SIAM J. COMPUT.
Vol. 5, No. 3, September 1976

A POWERDOMAIN CONSTRUCTION*

G. D. PLOTKIN†

Abstract. We develop a powerdomain construction, $\mathscr{P}[\cdot]$, which is analogous to the powerset construction and also fits in with the usual sum, product and exponentiation constructions on domains. The desire for such a construction arises when considering programming languages with nondeterministic features or parallel features treated in a nondeterministic way. We hope to achieve a natural, fully abstract semantics in which such equivalences as $(p\ par\ q) = (q\ par\ p)$ hold. The domain $(D \to \text{Truthvalues})$ is not the right one, and instead we take the (finitely) generable subsets of D. When D is discrete they are ordered in an elementwise fashion. In the general case they are given the coarsest ordering consistent, in an appropriate sense, with the ordering given in the discrete case. We then find a restricted class of algebraic inductive partial orders which is closed under $\mathscr{P}[\cdot]$ as well as the sum, product and exponentiation constructions. This class permits the solution of recursive domain equations, and we give some illustrative semantics using $\mathscr{P}[\cdot]$.

It remains to be seen if our powerdomain construction does give rise to fully abstract semantics, although such natural equivalences as the above do hold. The major deficiency is the lack of a convincing treatment of the fair parallel construct.

1. Introduction. When one follows the Scott–Strachey approach to the semantics of programming languages, various constructions on domains arise naturally. These include sum, product and exponentiation constructions. Their use is illustrated in [12], [18]. Encountering languages with nondeterministic and parallel programming features induces the desire for a powerdomain construction analogous to the powerset construction on sets. Unfortunately, domains of the form $(D \to \text{Truthvalues})$ will not do—as will be seen—and we present here an alternative, rather more complicated proposal.

Milner [10], [11] handled nondeterminism by using oracles. Unfortunately, the resulting semantics does not give some intuitive equivalences since; for example, the programs $(p\ or\ q)$ and $(q\ or\ p)$ have different meanings in general. Let us say that two well-formed program fragments are behaviorally equivalent iff whenever embedded in a context to form a program, they give rise to the same behavior. Behavior itself is to be defined in some operational way. Relative to some such notion we say that a semantics is fully abstract iff behavioral and denotational equivalence coincide [11], [13], [19]. We would expect that $(p\ or\ q)$ and $(q\ or\ p)$ would be behaviorally equivalent. So Milner's semantics is not fully abstract.

Milner asked if there was a generalization to relations of the notion of a continuous function. Rather than consider relations $R \subseteq D \times E$ directly we define $\mathscr{P}[E]$, the powerdomain of E, and use continuous functions $R : D \to \mathscr{P}[E]$. As a result, we obtain a semantics in which the programs $(p\ or\ q)$ and $(q\ or\ p)$ always do have the same meaning. But it remains an open question whether we thus achieve a fully abstract semantics.

We begin by considering a simple language with a "nondeterministic branch" feature. In this setting it is quite natural to consider sets when looking for a

* Received by the editors April 30, 1975, and in revised form December 10, 1975.

† Department of Artificial Intelligence, University of Edinburgh, EH1 9NW Scotland. This work was supported by an SRC Grant.

semantics. Indeed, a straightforward construction is available which we believe has intuitive appeal. This construction forms the basis of the subsequent more general powerdomain construction.

The need for that is demonstrated by considering a more elaborate language with a simple parallel processing facility. In particular, one wants to find domains satisfying recursion equations involving the hypothetical powerdomain construction $\mathcal{P}[\cdot]$, just as equations involving $+$, $\times$ and $\rightarrow$ were considered in the deterministic case. Our approach is to use consistency criteria to determine the ordering of $\mathcal{P}[D]$, and to consider only certain subsets of D as candidate members of $\mathcal{P}[D]$. This gives a rather indirect definition of $\mathcal{P}[D]$. The main body of the paper considers a wide class of domains D for which it is possible to establish a direct definition of $\mathcal{P}[D]$, and examines some of its properties. This class allows us to define continuous functions and predicates analogous to some standard ones on sets and also to solve recursive domain equations involving $\mathcal{P}[\cdot]$. The final part of the paper applies these results by giving an illustrative semantics for the simple language considered. We also give one for a language of Milner's which has more extended multiprocessing features.

2. Establishing a definition. The first programming language considered has simple nondeterministic choice points. The illustrative programs in Fig. 1 are written in it. A formal definition will be given later.

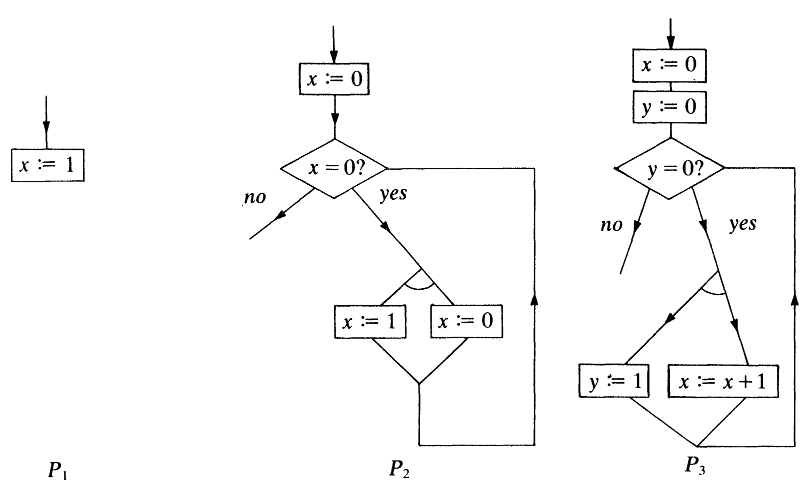

FIG. 1

In this language, states are integer vectors of the appropriate length, and it is clear how execution sequences are defined. From a given starting vector an execution can, in general, result in one of several possible final vectors and may even fail to terminate. For example, the first program P_1 always terminates with $x=1$; P_2 has many possible execution sequences. One does not terminate, but the others all terminate after varying amounts of time with $x=1$. An execution of P_3 either does not terminate or else terminates with $y=0$ and $x=$ an arbitrary number. It is therefore natural to let the meaning of a program, P, be a function.

DEFINITION. An inductive partial order (ipo) is a partial order with a bottom element in which every directed subset has a least upper bound.

Now let S be the set of possible states of the program P, and let $S_\perp = S \cup \{\perp\}$ be the ipo ordered by: $x \sqsubseteq y$ iff $x = \perp$ or $x = y$. The meaning of P will be a function p from S to $\mathcal{P}(S_\perp)$, the collection of subsets of $S_\perp$. We want $s' \in p(s)$ iff either $s' = \perp$ and, starting from s, P has a nonterminating execution sequence or else $s' \neq \perp$ and, starting from s, P has an execution sequence ending in s'.

Notice the use of ipo's here. Sometimes they are called cpo's—complete partial orders—instead. At the moment, we are using them for convenience instead of complete lattices: including a top element would, we believe, not permit so natural a development. Later, it seems to be essential to use ipo's instead of complete lattices. The construction of $S_\perp$ from S is an example of a general construction, which will be used again.

If the meaning of P_i is p_i $(i=1,2,3)$ then we have $p_1(\langle m, n\rangle) = \{\langle 1, n\rangle\}$, $p_2(\langle m, n\rangle) = \{\perp, \langle 1, n\rangle\}$ and $p_3(\langle m, n\rangle) = \{\perp\} \cup \{\langle j, 0\rangle \mid j \geq 0\}$. The use of functions rather than relations as meanings reflects the input-output asymmetry. Whichever is used, we feel it would be a mistake to take meanings in either $S \to \mathcal{P}(S)$ or $\mathcal{P}(S \times S)$ as we want to distinguish P_1 from P_2. That is, we do not want nondeterminism to mask nontermination. This difficulty with using the relational approach [1], [2], [7] to handle nondeterminism is noted by Milner.

How are we to define the ordering relation, $\sqsubseteq$, on meanings? Since each p is a function, $\sqsubseteq$ could be defined pointwise if only we had a definition of $\sqsubseteq$ on sets. Now in the deterministic case, the orderings arose because one introduced $\perp$ in order to be able to deal with partial functions as total functions. Then the order reflected the fact of partialness and was induced by: $\perp \sqsubseteq$ anything.

In our case the analogous course is to define the ordering on sets elementwise, just as it was defined coordinatewise on previous occasions. So, for sets X and Y, in $\mathcal{P}(S_\perp)$, we put $X \sqsubseteq Y$ iff Y is obtained from X by replacing $\perp$ in X by some nonempty set. That is, $X \sqsubseteq Y$ if ($\perp \notin X$ and $Y = X$) or else ($\perp \in X$ and $Y \supseteq (X - \{\perp\})$ and Y is nonempty). There is a neater definition. Let D be an arbitrary ipo. The *Milner ordering* $\sqsubseteq_M$ on $\mathcal{P}(D)$ is defined by:

$$X \sqsubseteq_M Y \quad \text{iff} \quad (\forall x \in X. \exists y \in Y. x \sqsubseteq y) \quad \text{and} \quad (\forall y \in Y. \exists x \in X. x \sqsubseteq y).$$

It is the ordering on subsets of D induced elementwise by the ordering of D; in particular it is the same as the one defined above for $\mathcal{P}(S_\perp)$ when $D = S_\perp$. For arbitrary ipo's $\sqsubseteq_M$ is only a quasi-order, for if $x \sqsubseteq y \sqsubseteq z$ are three distinct elements in D, then $\{x, z\} =_M \{x, y, z\}$, where $=_M$ is the equivalence induced by $\sqsubseteq_M$.

We pause to consider a useful fact about $\sqsubseteq_M$. Suppose $f: D \to E$ where D and E are ipo's.

Notation. If $X \subseteq D$, then $f(X) =_{\text{def}} \{f(x) \mid x \in X\}$.

Now it is not hard to see that if $X, Y \subseteq D$ and $X \sqsubseteq_M Y$, then $f(X) \sqsubseteq_M f(Y)$, as f is monotonic.

In the present case ($S_\perp$), it is tempting to allow $X \sqsubseteq Y$ if $X \subseteq Y$ although one can hardly then claim $\sqsubseteq$ as a "less defined than" ordering. But then $\{\perp, s\} \sqsubseteq \{s\}$, by the elementwise criterion and $\{s\} \sqsubseteq \{\perp, s\}$ by the subset one for any $s \in S$, and it follows that P_1 and P_2 have equivalent meanings—against our wishes. So the temptation will be resisted. However, both $\cup$ and $\{\ \}$ will be seen to be continuous functions. So a proof system based on $\sqsubseteq$ with symbols for $\cup$ and $\{\ \}$ can be used to prove both subset and membership relations if required, as $X \subseteq Y$ iff $X \cup Y = Y$ and $s \in X$ iff $\{s\} \subseteq X$.

Having considered the ordering on meanings, which meanings should be considered? Again we want to know which members of $\mathscr{P}(S_\perp)$ to allow. Since $p(s) \neq \emptyset$, even if P does not terminate on s, we exclude the empty set.

