \\ & = & \theta(c) \cdot (v \cdot \theta(b)) & (5) \end{array}$$

This proof extends to any equation (between applicative combinations of elements and variables); further, rather than extensionality, it would suffice if $\mathcal{U}$ was the combinatory algebra associated to a syntactical λ-model.

We now construct such an extensional model, using a term model construction. Let C be the ternary set of constants $\{c, a, b\}$. To make the first two equations true we wish to add reductions $c\boldsymbol{T} \to \boldsymbol{T}$ and $c\boldsymbol{F} \to \boldsymbol{F}$. For the third we want to add the reduction $a \to b$, but only "in the context of" an c. To this end we define two parallel one-step reduction relations. The first adds all reductions, ignoring c-contexts; the second uses the first and is sensitive to c-contexts.

For the first, let $\trianglerighteq_{(i)}$ be the relation on $\Lambda(C)$ given inductively by clauses 1-5 above and such that $c\boldsymbol{T} \trianglerighteq \boldsymbol{T}$, $c\boldsymbol{F} \trianglerighteq \boldsymbol{F}$ and $a \trianglerighteq b$. For the second, let $\trianglerighteq_{(ii)}$ be the relation on $\Lambda(C)$ given inductively by clauses 1-5 above and such that $c\boldsymbol{T} \trianglerighteq \boldsymbol{T}$, $c\boldsymbol{F} \trianglerighteq \boldsymbol{F}$ and:

8. if $M \trianglerighteq_{(i)} N$ then $cM \trianglerighteq_{(ii)} cN$.

Note that $\trianglerighteq_{(i)}$ includes $\trianglerighteq_{(ii)}$. It is easy to establish Substitution and then Diamond for $\trianglerighteq_{(i)}$. Using that, the corresponding properties for $\trianglerighteq_{(ii)}$ easily follow.

We may now form $\mathcal{U}$ as the collection of equivalence classes of $\Lambda(C)$-terms, where the equivalence is that generated by $\trianglerighteq_{(ii)}$ (with the usual definitions of application and k and s). This yields a non-trivial extensional combinatory algebra. Also, as a and b are normal forms (for the reduction relation associated to $\trianglerighteq_{(ii)}$), $[a]$ and $[b]$ are distinct. Finally $[a]$,$[b]$ and $[c]$ satisfy the above three equations, as we have the conversions: $c\boldsymbol{T} =_{(ii)} \boldsymbol{T}$, $c\boldsymbol{F} =_{(ii)} \boldsymbol{F}$ and $c(Ma) =_{(ii)} c(Mb)$ where $=_{(ii)}$ is the equivalence relation associated to $\trianglerighteq_{(ii)}$. We have proved:

Theorem 2 *$\mathcal{U}$ is a non-trivial extensional combinatory algebra with two absolutely inseparable elements.*

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References

[Abr] S. Abramsky.
Semantic Foundations for Applicative Multiprogramming. Proc. ICALP '83 (ed. J. Diaz), LNCS Vol.154, pp.1–15, 1983.

[Bar] H. Barendregt. *The Lambda Calculus.* North Holland, 1984.

[CK] C. C. Chang and H. J. Keisler. *Model Theory.* Studies in Logic, Vol.73. North Holland, 1973.

[FM] R. Flagg and J. Myhill. *A Type-Free System Extending (ZFC).* Annals of Pure and Applied Logic, (43), pp.79–97, 1989.

[Leh] D. Lehmann. *Categories for Fixpoint Semantics.* Proc. 17th. Annual Symp. on Foundations of Computer Science, IEEE, 1976 (also Warwick University Theory of Computation Report No.15.).

[Smy] M. Smyth. *Effectively Given Domains.* Th. Comp. Sci., (5), pp.257–274, 1977.