Now the set of all initial segments of execution sequences of a given nondeterministic program P, starting from a given state, will form a tree. The branching points will correspond to the choice points in the program. Since there are always only finitely many alternatives at each such choice point, the branching factor of the tree is always finite. That is, the tree is finitary. Now Konig's lemma says that if every branch of a finitary tree is finite, then so is the tree itself. In the present case this means that if every execution sequence of P terminates, then there are only finitely many execution sequences. So if an output set of P is infinite it must contain $\perp$. Therefore we require infinite sets in $\mathscr{P}[S_\perp]$ to contain $\perp$.

DEFINITION. $\mathscr{P}[S_\perp]$, the powerdomain of $S_\perp$, is the set $\{X \subseteq S_\perp | X \neq \emptyset$, and X is finite or contains $\perp\}$ ordered by $\sqsubseteq_M$.

Notice that we can define $\mathscr{P}[D]$ for any denumerable discrete ipo D analogously:

DEFINITION. An ipo D is *ω-discrete* iff it is denumerable and for x, y in D, $x \sqsubseteq y$ iff $x = \perp$ or $x = y$. The ω-discrete ipo's are just those of the form $X_\perp$, where X is a denumerable set.

DEFINITION. $\mathscr{P}[D]$ is the set $\{X \subseteq D | X \neq \emptyset$, and X is finite or contains $\perp\}$ ordered by $\sqsubseteq_M$, when D is ω-discrete.

Clearly $S_\perp$ is ω-discrete and so, for example, is $N_\perp$ defined similarly from the set of nonnegative integers. Other examples are $\mathbb{1} = \{\perp\}$, the one-point lattice and $\mathbb{O} = \{\perp, \top\}$ the two-point complete lattice and $\mathbb{T} = \text{Truthvalues} = \{\perp, \textit{true}, \textit{false}\}$. The elements of ω-discrete ipo's can often be taken to correspond to discrete items of output. As such, the same justification given for the definition of $\mathscr{P}[S_\perp]$ applies also to that of $\mathscr{P}[D]$ when D is ω-discrete.

THEOREM 1. *$\mathscr{P}[D]$ is an* ipo *in which every element is the limit of an increasing sequence of finite elements. The functions $\cup$ and $\{\cdot\}$ are continuous.*

Proof. First $\sqsubseteq_M$ is a partial order, for if $X =_M Y$ and $x \in X$, then either $x \neq \perp$ and $x \in Y$ as $X \sqsubseteq_M Y$ or $x = \perp$ and $x \in Y$ as $X \sqsupseteq_M Y$. Therefore $X \subseteq Y$ and similarly $Y \supseteq X$, which proves antisymmetry. Transitivity and reflexivity are clear. The set $\{\perp\}$ is the $\perp$ of $\mathscr{P}[D]$.

We now prove that if X, Y are sets in $\mathscr{P}[D]$ with an upper bound Z and X does not contain $\perp$, then $X \sqsupseteq_M Y$. For if $y \in Y$, there is a $z \in Z$ and an $x \in X$ such that $y \sqsubseteq z \sqsupseteq x$. As $x \neq \perp$, $y \sqsubseteq x$. Similarly if $x \in X$, there is a $z \in Z$ and a $y \in Y$ such that $x \sqsubseteq z \sqsupseteq y$. As $x \neq \perp$, $x \sqsupseteq y$. By the definition of $\sqsubseteq_M$, $X \sqsupseteq_M Y$.

Now suppose $\mathscr{X} \subseteq \mathscr{P}[D]$ is directed. If there is a set $X \in \mathscr{X}$ which does not contain $\perp$, then, by the above remarks, $X = \bigsqcup \mathscr{X}$.

Otherwise every set in $\mathscr{X}$ contains $\perp$. Let $X^* = \bigcup \mathscr{X}$. If $X \in \mathscr{X}$, then as $X \subseteq X^*$ and $\perp \in X$, $X \sqsubseteq_M X^*$. If Y is any upper bound of $\mathscr{X}$, then $X^* \sqsubseteq_M Y$. For if $x \in X^*$, then there is an X in $\mathscr{X}$ which contains x and so, as $X \sqsubseteq_M Y$, there is a $y \in Y$ such that $x \sqsubseteq y$. Conversely, if $y \in Y$, then $y \sqsupseteq \perp \in X^*$. Thus we have proved that $X^* = \bigsqcup \mathscr{X}$. So $\mathscr{P}[D]$ is an ipo.

If $X \in \mathscr{P}[D]$, and is not finite, let $X_n = \{\perp, s_1, \cdots, s_n\}$ where $s_1, \cdots$ is a listing of X. Then $X = \bigcup_{n \geqq 0} X_n = \bigsqcup_{n \geqq 0} X_n$. It is easy to verify that the functions $\cup$ and $\{\ \}$ considered as members of $\mathscr{P}^2[D] \to \mathscr{P}[D]$ and $D \to \mathscr{P}[D]$, respectively, are continuous, which concludes the proof. □

It should be emphasized that if $\mathscr{X}$ is a finite directed subset of $\mathscr{P}[D]$, $\bigsqcup \mathscr{X}$ and $\bigcup \mathscr{X}$ need not be equal.

So we have our powerdomain construction for ω-discrete domains, such as $S_\perp$. Some subsidiary questions arise. First, if we had excluded $\varnothing$ but included everything else, an ipo would still have been obtained although not every element would have been a limit of finite sets. At the present level the choice of $\mathscr{P}[D]$ for ω-discrete D is governed by considerations of economy. We shall see later that a similar choice in the general case gives rise to a difficulty in our theory. Notice by the way that $\mathscr{P}[D]$ could not have been taken as $(D \to \mathbb{T})$, where a function f represented the set $\{d \in D \mid f(d) = \mathit{true}\}$ as no function would have represented $\{\perp\}$.

Let us put these ideas together to give the semantics for our simple language. Its grammar is specified by:

$$\nu ::= x_1 \mid \cdots \mid x_n$$

$$\tau ::= \nu \mid 0 \mid 1 \mid (\tau_1 + \tau_2) \mid (\tau_1 \dot{-} \tau_2)$$

$$\pi ::= (\nu := \tau) \mid (\pi_1; \pi_2) \mid (\pi_1 \ \mathit{or}\ \pi_2) \mid (\mathit{if}\ \nu\ \mathit{then}\ \pi_1\ \mathit{else}\ \pi_2) \mid (\mathit{while}\ \nu\ \mathit{do}\ \pi).$$

following the style of Scott and Strachey [18]. Here ν ranges over the set of variables, τ over terms and π over statements.

We take $S = \{\langle m_1, \cdots, m_n \rangle \mid m_i \geqq 0\}$.

For $p: S \to \mathscr{P}[S_\perp]$ let $p_\perp : (S_\perp \to \mathscr{P}[S_\perp])$ be defined by

$$p_\perp(s) = \begin{cases} \{\perp\}, & (s = \perp), \\ p(s), & (s \neq \perp). \end{cases}$$

Notice here that although S is just a set, $S \to \mathscr{P}[S_\perp]$ can be considered to be an ipo with the induced pointwise ordering. Here and elsewhere we always intend the interpretation as an ipo. When D and E are taken to be ipo's, $D \to E$ is always to be the ipo of all continuous functions from D to E.

It is not hard to show that the function which sends p to $p_\perp$ is continuous.

The combinator $*$: $(S \to \mathscr{P}[S_\perp])^2 \to (S \to \mathscr{P}[S_\perp])$ is defined by: $p^*q = \lambda s\{s'' \mid \exists s' \in p(s).s'' \in q_\perp(s')\}$. It is tedious to show directly that $*$ is continuous, although it is clearly well-defined. An indirect proof of continuity will be given in § 6.

The combinator COND: $N_\perp \to \mathscr{P}[S_\perp] \to \mathscr{P}[S_\perp] \to \mathscr{P}[S_\perp]$ is defined by:

$$\text{COND}\ x\ X\ Y = \begin{cases} \{\perp\}, & (x = \perp), \\ X, & (x = 0), \\ Y, & (x > 0). \end{cases}$$

The semantics of the language is then given by two functions $\mathcal{V}$: Terms $\rightarrow (S \rightarrow N)$ and $\mathcal{M}$: Statements $\rightarrow (S \rightarrow \mathcal{P}[S_\perp])$ where:

$\mathcal{V}[\![x_i]\!](s) = (s)_i$,
$\mathcal{V}[\![0]\!](s) = 0$,
$\mathcal{V}[\![1]\!](s) = 1$,
$\mathcal{V}[\![(\tau_1 + \tau_2)]\!](s) = \mathcal{V}[\![\tau_1]\!](s) + \mathcal{V}[\![\tau_2]\!](s)$,
$\mathcal{V}[\![(\tau_1 \dot{-} \tau_2)]\!](s) = \mathcal{V}[\![\tau_1]\!](s) \dot{-} \mathcal{V}[\![\tau_2]\!](s)$,
$\mathcal{M}[\![(x_i := \tau)]\!] = \lambda s \in S.\{\langle (s)_1, \cdots, (s)_{i-1}, \mathcal{V}[\![\tau]\!](s), (s)_{i+1}, \cdots, (s)_n\rangle\}$,
$\mathcal{M}[\![(\pi_1; \pi_2)]\!] = \mathcal{M}[\![\pi_1]\!]^* \mathcal{M}[\![\pi_2]\!]$,
$\mathcal{M}[\![(\pi_1 \text{ or } \pi_2)]\!] = \lambda s \in S.\mathcal{M}[\![\pi_1]\!](s) \cup \mathcal{M}[\![\pi_2]\!](s)$,
$\mathcal{M}[\![\text{if } x_i \text{ then } \pi_1 \text{ else } \pi_2]\!] = \lambda s \in S.\text{COND}(s)_i(\mathcal{M}[\![\pi_1]\!](s))(\mathcal{M}[\![\pi_2]\!](s))$,
$\mathcal{M}[\![\text{while } x_i \text{ do } \pi]\!] = Y(\lambda p \in (S \rightarrow \mathcal{P}[S_\perp]).\lambda s \in S.\text{COND}((s)_i)((\mathcal{M}[\![\pi]\!]^* p)(s))(\{s\}))$.

A thorough treatment would now investigate the connection between this semantics and an operational one, as envisaged above, with extensions to more elaborate languages. One would also like to see proof systems based on $\sqsubseteq$ and consider their relationship, both practical and theoretical, to those based on the relational approach [1], [2], [7]. However, we press on to situations requiring a more involved $\mathcal{P}$.

Now suppose we introduce a parallel operation into the language by adding the clauses, $\pi ::= (\pi_1 \text{ par } \pi_2)$.

The execution of a program is now conceived of as the performance of a sequence of elementary operations on the state. Statements of the form $(\pi_1 \text{ par } \pi_2)$ perform an arbitrary interleaving of the elementary operations of π_1 and π_2.

The meanings of statements can no longer be functions. For example if $\pi_1 = ((x := 0); (x := x+1))$, and $\pi_2 = (x := 1)$, then although π_1 and π_2 have the same meaning as functions, $(\pi_1 \text{ par } \pi_2)$ and $(\pi_2 \text{ par } \pi_2)$ have quite different meanings. (This example is taken from [10], [11].)

Suppose meanings are entities $r \in R$, say, and π has meaning r. The execution of π on $s \in S$ either results in a final state, or results in some state and reaches a point in π from which computation can be resumed. From a given s, several of these possibilities can obtain. This is modeled by assuming that $r(s) \in S$ or $r(s) \in S \times R$ and we want $R \cong S \rightarrow \mathcal{P}[S_\perp + (S_\perp \times R)]$.

This domain R of resumptions is inspired by Milner's domain of processes. It is simpler, and is useful when side-effects have a fixed limited form arising from, say, a predetermined number of common registers or buffers (when a slightly different S is needed).

So we do indeed need to solve recursive domain equations involving $\mathcal{P}$. As another example, we will be able to eliminate the use of oracles in giving the semantics of Milner's multiprocessing language. This will use a domain, P, of nondeterministic processes, satisfying the equation:

$$P = V \rightarrow \mathcal{P}[L \times V \times P],$$

where V is a domain of values and L of L-values.

The method of investigation is to find a wide class of ipo's D for which $\mathcal{P}[D]$ can be defined and in which such equations can be solved. This is done by defining first the members of $\mathscr{P}[D]$, then the ordering of $\mathscr{P}[D]$ for an arbitrary ipo D and then finding a class of ipo's in which this definition leads to pleasant and useful properties.

In the case of the $\rightarrow$ construction, continuity successfully cut down the cardinality of $D \rightarrow E$ and gave a smooth mathematical theory into which computability fitted nicely [17], [18]. For $\mathscr{P}[\cdot]$ we want an abstract notion of generable set. When $D = S_\perp$ we expect this will coincide with the finite nonempty subsets of $S_\perp$ and the infinite ones containing $\perp$, at least if the finitary tree nature of the execution sequences is a general feature. Not only ω-discrete domains figure as output domains. Suppose one added a print instruction to the language. Output would then be a sequence of integers, and the possible sequences of a given program form the branches of a finitary tree. Here outputs belong to N^ω the ipo of finite and infinite sequences of nonnegative integers with the subsequence ordering.

Such considerations suggest a definition. Let Ω be the ipo of finite and infinite sequences of 0's and 1's, with the subsequence ordering. This is the oracle ipo [8]; it is the infinite binary tree with limit points added.

DEFINITION. A subset X of D is *finitely generable* iff there is a continuous function $f : \Omega \rightarrow D$ such that $X = Bd(f) =_{\text{def}} \{f(\omega) | \omega \text{ infinite}\}$.

There is, actually, a connection with the oracle idea. Suppose we have a semantics in which the denotation of a program takes as argument an oracle ω, in Ω, which is used to determine the direction to take at choice points. Then the program will deliver a result, $f(\omega)$, in its output domain D. In the kind of semantics advocated here, the denotation of that program would not take an oracle as argument but would be, essentially, $Bd(f)$. (Certain complications, considered below, prevent it being exactly $Bd(f)$.)

When D is ω-discrete, its finitely generable subsets form the domain of $\mathscr{P}[D]$. For one can see that all the sets in $\mathscr{P}[D]$ are finitely generable. Conversely suppose $X = Bd(f)$ is a finitely generable subset of D. Let T be the finitary tree consisting of these finite elements, ω, of Ω such that $f(\omega) = \perp$. If T is finite, then X is finite. If T is infinite, it has an infinite branch by König's lemma, and so $\perp \in X$. In either case X is in $\mathscr{P}[D]$. When D is N^ω, the output sets considered above are finitely generable as can be seen from the connection with oracles.

The definition does not lose any sets by considering only Ω. It can be shown without much difficulty that if T is any finitary tree with limit points added, and $f : T \rightarrow D$, then $Bd(f) =_{\text{def}} \{f(\omega) | \omega \text{ a maximal member of } T\}$ is finitely generable. It follows that every finite nonempty subset of any ipo D is finitely generable, as is every denumerable nonempty subset containing a lower bound of itself.

Notice that if $X \subseteq D$ is finitely generable and $g : D \rightarrow E$, then $g(X)$ is also finitely generable. That is, the definition is consistent with that of $\mathscr{P}[E]$ for ω-discrete E in that, starting with finitely generable sets and applying suitable continuous functions, only members of $\mathscr{P}[E]$ are reached. Unfortunately, that does not determine the finitely generable sets. For example, if $N_\top = \{\perp, \top, 0, 1, \cdots\}$ is the lattice formed from N by adding $\perp$ and $\top$, then $X = \{\top, 0, 1, \cdots\}$ is not finitely generable, but if E is ω-discrete and $f : N_\top \rightarrow E$, then $f(X)$ is finite.

It is not unreasonable therefore to include only the finitely generable sets in the domain $\mathcal{P}[D]$. However there are reasons for considering other sets. Consider the program:

$$P = ((y := 0); (x := 0)); ((y := 1)\ par\ (while\ y\ do\ x := x+1)).$$

According to the parallel construct sketched above, P can either stop with x set to an arbitrary integer or fail to terminate. Now postulate a *fair* parallel construct which at any point will never restrict all its computation to one branch, unless the other has terminated. Clearly, every fair computation sequence of P terminates, but the set of results of the fair computation sequences is not finitely generable, being $\{\langle 0, 1\rangle, \langle 1, 1\rangle, \langle 2, 1\rangle, \cdots\}$. It may be possible to handle this by considering the *denumerably generable* sets instead of the finitely generable ones. These are obtained by replacing Ω in the definition by N^ω. In the ω-discrete case this leads to the alternative mentioned above. Another approach is not to consider *all* fair computation sequences, but rather parallel constructs which give rise to finitely generable sets of fair computation sequences. The importance of the problem is that the assumption of fairness is needed to prove the absence of deadlock of certain co-operating processes [5]. This poses one of the more interesting problems left unresolved in this paper.

At any rate we will form $\mathcal{P}[D]$ from $\mathcal{F}[D]$ the collection of finitely generable subsets of D, and an as yet undetermined ordering, $\sqsubseteq$. Since $\sqsubseteq$ will be only a quasi-ordering in general, $\mathcal{P}[D]$ will actually be the collection of equivalence classes with the induced partial ordering.

It remains to define this ordering. In the cases of $+$, $\times$ and $\rightarrow$ not only is the induced ordering natural, it is also, in a sense, necessary. For example in the case of $D_1 \times D_2$ one expects that the projection functions $\pi_i : D_1 \times D_2 \rightarrow D_i$ $(i = 1, 2)$ are monotonic. So if $\langle d_1, d_2\rangle \sqsubseteq \langle e_1, e_2\rangle$, then $d_i = \pi_i(\langle d_1, d_2\rangle) \sqsubseteq \pi_i(\langle e_1, e_2\rangle) = e_i$ $(i = 1, 2)$.

Notation. Let D and E be ipo's and suppose E is ω-discrete. If $f : D \rightarrow \mathcal{P}[E]$, then $\tilde{f} : \mathcal{F}[D] \rightarrow \mathcal{P}[E]$ is defined by:

$$\tilde{f}(X) = \bigcup \{f(x) \mid x \in X\} \qquad (X \in \mathcal{F}[D]).$$

We omit the justification of the definition of $\tilde{f}$ for the moment.

DEFINITION. Let D be an ipo. The quasi-order $\sqsubseteq$ on $\mathcal{F}[D]$ is defined by:

$$X \sqsubseteq Y \text{ iff } (\forall \omega\text{-discrete } E. \forall f : D \rightarrow \mathcal{P}[E]. \tilde{f}(X) \sqsubseteq_M \tilde{f}(Y)), \qquad (X, Y \in \mathcal{F}[D]).$$

In the case of $\mathcal{P}[D]$ we would expect that if $f : D \rightarrow \mathcal{P}[E]$, where E is ω-discrete, then $\tilde{f} : \mathcal{F}[D] \rightarrow \mathcal{P}[E]$ would be monotonic.

So if $\sqsubseteq$ is our hypothetical ordering we expect that, for X, Y in $\mathcal{F}[D]$,

$$(1) \qquad (X \sqsubseteq_M Y \text{ implies } X \sqsubseteq Y) \quad \text{and} \quad (X \sqsubseteq Y \text{ implies } X \sqsubseteq Y).$$

The first part arises from the naturalness of the elementwise criterion. The second part arises from the above discussion and our expectation that in the case of ω-discrete domains, $\sqsubseteq$ will be $\sqsubseteq_M$.

We pause here to verify that $\tilde{f}$ is indeed well-defined. That is if X is finitely generable, then $\tilde{f}(X)$ is in $\mathcal{P}[D]$.

Suppose $X = Bd(g)$ for some $g : \Omega \to D$. Let T be the finitary tree consisting of those finite ω in Ω such that $\bot \in f(g(\omega))$. If T is finite, then $\tilde{f}(X)$ is finite. If T is infinite then, by König's lemma T has an infinite branch $\omega_0, \omega_1, \omega_2, \cdots$. Since $\bot \in f(g(\omega_i))$ for all $i \geqq 0$, it follows that $\bot \in \bigsqcup_{i \geqq 0} f(\omega_i) = f(\bigsqcup_{i \geqq 0} \omega_i) \subseteq \tilde{f}(X)$. As $\tilde{f}(X) \neq \emptyset$, it follows that in either case $\tilde{f}(X) \in \mathscr{P}[D]$.

In some cases the second part of (1) can be computationally justified. If E is the output domain, in some programming language, and D is part of a value domain, then a set $X \subseteq D$ might arise as the set of possible values of a variable at a stage in the computation. With everything else fixed, we get a continuous function $f : D \to \mathscr{P}[E]$ where $f(d)$ is the set of possible outputs arising from $d \in X$. Then $\tilde{f}(X)$ is interpreted as the set of all possible outputs, which should vary monotonically with X.

Here are some elementary facts about $\sqsubseteq$:

THEOREM 2. (i) *For X, Y in $\mathscr{F}[D]$, $X \sqsubseteq Y$ iff $\forall f : D \to \mathbb{O}. f(X) \sqsubseteq_M f(Y)$.*

(ii) *If $X \sqsubseteq Y$ and $f : D \to E$, then $f(X) \sqsubseteq f(Y)$ (X, Y in $\mathscr{F}[D]$).*

(iii) *If $X \sqsubseteq_M Y$, then $X \sqsubseteq Y$ (X, Y in $\mathscr{F}[D]$).*

(iv) *If D is ω-discrete and $X \sqsubseteq Y$, then $X \sqsubseteq_M Y$ (X, Y in $\mathscr{F}[D]$).*

Proof. (i) Let X, Y be in $\mathscr{F}[D]$. Suppose $f : D \to \mathbb{O}$. Define $g : D \to \mathscr{P}[\mathbb{O}]$ by $g(d) = \{f(d)\}$. Then if $X \sqsubseteq Y$, $\tilde{g}(X) \sqsubseteq_M \tilde{g}(Y)$, so $f(X) \sqsubseteq_M f(Y)$.

Conversely suppose $f(X) \sqsubseteq_M f(Y)$ for all $f : D \to \mathbb{O}$ and $g : D \to \mathscr{P}[E]$ where E is ω-discrete. Suppose e is in $\tilde{g}(X)$ and $e \neq \bot$. Define $h : \mathscr{P}[E] \to \mathbb{O}$ by:

$$h(Z) = \begin{cases} \bot & (e \notin Z), \\ \top & (e \in Z), \end{cases} \quad (Z \text{ in } \mathscr{P}[E]).$$

This does define a continuous function. Taking $f = h \circ g$, we find that $\top \in f(X) \sqsubseteq_M f(Y) = h(g(Y))$. Therefore e is in $\tilde{g}(Y)$ which is half the proof that $\tilde{g}(X) \sqsubseteq_M \tilde{g}(Y)$. If $\bot \in \tilde{g}(X)$, we are finished. Otherwise $\tilde{g}(X)$ is finite since it is in $\mathscr{P}[E]$ and so for each x in X, $g(x)$ is finite and $\bot \notin g(x)$.

Define $h : \mathscr{P}[E] \to \mathbb{O}$ by:

$$h(Z) = \begin{cases} \top & (Z = g(x) \text{ for some } x \in X), \\ \bot & (\text{otherwise}), \end{cases} \quad (Z \text{ in } \mathscr{P}[E]).$$

Then h is continuous and with $f = h \circ g$, $\{\top\} = f(X) \sqsubseteq_M f(Y) = h(g(Y))$. Thus $\tilde{g}(Y) \subseteq \tilde{g}(X)$ which concludes the proof of part (i).

(ii) Suppose $X \sqsubseteq Y$ for X, Y in $\mathscr{F}[D]$ and $f : D \to E$. If $g : E \to \mathbb{O}$, then $g(f(X)) \sqsubseteq_M g(f(Y))$ by part (i). So $f(X) \sqsubseteq f(Y)$ by part (i).

(iii) Suppose $X \sqsubseteq_M Y$ for X, Y in $\mathscr{F}[D]$. If $f : D \to E$, then $f(X) \sqsubseteq_M f(Y)$. Therefore $X \sqsubseteq Y$ by part (i).

(iv) Suppose D is ω-discrete and $X \sqsubseteq Y$ for X, Y in $\mathscr{F}[D]$. Define $f : D \to \mathscr{P}[D]$ by: $f(d) = \{d\}$. Then

$$\begin{aligned} X &= \tilde{f}(X) \\ &\sqsubseteq_M \tilde{f}(Y) \quad (\text{by the definition of } \sqsubseteq) \\ &= Y. \end{aligned}$$

□

Unfortunately (1) does not determine $\sqsubseteq$. For example let $X_0 = \{\perp\} \cup \{0^n 1^\omega | n \geqq 0\}$, $X_1 = X_0 \cup \{0^\omega\}$, using an obvious notation, define finitely generable subsets of Ω. We claim that $X_0 \simeq X_1$ where $\simeq$ is the equivalence induced by $\sqsubseteq$. But clearly, $X_0 \neq_M X_1$. So $X \sqsubseteq Y$ does not imply $X \sqsubseteq_M Y$, in general.

To see that $X_0 \simeq X_1$, let $f: D \to \mathbb{O}$. Clearly $f(X_0) \subseteq f(X_1)$. If $\perp \in f(X_1)$, then $\perp = f(\perp) \in f(X_0)$; if $\top \in f(X_1)$, then $\top = f(d)$ for some d in X_1. If $d \in X_0$, $\top \in f(X_0)$. Otherwise, $d = 0^\omega$. Then for some n, $\top = f(0^n)$. Therefore $\top \in f(X_0)$ in this case too. Therefore $f(X_0) = f(X_1)$ and so, by the theorem, $X_0 \simeq X_1$.

So our analysis has left us in rather a quandary as to the definition of $\sqsubseteq$ on $\mathscr{F}[D]$. One expedient would be to search for other necessary conditions on $\sqsubseteq$ by looking for functions other than those of the form $\tilde{f}$. These might be provided by considering programming features which allow some inspection of all possible execution sequences of subprograms. We do not consider such "AND" programming here (but see Manna [9]).

Since we would like our semantics to be fully abstract, we choose the coarser relation $\sqsubseteq$. By Theorem 2(iv) this is consistent with our earlier definition of $\mathscr{P}[D]$ for D ω-discrete. We conjecture that choosing $\sqsubseteq_M$ instead would not even give an ipo, but lack a counter-example. We were not able to develop a satisfactory independently justified definition of $\sqsubseteq$ for sequence domains like Ω or N^ω. However $\sqsubseteq$ does have some pleasant properties for these domains. Suppose X, Y are two members of $\mathscr{F}[D]$, where D is such a sequence domain. If every member of Y is infinite then $X \sqsubseteq Y$ iff $X \sqsubseteq_M Y$; if every member of X is infinite, $X \sqsubseteq Y$ iff X and Y are identical. The second claim follows from the first which will be proved later.

We now have to within isomorphism a definition of $\mathscr{P}$:

DEFINITION. $\mathscr{P}[D] \cong \langle \mathscr{F}[D]/\simeq, \sqsubseteq/\simeq \rangle$ where $\mathscr{F}[D]/\simeq$ is the set of finitely generable subsets of D modulo $\simeq$, the equivalence induced by the quasi-order $\sqsubseteq$, and $\sqsubseteq/\simeq$ is the induced partial order.

When D is ω-discrete we retain the earlier definition as the standard one; later we will fix on a standard one for other cases.

As yet we have no guarantee that $\mathscr{P}[D]$ is an ipo when D is not discrete, let alone that we may solve recursive domain equations using $\mathscr{P}$. We need a suitable class of ipo's in which such guarantees can be obtained. We begin by showing that $\mathscr{P}[D]$ is a well-defined ipo if D is finite. That is carried out in § 3. By considering limits of infinite sequences of finite ipo's we arrive at a category SFP. This category is described in § 4. It turns out that if D is an SFP object so is $\mathscr{P}[D]$. Indeed $\mathscr{P}$ is a functor on SFP. We also obtain good internal descriptions of $\mathscr{P}[D]$. This work occupies § 5 and part of § 6. In § 6 we show that various associated functions are continuous. These functions are useful for defining various denotational semantics. Finally in § 7 we can show that a wide variety of recursive domain equations are solvable in SFP. Our main method is categorical, and we also have a universal domain method. It is possible to use Scott's $\mathscr{P}(\omega)$, [17], but not in a particularly satisfactory way. The last section, § 8, applies this work by giving some illustrative semantics as mentioned above.

3. Finite powerdomains. If D is finite, then $\mathscr{P}[D]$ as described in the last section is certainly a partial order. It has a least element which is the equivalence class of $\{\bot\}$ under $\simeq$. Since D is finite it follows that $\mathscr{P}[D]$ is indeed an ipo. However we really want a more pleasant description of $\mathscr{P}[D]$. This is given by the next theorem.

DEFINITION. Let E be an ipo. A subset X of E is *convex* iff $(\forall x, y, z \in E.\ (x \sqsubseteq y \sqsubseteq z$ and $x \in X$ and $z \in X)$ implies $y \in X)$.

The convex closure operator, Con, is defined on subsets of E by, $\mathrm{Con}(X) =_{\mathrm{def}} \{y \in E \mid \exists x, z \in X. x \sqsubseteq y \sqsubseteq z\}$ $(X \subseteq E)$.

THEOREM 3. (i) Con *is a closure operator on any* ipo E. *For any* $X, Y \subseteq E$, $\mathrm{Con}(X)$ *is the least convex set containing* X, $X =_M \mathrm{Con}(X)$, *and* $X =_M Y$ *iff* $\mathrm{Con}(X) = \mathrm{Con}(Y)$.

(ii) *If* D *is finite*, $X \sqsubseteq Y$ *iff* $X \sqsubseteq_M Y$ $(X, Y \in \mathscr{F}[D])$.

(iii) *If* D *is finite*, $\mathscr{P}[D] \cong \langle \{\mathrm{Con}(X) \mid X \subseteq D, X \neq \varnothing\}, \sqsubseteq_M \rangle$.

Proof. (i) The proof is straightforward.

(ii) By Theorem 2(iii) we know that $X \sqsubseteq_M Y$ implies $X \sqsubseteq Y$. Suppose $X \sqsubseteq Y$ and $x \in X$. Define $f: D \to \Phi$ by:

$$f(d) = \begin{cases} \top, & (d \sqsupseteq x), \\ \bot, & (d \not\sqsupseteq x). \end{cases}$$

Then f is continuous. As $\top \in f(X) \sqsubseteq_M f(Y)$ by Theorem 2(i), $y \sqsupseteq x$ for some y in Y.

Suppose $y \in Y$. Now define $f: D \to \mathbb{O}$ by

$$f(d) = \begin{cases} \bot, & (d \sqsubseteq y), \\ \top, & (d \not\sqsubseteq y). \end{cases}$$

Again f is continuous and now $\bot \in f(Y) \sqsupseteq_M f(X)$. So $x \sqsubseteq y$ for some x in X. Therefore $X \sqsubseteq_M Y$ as required.

(iii) By definition, $\mathscr{P}[D] \cong \langle \mathscr{F}[D]/\simeq, \sqsubseteq/\simeq \rangle$. Clearly the finitely generable subsets of D are the nonempty ones. By (i) and (ii), Con assigns to each X in $\mathscr{F}[D]$ the maximal member of its equivalence class under $\simeq$. It therefore induces a 1–1 correspondence between $\mathscr{F}[D]/\simeq$ and $\{\mathrm{Con}(X) \mid X \subseteq D, X \neq \varnothing\}$ which is an isomorphism of the partial orders. □

From now on we will take $\langle \{\mathrm{Con}(X) \mid X \subseteq D, X \neq \varnothing\}, \sqsubseteq_M \rangle$ as our standard definition of $\mathscr{P}[D]$ when D is finite. This is consistent with our previous standard definition for the discrete case.

Let us consider the extension of functions to finite powerdomains. Suppose $f: D \to E$ for finite ipo's D and E. Define $\hat{f}: \mathscr{P}[D] \to \mathscr{P}[E]$ by:

$$\hat{f}(X) = \mathrm{Con}(f(X)), \qquad (X \in \mathscr{P}[D]).$$

The function $\hat{f}$ is monotonic and therefore continuous; for, if $X \sqsubseteq_M Y$, $\hat{f}(X) =_M f(X) \sqsubseteq_M f(Y) =_M \hat{f}(Y)$ by Theorem 2.2. Note that $\hat{I}_D$ is $I_{\mathscr{P}[D]}$ where I_D and $I_{\mathscr{P}[D]}$ are the identities on D and $\mathscr{P}[D]$, respectively. Finally, extension commutes with composition; for if D, E, F are finite ipo's, $f: D \to E$ and $g: E \to F$, then for $X \in \mathscr{P}[D]$, $\hat{g}(X) =_M g(X)$, so $\hat{f}(\hat{g}(X)) =_M \hat{f}(g(X)) = \mathrm{Con}(f(g(X))) = \widehat{f \circ g}(X)$.

The finite ipo's allow us to show that even if D is a lattice, $\mathcal{P}[D]$ need not be. Let $\mathbb{T}_\top = \{\bot, \text{true}, \text{false}, \top\}$ be the lattice of truth values and take $D = \mathbb{T}_\top \times \mathbb{T}_\top$. Let $a = \langle \mathit{true}, \bot \rangle$, $b = \langle \mathit{false}, \bot \rangle$, $c = \langle \bot, \mathit{true} \rangle$ and $d = \langle \bot, \mathit{false} \rangle$. Let $A = \{a, b\}$, $B = \{c, d\}$, $C = \{a \sqcup c, c \sqcup d\}$ and $D = \{a \sqcup d, b \sqcup c\}$. Then one can check that C and D are incomparable minimal upper bounds of A and B in $\mathcal{P}[D]$. Therefore $\mathcal{P}[D]$ is not a lattice. Indeed it is not even a semilattice, which we take to be a closed subset of a lattice, as in [17]. So converting $\mathcal{P}[D]$ into a lattice would require one to add many points—not just a top element. It is not clear to the author how to keep these separate from the bona fide elements.

4. The category SFP. In this section we present the class of domains over which our powerdomain construction works. They are certain limits in the category IPO. They will form a category SFP and $\mathcal{P}$ will be a functor from SFP to SFP. An alternate characterization of the SFP objects in terms of their order structure will provide'a priori reasons for their usefulness.

Perhaps we could comment on our use of category theory. No deep theorems of category theory are used. Rather, it allows a systematic development of the material. We cannot give such a development entirely within Scott's $\mathcal{P}(\omega)$, [17].

We begin by considering the relevant limits in IPO.

DEFINITION. IPO-P is the category whose objects are the ipo's and whose morphisms $p : D \to E$ are pairs $\langle \varphi, \psi \rangle$ where $\varphi : D \to E$ and $\psi : E \to D$. Composition is defined by:

$$q \circ p = \langle (q)_1 \circ (p)_1, (p)_2 \circ (q)_2 \rangle.$$

The identity on D is $\langle I_D, I_D \rangle$ where $I_D : D \to D$ is the usual identity. It will also be called I_D and we rely on context to settle the ambiguity. Composition is continuous with respect to the induced componentwise ordering on morphisms.

The category IPO-P is useful because it has the interesting subcategory, IPO-PR. If $p : D \to E$ let $p^\dagger = \langle p_2, p_1 \rangle$. We have the law, $(q \circ p)^\dagger = p^\dagger \circ q^\dagger$, and also $I_D^\dagger = I_D$. A pair $p : D \to E$ is a *projection* (of E onto D) iff $p^\dagger \circ p = I_D$ and $p \circ p^\dagger \sqsubseteq I_E$, under the induced componentwise ordering. This agrees with the usual definition of projection. The composition of two projections is a projection as are the identities. Note that the set of projections between two ipo's forms an ipo under the induced ordering.

DEFINITION IPO-PR is the subcategory of IPO-P with the same objects and with projections as morphisms.

If D and E are isomorphic in IPO-PR, then they are isomorphic in IPO, that is, as ipo's.

The use of these derived categories is suggested by the work of Reynolds [14], [15] and Wand [20], [21].

The category IPO-PR has direct limits of sequences. We give a definition which is, essentially, taken from [6].

DEFINITION. A *directed sequence* (in IPO-PR) is a family $\mathcal{D} = \langle D_m, p_{mn} \rangle$ of ipo's D_m $(m \geqq 0)$ together with projections $p_{mn} : D_m \to D_n$ $(0 \leqq m \leqq n)$ such that $p_{mm} = I_{D_m}$ and $p_{mn} \circ p_{lm} = p_{ln}$ when $0 \leqq l \leqq m \leqq n$.

DEFINITION. A *cone* from a directed sequence $\mathcal{D} = \langle D_m, p_{mn}\rangle$ to an ipo D is a family $\langle r_m\rangle$ of projections $r_m : D_m \to D$ such that:

$$r_m \circ p_{lm} = r_l \qquad (0 \leqq l \leqq m).$$

Such a cone is *universal* iff whenever $\langle r'_m\rangle$ is a cone from $\mathcal{D}$ to an ipo D' then there is a unique mediating projection $r' : D \to D'$ such that

$$r'_m = r' \circ r_m \qquad (m \geqq 0).$$

In this case, D is a *direct limit* of D and we write: $D = \lim_{\to} \mathcal{D}$.

It follows from the uniqueness of the mediating projections that a direct limit is unique, up to isomorphism, and the mediating projection from one direct limit to another is an isomorphism.

LEMMA 1. *Let $\mathcal{D} = \langle D_m, p_{mn}\rangle$ be a directed sequence and let $\langle r_m\rangle$ and $\langle r'_m\rangle$ be cones from $\mathcal{D}$ to D and D', respectively. Then $\langle r'_m \circ r^\dagger_m\rangle$ is an increasing sequence. The cone $\langle r_m\rangle$ is universal iff $\bigsqcup_{n\geqq 0} r_n \circ r^\dagger_n = I_D$. If $\langle r_m\rangle$ is universal, then the unique mediating morphism $r' : D \to D'$ is $r' = \bigsqcup_{n\geqq o} r'_n \circ r^\dagger_n$.*

Proof. Suppose $n \geqq m$. Then $r'_m \circ r^\dagger_m = (r'_n \circ p_{mn}) \circ (r_n \circ p_{mn})^\dagger = r'_n \circ p_{mn} \circ p^\dagger_{mn} \circ r_n \sqsubseteq r'_n \circ I_{D_m} \circ r_n = r'_n \circ r_n$. Therefore $\langle r'_m \circ r^\dagger_m\rangle$ is an increasing sequence.

Suppose $\langle r_m\rangle$ is universal. Since $\langle r_m\rangle$ is a cone from $\mathcal{D}$ to D there is a unique mediating projection $r : D \to D$ such that $r_m = r \circ r_m$ $(m \geqq 0)$. Since I_D is a mediating projection, $r = I_D$. We show that $\bigsqcup_{n\geqq 0} r_n \circ r^\dagger_n$ is also a mediating projection. It then follows that it too is I_D.

Now, $(\bigsqcup_{n\geqq 0} r_n \circ r^\dagger_n) \circ r_m = \bigsqcup_{n\geqq m} r_n \circ r^\dagger_n \circ (r_n \circ p_{mn}) = \bigsqcup_{n\geqq m} r_n \circ p_{mn} = r_m$. Therefore $\bigsqcup_{n\geqq 0} r_n \circ r^\dagger_n$ is indeed the mediating projection I_D.

Conversely, suppose $\bigsqcup_{n\geqq 0} r_n \circ r^\dagger_n = I_D$ and let $r' : D \to D'$ be $\bigsqcup_{n\geqq 0} r'_n \circ r^\dagger_n$. We show that r' is a mediating projection from D to D' and that if r'' is any mediating projection, then $r'' = r'$.

First,

$$\begin{aligned} r' \circ r_m &= \Big(\bigsqcup_{n\geqq 0} r'_n \circ r^\dagger_n\Big) \circ r_m = \bigsqcup_{n\geqq m} r'_n \circ (r^\dagger_n \circ r_n \circ p_{mn}) \\ &= \bigsqcup_{m\geqq n} r'_n \circ p_{mn} \\ &= r'_m \qquad (m \geqq 0). \end{aligned}$$

Therefore r' is a mediating projection from D to D'.

Next, suppose r'' is such a mediating projection. Then,

$$\begin{aligned} r'' &= r'' \circ \Big(\bigsqcup_{n\geqq 0} r_n \circ r^\dagger_n\Big) \\ &= \bigsqcup_{n\geqq 0} (r'' \circ r_n) \circ r^\dagger_n \\ &= \bigsqcup_{n\geqq 0} r'_n \circ r^\dagger_n \\ &= r'. \end{aligned}$$

□

We can now show that IPO-PR has direct limits of sequences.

THEOREM 4. *Let $\mathscr{D}=\langle D_m, p_{mn}\rangle$ be a directed sequence where $p_{mn}=\langle \varphi_{mn}, \psi_{nm}\rangle$. Let D_∞ be the set of vectors, $\{\langle d_m\rangle | m \geqq 0$ and $d_m=\psi_{nm}(d_m)$ if $0 \leqq m \leqq n\}$ with the pointwise ordering: $d \sqsubseteq e$ iff $\forall m \geqq 0.(d)_m \sqsubseteq (e)_m$ $(d, e \in D)$. Then D is an ipo. Define $i_m : D_m \to D$, $j_m : D \to D_m$ $(m \geqq 0)$ by:*

$$(i_m(d_m))_n = \begin{cases} \varphi_{mn}(d_m), & (n \geqq m), \\ \psi_{mn}(d_m), & (n < m), \end{cases}$$
$$i_m(d) = (d)_m.$$

Then i_m and j_m are indeed continuous and $r_m = \langle i_m, j_m\rangle$ is a projection. Further, $\langle r_m\rangle$ is a universal cone and so $D = \lim_{\to} \mathscr{D}$.

Proof. The proof is a straightforward point-by-point verification. Universality is proved using the criterion given by Lemma 1. We omit the details which can be taken over from the usual proofs for complete lattices, as given in [16], [14], [21]. □

DEFINITION. The category SFP (Sequences of Finite inductive Partial orders) has as objects those ipo's $D = \lim_{\to} \mathscr{D}$, where $\mathscr{D} = \langle D_m, p_{mn}\rangle$ is a directed sequence of finite ipo's D_m. Its morphisms $f: D \to E$ are the continuous functions with the usual composition.

Every finite ipo is an SFP object as it is a trivial limit of finite ipo's.

The categories SFP-P and SFP-PR are defined analogously to IPO-P and IPO-PR and are actually full subcategories of them. In other words if D and E are SFP-P (SFP-PR) objects, then the set of morphisms from D to E is the same in SFP-P (SFP-PR) as it is in IPO-P (IPO-PR). The notions of directed sequence, cone, universality and limit in SFP-PR are defined analogously to the corresponding notions in IPO-PR.

We now turn to an alternate characterization of SFP objects.

DEFINITION. An element d in an ipo D is *finite* iff whenever $X \subseteq D$ is directed and $d \sqsubseteq \bigsqcup X$, then $d \sqsubseteq x$ for some x in X.

DEFINITION. An ipo D is *algebraic* iff for any x in D the set $\{d | d \sqsubseteq x$ and d is finite$\}$ is directed and has least upper bound x. D is *ω-algebraic* iff it is algebraic and has denumerably many finite elements.

DEFINITION. Let X be a subset of an ipo D. An element u of D is a *minimal upper bound* of X iff it is an upper bound of X and it is not strictly greater than any other upper bound of X; $\mathscr{U}(X)$ is defined to be the set of all minimal upper bounds of X. $\mathscr{U}(X)$ is a *complete* set of upper bounds of X iff whenever u is an upper bound of X, then $u \sqsupseteq v$ for some v in $\mathscr{U}(X)$. $\mathscr{U}^*(X)$ is defined to be the least set containing X and closed under $\mathscr{U}$.

If an ipo D is algebraic and X is a finite set of finite elements, then every element of $\mathscr{U}(X)$ is finite. For suppose $u \in \mathscr{U}(X)$. Since X is a subset of $\{d | d \sqsubseteq u$ and d finite$\}$ and that set is directed there is an upper bound, u', of X in the set. Since u is a minimal upper bound of X it must be u' and so u is finite as u' is.

THEOREM 5. (i) *An* ipo *D is an* SFP-*object iff it is ω-algebraic, and, whenever X is a finite set of finite elements of X, then $\mathscr{U}(X)$ is a complete set of upper bounds of X and $\mathscr{U}^*(X)$ is finite.*

(ii) *The category* SFP-PR *has direct limits of directed sequences.*

Proof. (i) Every finite ipo clearly satisfies the conditions. We show that they are preserved under direct limits of directed sequences in IPO-PR. Suppose $\mathcal{D} = \langle D_m, p_{mn}\rangle$ is a directed sequence in IPO-PR, and the D_m satisfy the conditions. Let D_∞ and $r_m = \langle i_m, j_m\rangle$ be as described in Theorem 4.

First we show that D_∞ is ω-algebraic. Each element of the form $i_m(d_m)$, where d_m is a finite element of D_m, is finite. For if $X \subseteq D$ is directed and $i_m(d_m) \sqsubseteq \bigsqcup X$, then $d_m = j_m \circ i_m(d_m) \sqsubseteq \bigsqcup_{x\in X} j_m(x)$. So as d_m is finite there is an $x \in X$ such that $d_m \sqsubseteq j_m(x)$. Then $i_m(d_m) \sqsubseteq i_m \circ j_m(x) \sqsubseteq x$. So $i_m(d_m)$ is indeed finite.

Further every element in D_∞ is a least upper bound of a directed set of such elements. For if $d \in D_\infty$, then

$$
\begin{aligned}
d &= \bigsqcup_{m\geqq 0} i_m(j_m(d))\\
&= \bigsqcup_{m\geqq 0} i_m(\bigsqcup\{d_m \in D_m \mid d_m \text{ finite and } d_m \sqsubseteq j_m(d)\}) \qquad \text{(as } D_m \text{ is algebraic)}\\
&= \bigsqcup\{i_m(d_m) \mid m \geqq 0,\ d_m \in D_m,\ d_m \text{ finite and } d_m \sqsubseteq j_m(d)\},
\end{aligned}
$$

and the set on the right is a directed set of finite elements of D_∞.

So D_∞ must be algebraic and its finite elements are those of the form $i_m(d_m)$ where $m \geqq 0$ and d_m is a finite member of D_m. Since each D_m is ω-algebraic, there are denumerably many such elements and so D_∞ is also ω-algebraic.

We now consider the operation $\mathcal{U}$. Let it be $\mathcal{U}_m$ in D_m $(m \geqq 0)$ and $\mathcal{U}_\infty$ in D_∞. Then if $X_m \subseteq D_m$, $\mathcal{U}_\infty(i_m(X_m)) = i_m(\mathcal{U}_m(X_m))$. (We are using a notation for function application which was defined in § 2.) For it is straightforward to check that if u is a minimal upper bound of $i_m(X_m)$, then so is $j_m(u)$ of X_m and $u = i_m \circ j_m(u)$. Conversely if u is a minimal upper bound of X_m, then so is $i_m(u)$ of $i_m(X_m)$.

Now suppose $X \subseteq D$ is a finite set of finite elements of D. Then $X = i_m(X_m)$ for some finite set of finite elements of some D_m. Therefore, $\mathcal{U}_\infty(X) = i_m(\mathcal{U}_m(X_m))$. If u in D is an upper bound of X, then $j_m(u)$ is an upper bound of X_m. Therefore there is a v in $\mathcal{U}_m(X_m)$ such that $v \sqsubseteq j_m(u)$. Then $i_m(v) \in \mathcal{U}_\infty(X)$ and $i_m(v) \sqsubseteq u$. So $\mathcal{U}_\infty(X)$ is complete.

Next we show that $\mathcal{U}^*_\infty(X)$ is finite. First we define $\mathcal{U}^r$ for any ipo E and $r \geqq 0$:

$$
\begin{aligned}
&\mathcal{U}^0(Y) = \varnothing,\\
&\mathcal{U}^{r+1}(Y) = \bigcup\{\mathcal{U}(Y') \mid Y' \subseteq \mathcal{U}^r(Y)\} \cup Y \qquad (Y \subseteq E).
\end{aligned}
$$

Then for $Y \subseteq E$ we have $\mathcal{U}^r(Y) \subseteq \mathcal{U}^{r+1}(Y) \subseteq \mathcal{U}^*(Y)$ $(r \geqq 0)$. If $\mathcal{U}^r(Y) = \mathcal{U}^{r+1}(Y)$ for some $r \geqq 0$, then $\mathcal{U}^*(Y) = \mathcal{U}^r(Y)$.

We now show by induction on r that $\mathcal{U}^r_\infty(i_m(Y)) = i_m(\mathcal{U}^r_m(Y))$ for all $Y \subseteq D_m$. It is clear for $r = 0$. For $r+1$ we have,

$$
\begin{aligned}
\mathcal{U}^{r+1}_\infty(i_m(Y)) &= \bigcup\{\mathcal{U}_\infty(Y') \mid Y' \subseteq \mathcal{U}^r_\infty(i_m(Y))\} \cup i_m(Y)\\
&= \bigcup\{\mathcal{U}_\infty(Y') \mid Y' \subseteq i_m(\mathcal{U}^r_m(Y))\} \cup i_m(Y) \qquad \text{(by induction hypothesis)}
\end{aligned}
$$

$$= \bigcup\{\mathcal{U}_\infty(i_m(Y'))\,|\,Y' \subseteq \mathcal{U}^r_m(Y)\} \cup i_m(Y)$$

$$= \bigcup\{i_m(\mathcal{U}_m(Y'))\,|\,Y' \subseteq \mathcal{U}^r_m(Y)\} \cup i_m(Y)$$

(by a previous remark)

$$= i_m(\mathcal{U}^{r+1}_m(Y)) \qquad (Y \subseteq D_m).$$

Now, since $\mathcal{U}^*_m(X_m)$ is finite there is an r such that $\mathcal{U}^r_m(X_m) = \mathcal{U}^{r+1}_m(X_m) = \mathcal{U}^*_m(X_m)$. Therefore $\mathcal{U}^r_\infty(X) = i_m(\mathcal{U}^r_m(X_m)) = i_m(\mathcal{U}^{r+1}_m(X_m)) = \mathcal{U}^{r+1}_\infty(X)$. Therefore $\mathcal{U}^*_\infty(X) = \mathcal{U}^r_\infty(X) = i_m(\mathcal{U}^*_m(X_m))$, by the above remarks, and so $\mathcal{U}^*_\infty(X)$ is finite. We have therefore shown that D satisfies all the conditions.

We now show that if D satisfies all the conditions, then it is the limit of a directed sequence of finite ipo's.

Let D have finite elements, $\bot = e_0, e_1, \cdots$.

Let $D_m = \langle \mathcal{U}^*_D(\{e_0, e_1, \cdots, e_m\}), \sqsubseteq\rangle$ where $\sqsubseteq$ is inherited from D ($m \geqq 0$). Then D_m is a finite ipo. Define $p_{mn} : D_m \to D_n$ $(0 \leqq m \leqq n)$ by:

$$\varphi_{mn}(d_m) = d_m (d_m \in D_m),$$

$$\psi_{nm}(d_n) = \bigsqcup\{x \in D_m | x \sqsubseteq d_n\} \qquad (d_n \in D_n),$$

$$p_{nm} = \langle \varphi_{mn}, \psi_{nm}\rangle \qquad (0 \leqq m \leqq n).$$

Then ψ_{nm} is well-defined as the set on the RHS (right-hand side) is directed as it contains $\bot$ and by the properties of $\mathcal{U}$. Both φ_{mn} and ψ_{nm} are monotonic and therefore continuous. One can check that each p_{mn} is a projection and $\langle D_m, p_{mn}\rangle$ is a directed sequence. Define $r_m : D_m \to D$ by:

$$i_m(d_m) = d_m \qquad (d_m \in D_m),$$

$$j_m(d) = \bigsqcup\{x \in D_m | x \sqsubseteq d\} \qquad (d \in D,\ m \geqq 0),$$

$$r_m = \langle i_m, j_m\rangle.$$

As before j_m is well-defined and is monotonic and therefore continuous. Clearly i_m is well-defined and continuous. One can check that $\langle r_m\rangle$ is a cone from $\mathcal{D} = \langle D_m, p_{mn}\rangle$ to D.

Further,

$$\bigsqcup_{m \geqq 0} i_m \circ j_m(d) = \bigsqcup_{m \geqq 0} \bigsqcup\{x \in D_m | x \sqsubseteq d\} = d \qquad (d \in D).$$

Therefore by Lemma 1, $D = \lim_\to \mathcal{D}$, as required.

(ii) Suppose $\mathcal{D}$ is a directed sequence in SFP-PR. Then it is also a directed sequence in IPO-PR. It therefore has a direct limit, D, in IPO-PR. The argument in part (i) of the proof shows that D is actually an SFP object. As SFP-PR is a full subcategory of IPO-PR, D is also the direct limit in SFP-PR of $\mathcal{D}$. □

The internal characterization of SFP objects given by Theorem 5 allows us to argue for the reasonableness of the category SFP. First, all denotational semantics for programming languages, published till now, use ω-algebraic ipo's. The example given in § 3 shows that we should probably not expect consistent completeness. Our axioms for $\mathcal{U}$ give a kind of substitute.

It may be that at some time we shall want a concept of continuous ipo and a powerdomain construction for it. Universal domains provide a candidate construction, as will be seen. However the author does not know a good definition of the notion of a continuous ipo. Finally, considering ω-algebraic ipo's rather than just algebraic ones provides a framework for computability considerations.

5. Powerdomains in the category SFP. In this section we obtain sufficient conditions for $\mathscr{P}[D]$ to be an ipo. Indeed we show that if D is an SFP object, so is $\mathscr{P}[D]$. As a byproduct it turns out that instead of taking equivalence classes in $\mathscr{P}[D]$ we can choose maximal members of the equivalence classes. With the aid of a topological closure operator and the convex closure operator, Con, defined in § 3, we get an internal characterization of the maximal equivalence classes. We then go on to obtain similar characterizations of the finite elements of $\mathscr{P}[D]$ and l.u.b.'s of sequences in $\mathscr{P}[D]$. Finally we can provide evidence that the $\sqsubseteq$ ordering on sets of sequences (of integers, say) is not too unnatural. It turns out that, in many cases, $\sqsubseteq$ is just $\sqsubseteq_M$.

Let D be any SFP object. Then by definition we have a directed sequence $\mathscr{D}=\langle D_m, p_{mn}\rangle$ of finite ipo's such that $D=\lim_{\rightarrow}\mathscr{D}$. Let $\langle r_m\rangle=\langle\!\langle i_m, j_m\rangle\!\rangle$ be the universal cone from $\mathscr{D}$ to D. Let $\hat{\mathscr{D}}=\langle\mathscr{P}[D_m], \hat{p}_{mn}\rangle$ where $\hat{p}_{mn}=\langle\hat{\varphi}_{mn}, \hat{\psi}_{nm}\rangle$ $(m\leqq n)$. Let $E=\lim_{\rightarrow}\hat{\mathscr{D}}$ with universal cone, $\langle\tilde{r}_m\rangle=\langle\tilde{i}_m, \tilde{j}_m\rangle$, say. By establishing an isomorphism between E and $E'=\langle\mathscr{F}[D]/\simeq, \sqsubseteq/\simeq\rangle$ we shall see, by Theorem 5, that $\mathscr{P}[D]$ is an SFP object. The technique used is to define functions $\xi: E\rightarrow\mathscr{F}[D]$ and $\eta:\mathscr{F}[D]\rightarrow E$ which will induce the isomorphism. They are defined by:

$$\xi(e)=\{d\in D\,|\,\forall m\geqq 0. j_m(d)\in e_m\}\qquad (e\in E),$$
$$\eta(X)=\langle\mathrm{Con}\,(j_m(X))\rangle_{m=0}^{\infty}\qquad (X\in\mathscr{F}[D]).$$

We should show that they are well-defined. Suppose $e\in E$. Consider the finitary tree whose nodes are sequences $\langle d^{(1)},\cdots,d^{(n)}\rangle$ $(n\geqq 0)$ with $d^{(m)}\in e_m$ $(0<m\leqq n)$ and $\psi_{(m+1)m}(d^{(m+1)})=d^{(m)}$ $(0<m<n)$. If T is its completion, considered as an ipo, define $f: T\rightarrow E$ by:

$$f(\omega)=\begin{cases}\perp & (\omega \text{ has length } 0),\\ i_n((\omega)_n & (\omega \text{ has length } n>0),\\ \bigsqcup_{n>0} i_n((\omega)_n) & (\omega \text{ is infinite}).\end{cases}$$

Then $\xi(e)$ is $Bd(f)$, the boundary of f, and so is finitely generable.

As for η, we have:

$$\begin{aligned}\hat{\psi}_{mn}(\eta(X)_m)&=\mathrm{Con}\,(\psi_{mn}(j_m(X)))\\ &=\mathrm{Con}\,(j_n(X))\\ &=\eta(X)_n\quad\text{for } m\geqq n.\end{aligned}$$

In the arguments that follow, we often use König's lemma in a particular way. Suppose $\langle X_n\rangle_{n=0}^{\infty}\in E$ for $n\geqq 0$ and $u^{(n)}\in X_n$ for $n\geqq 0$. Then there is an x in D such that for all $n\geqq 0$, $j_n(x)\in X_n$ and there is an $m\geqq n$ such that $\psi_{mn}(u^{(m)})=j_n(x)$. To see this, consider the finitary tree whose nodes are sequences $\langle d^{(1)},\cdots,d^{(n)}\rangle$ $(n\geqq 0)$ with $d^{(m)}$ in X_m $(0<m\leqq n)$, with $\psi_{(m+1)m}(d^{(m+1)})=d^{(m)}$.$(0<m<n)$ and

such that there is a $p \geqq n$ such that $d^{(n)} = \psi_{pn}(u^{(p)})$. As it is infinite, it follows from König's lemma that it has an infinite branch, $\omega_0, \omega_1, \cdots$. Let $x = \bigsqcup_{n>0} i_n((\omega_n)_n)$. Then x is well-defined and has the required properties.

LEMMA 2. (i) *If* $e \sqsubseteq e'$, *then* $\xi(e) \sqsubseteq_M \xi(e')$ $(e, e' \in E)$.

(ii) *If* $X \sqsubseteq Y$, *then* $\eta(X) \sqsubseteq \eta(Y)$ $(X, Y \in \mathscr{F}[D])$.

(iii) $\eta \circ \xi = I_E$.

(iv) *If* $X \in \mathscr{F}[D]$, *then* $\xi \circ \eta(X) \supseteq X$, $\xi \circ \eta(X) \sqsupseteq_M X$ and $\xi \circ \eta(X) \simeq X$.

Proof. (i) Suppose $x \in \xi(e)$. Then for all $n \geqq 0$ there are $u^{(n)} \in (e')_n$ such that $u^{(n)} \sqsupseteq j_n(x)$. Applying König's lemma in the form described we obtain a $y \in D$ such that for all $n \geqq 0$, $y_n \in (e')_n$ and there is an $m \geqq n$ such that $y_n = \psi_{mn}(u^{(m)})$. As $u^{(m)} \sqsupseteq j_m(x)$, it follows that $y_n \sqsupseteq x_n$. Therefore $y \in \xi(e')$ and $y \sqsupseteq x$.

Suppose $y \in \xi(e')$. Then for all $n \geqq 0$ there are $u^{(n)} \in (e)_n$ such that $u^{(n)} \sqsubseteq y_n$. A similar argument using König's lemma provides an $x \in \xi(e)$ such that $x \sqsubseteq y$.

(ii) Here,

$$\begin{aligned}\eta(X) &= \langle \text{Con}\,(j_m(X))\rangle_{m=0}^{\infty} \\ &\sqsubseteq \langle \text{Con}\,(j_m(Y))\rangle_{m=0}^{\infty} \quad (\text{as } X \sqsubseteq Y,\, j_m(X) \sqsubseteq_M j_m(Y) \\ &\qquad\qquad\qquad\qquad\qquad \text{by Theorems 2.2 and 3.2}) \\ &= \eta(Y).\end{aligned}$$

(iii) We must show that $\eta(\xi(e)) = e$ for any $e \in E$. That is, we must show that $e_m = \text{Con}\,(j_m\{d \in D | \forall m \geqq 0. j_m(d) \in e_m\})$ for all $m \geqq 0$. Clearly RHS $\subseteq$ LHS (left-hand side). Take $x_m \in e_m$. Since $\hat{\psi}_{nm}(e_n) = e_m$ for $n \geqq m$, there are $u^{(n)}$, $v^{(n)}$ in e_n such that $\psi_{nm}(u^{(n)}) \sqsupseteq x_m \sqsupseteq \psi_{nm}(v^{(n)})$ for $n \geqq m$. Define $u^{(n)} = v^{(n)} = \psi_{mn}(x_m)$ for $n < m$. Applying König's lemma twice in the form prescribed above we obtain u, v in D such that $u_n, v_n \in e_n$ for all n and $u_m \sqsupseteq x_m \sqsupseteq y_m$, which shows that $x_m \in$ RHS.

(iv) We have, $\xi \circ \eta(X) = \{d \in D | \forall m \geqq 0. j_m(d) \in \text{Con}\,(j_m(X))\}$. If $d \in X$, then $j_m(d) \in j_m(X)$ and so $\xi \circ \eta(X) \supseteq X$. To show that $X \sqsubseteq_M \xi \circ \eta(X)$ it only remains to prove that if $y \in \xi \circ \eta(X)$, then $x \sqsubseteq y$, for some $x \in X$. So suppose $y \in \xi \circ \eta(X)$. Then for all $m \geqq 0$, $j_m(y)$ is in Con $(j_m(X))$. There is an $x^{(m)}$ in X such that $j_m(x^{(m)}) \sqsubseteq j_m(y)$. Therefore for all $n \geqq m$, $j_m(x^{(n)}) \sqsubseteq j_m(y)$.

Since X is finitely generable there is a $g : \Omega \to D$ such that $X = Bd(g)$. Let T be the finitary subtree of Ω consisting of the finite ω in Ω such that for infinitely many $n \geqq 0$, $g(\omega) \sqsubseteq x^{(n)}$. Clearly the empty sequence is a node in T. If ω is a node in T, then one of its two successors in Ω is in T. Therefore T has infinitely many nodes and we may apply König's lemma to obtain an infinite branch $\omega_0, \omega_1, \cdots$.

Now if ω is a node in T, then if $m \geqq 0$, there is an $n \geqq m$ such that $g(\omega) \sqsubseteq x^{(n)}$. Then $j_m(g(\omega)) \sqsubseteq j_m(x^{(n)}) \sqsubseteq j_m(y)$ by the definition of T and the properties of the $x^{(n)}$. Therefore $g(\omega) \sqsubseteq y$. Now if we put $x = \bigsqcup_{n \geqq 0} g(\omega_i)$, then $x \in X$ and $x \sqsubseteq y$, which concludes the proof that $X \sqsubseteq_M \xi \circ \eta(X)$.

To show that $\xi \circ \eta(X) \simeq X$ it only remains to show that $\xi \circ \eta(X) \sqsubseteq X$. We use Theorem 2(i). Let $f : D \to \mathbb{O}$. We must show that $f(\xi \circ \eta(X)) \sqsubseteq_M f(X)$. If $y \in f(X)$, then $y = f(x)$ for some x in $X \subseteq \xi \circ \eta(X)$. Therefore $y \in f(\xi \circ \eta(X))$. So we must show that if $x \in f(\xi \circ \eta(X))$, then for some $y \in f(X)$, $x \sqsubseteq y$. The case $x = \bot$ is trivial. If $x = \top$, then, by the continuity of f, there is a d in $\xi \circ \eta(X)$ and an $m \geqq 0$ such that $\top = f(i_m \circ j_m(d))$. As $j_m(d) \in \text{Con}\,(j_m(X))$, there is an $x \in X$ such that $j_m(d) \sqsubseteq j_m(x)$. Then $\top = f(x) \in f(X)$, as required. $\square$

THEOREM 6. *$\mathscr{P}[D] \cong E$ and is, therefore, an* SFP *object.*

Proof. Let $[X]$ be the equivalence class of X in $\mathscr{F}[D]$ modulo $\simeq$. Define $\xi/\simeq : E \to E'$ by:

$$(\xi/\simeq)(e) = [\xi(e)]$$

and $(\eta/\simeq) : E' \to E$ by:

$$(\eta/\simeq)([X]) = \eta(X).$$

By Lemma 2(ii), $\eta/\simeq$ is well-defined. By Lemmas 2(i) and 2(ii) and Theorem 2(iii), $\xi/\simeq$ and $\eta/\simeq$ are monotonic. By Lemmas 2(iii) and 2(iv) they are mutual inverses.

It is possible to choose maximal members of the equivalence classes to get a better description of $\mathscr{P}[D]$. Define an operation on $\mathscr{F}[D]$: by: $X^* = \xi \circ \eta(X)$.

COROLLARY 1. (i) *The operation X^* is a closure operation on $\mathscr{F}[D]$.*

(ii) *For X, Y in $\mathscr{F}[D]$, $X \sqsubseteq Y$ iff $X^* \sqsubseteq_M Y^*$, and $X \simeq Y$ iff $X^* = Y^*$.*

(iii) *$\mathscr{P}[D] \cong \langle \{X^* | X \in \mathscr{F}[D]\}, \sqsubseteq_M \rangle$.*

Proof. (i) We show that if $X \subseteq Y$, then $X^* \subseteq Y^*$; that $X \subseteq X^*$, and that $X^{**} = X^*$. The first is immediate from the definitions of ξ and η, the second from Lemma 2(iv) and the third from Lemma 2(iii).

(ii) That $X \sqsubseteq Y$ implies $X^* \sqsubseteq_M Y^*$ follows from Lemmas 2(i) and 2(iii). The converse follows from Lemma 2(iv) and Theorem 2(iii). The other part is similar.

(iii) We can define $\alpha : E' \to \langle \{X^* | X \in \mathscr{F}[D]\}, \sqsubseteq_M \rangle$ and $\beta : \langle \{X^* | X \in \mathscr{F}[D]\}, \sqsubseteq_M \rangle \to E'$ by:

$$\alpha([X]) = X^* \qquad (X \in \mathscr{F}[D]),$$
$$\beta(X) = [X] \qquad (X = X^* \in \mathscr{F}[D]).$$

Part (ii) shows that α is well-defined and monotonic and that β is monotonic. By part (i) and Lemma 2(iv) they are mutual inverses. Therefore they are isomorphisms.

We will take this definition of $\mathscr{P}[D]$ as our standard one. It can be seen that this agrees with the previous definition when D is finite. For by Lemma 2(iv), X^* is the maximal member of $[X]$, and by Theorem 3, so is Con (X) when D is finite. It also agrees with the earlier standard definition when D is discrete.

From the proof of Theorem 6 and that of Corollary 1 we see that if we define $\eta^* : \mathscr{P}[D] \to E$ and $\xi^* : E \to \mathscr{P}[D]$ by $\eta^* = (\eta/\simeq) \circ \beta$ and $\xi^* = \alpha \circ (\xi/\simeq)$, then η^* and ξ^* are isomorphisms. For they are mutual inverses. One sees too that $\eta^*(X) = \eta(X)$, for all X in $\mathscr{P}[D]$ and, using Lemma 2(iii), $\xi^*(e) = \xi(e)$ for e in E.

With the aid of a lemma we can get a picture of the finite elements of $\mathscr{P}[D]$.

LEMMA 3. (i) *For $X \in \mathscr{P}[D]$, $\tilde{j}_n \circ \eta^*(X) = \mathrm{Con}\,(j_n(X))$ $(n \geqq 0)$.*

(ii) *For $X \in \mathscr{P}[D_n]$, $\xi^* \circ \tilde{i}_n(X) = \mathrm{Con}\,(i_n(X))$ $(n \geqq 0)$.*

Proof. (i) The proof is immediate from the definition of η^*.

(ii) For $X \in \mathscr{P}[D_n]$,

$$\begin{aligned} \xi^* \circ \tilde{i}_n(X) &= \{d \in D \mid j_m(d) \in (\tilde{i}_n(X))_m,\ m \geqq 0\} \\ &= \{d \in D \mid j_m(d) \in \hat{\varphi}_{nm}(X),\ m \geqq n\}. \end{aligned}$$

This clearly includes $i_n(X)$ and therefore since it is closed it includes Con $(i_n(X))$. Now if $d \in \xi^* \circ \tilde{i}_n(X)$, then for all $m \geqq n$, $j_m(d) \in \hat{\varphi}_{nm}(X)$ and therefore there are

$u^{(m)}$, $v^{(m)} \in X$ such that $i_n(u^{(m)}) \sqsubseteq i_m \circ j_m(d) \sqsubseteq i_n(v^{(m)})$ for $m \geqq n$. Since X is finite, infinitely many of the $u^{(m)}$ are identical, as are infinitely many of the $v^{(m)}$. Therefore there are u, v in X such that $i_n(u) \sqsubseteq d \sqsubseteq i_n(v)$, showing that d is in Con $(i_n(X))$ as required.

From the proof of Theorem 5, we see that the finite elements of E are those of the form $\tilde{i}_n(X)$ where $X \in \mathscr{P}[D_n]$. Therefore the finite elements of $\mathscr{P}[D]$ have the form, $\xi^* \circ \tilde{i}_n(X)$ or, by the lemma, Con $(i_n(X))$ where $X \in \mathscr{P}[D_n]$. That is, they are the convex closures of finite nonempty sets of finite elements of D.

One can then obtain another interesting characterization of $\mathscr{P}[D]$: it is the completion in IPO of the partial order

$$\langle \{\text{Con}\ (X) | X \text{ is a finite, nonempty set of finite elements of } D\}, \sqsubseteq_M \rangle.$$

The proof is straightforward. Since we do not use the result, we omit the proof.

The description of the closure operation X^* in terms of ξ and η is external to D, involving an arbitrary choice of a sequence $\mathscr{D}$ whose limit is D. Considering the example of subsets of Ω which are $\simeq$ but not $=_M$ suggests looking for a suitable notion of limit. This is provided by a topology of positive and negative information.

DEFINITION. The *Cantor topology* on D has, as sub-basis, the sets $P_e = \{x | x \sqsupseteq e\}$ and $N_e = \{x | x \not\sqsupseteq e\}$ for finite e.

It can be shown that when D is a lattice, the Cantor topology is the order convergence topology defined by Birkhoff in [3].

DEFINITION. An *information* α is a pair with $(\alpha)_1$ a finite member of D and $(\alpha)_2$ a finite set of finite elements of D.

$$\mathcal{N}_\alpha =_{\text{def}} P_{\alpha_1} \cap \bigcap_{e \in \alpha_2} N_e.$$

Each $\mathcal{N}_\alpha$ is open and closed and the $\mathcal{N}_\alpha$ form a basis for the topology, for

$$P_e \cap P_{e'} = \bigcup \{P_d | d \in \mathscr{U}_D(\{e, e'\})\}.$$

Notice that if D is finite, this is just the discrete topology. In fact in general it is the topology inherited from the discrete topologies on the D_n. For, as a set, D is the limit of the inverse system of sets $\langle D_n, \psi_{mn} \rangle$ and since the topologies in the D_n are discrete, the ψ_{mn} are continuous, in the topological sense. Therefore, by [4], D has an inherited topology with basis $\{j_n^{-1}(X) | X \subseteq D_n\}$, and this is just $\{\mathcal{N}_\alpha\}$. Now as topological spaces, the D_n are all compact. Therefore, by [4], so is D. Consequently D is Hausdorff, and every finite subset of D is closed.

We called the topology the Cantor topology since under the natural injection $\varepsilon : D \to \mathscr{P}(\omega)$, it can be viewed as a subspace of Cantor space. Here $\varepsilon(d) = \{n | e_n \sqsubseteq d\}$ $(d \in D)$, where $e_0, e_1, \cdots$ is an enumeration of the finite elements of D [17]. It is easily seen to be continuous as a map from D under the Cantor topology to Cantor space. It is not hard to show that $\{\varepsilon(d) | d \in D\}$ is a closed set in Cantor space and so we have another proof that the topology is compact.

Recall the usual definition of a limit of a sequence:

$$\lim \langle x_n \rangle_{n=0}^{\infty} = a \text{ iff every neighborhood of } a \text{ contains almost all the } x_n.$$

This can be reformulated using the sub-basis:

$$\lim \langle x_n\rangle_{n=0}^{\infty} = a \text{ iff.}(\forall P_e.a \in P_e \text{ implies almost all the } x_n \text{ are in } P_e) \text{ and } (\forall N_e.a \in N_e \text{ implies almost all the } x_n \text{ are in } N_e).$$

Then the elementary properties of lim are:

c0: $\lim \vec{x} = a$ and $\lim \vec{x} = b$ implies $a = b$.

c1: If $\vec{x}$ is eventually constant, its limit is that constant.

c2: If $\lim \vec{x} = a$ and $\vec{y}$ is a subsequence of $\vec{x}$, $\lim \vec{y} = a$.

Example. In Ω, $\lim \langle 0^n 1^\omega\rangle_{n=0}^{\infty} = 0^\omega$.

Since D is compact, every sequence has a convergent subsequence. Since D has a denumerable basis, a set is closed iff it is closed under limits of sequences. This is all the directly topological information required about D.

LEMMA 4. (i) *Suppose* $x_0 \sqsubseteq x_1 \sqsubseteq x_2 \cdots$ *is an increasing sequence. Then* $\lim \langle x_n\rangle_{n=0}^{\infty} = \bigsqcup_{n\geqq 0} x_n$.

(ii) *Let* $\langle x_n\rangle_{n=0}^{\infty}$ *be a sequence in D. Suppose* $\lim \langle i_n \circ j_n(x_n)\rangle_{n=0}^{\infty} = x$. *Then* $\lim \langle x_n\rangle_{n=0}^{\infty} = x$.

(iii) *Let* $\langle x_n\rangle_{n=0}^{\infty}$ *and* $\langle y_n\rangle_{n=0}^{\infty}$ *be sequences in D. Suppose* $\lim \langle x_n\rangle_{n=0}^{\infty} = a$, $\lim \langle y_n\rangle_{n=0}^{\infty} = b$ *and* $x_n \sqsubseteq y_n$ *for all n. Then* $a \sqsubseteq b$.

(iv) (Milner) *Every nonempty closed subset of D is finitely generable.*

Proof. (i) If $\bigsqcup_{n\geqq 0} x_n \in P_e$, then some and hence almost all the x_n are in P_e. If $\bigsqcup_{n\geqq 0} x_n \in N_e$, then every $x_n \in N_e$.

(ii) If $x \in P_e$, then almost all the $i_n \circ j_n(x_n)$ are in P_e. As $x_n \sqsupseteq i_n \circ j_n(x_n)$ for all n, it follows that almost all the x_n are in P_e. If $x \in N_e$, then all the $i_n \circ j_n(x_n)$ are in N_e for $n \geqq n_0$, say. For $n \geqq n_1$, say $e = i_n \circ j_n(e)$. Then for $n \geqq \max(n_0, n_1)$, $x_n \in N_e$ as otherwise $x_n \sqsupseteq e$ and $i_n \circ j_n(x_n) \sqsupseteq i_n \circ j_n(e) = e$.

(iii) If a is greater than some finite e, then $a \in P_e$ so almost all the x_n are in P_e, so almost all the y_n are in P_e. Therefore, as D is algebraic, $b \sqsupseteq a$.

(iv) Let $X \subseteq D$ be closed and nonempty. Let T be the completion of the finitary tree with nodes of the form $\langle i_1 \circ j_1(x), \cdots, i_n \circ j_n(x)\rangle$ for $x \in D$ and $n \geqq 0$, with the subsequence ordering. Define $g: T \to D$ by

$$g(\omega) = \begin{cases} \perp & (\omega = \langle\cdot\rangle), \\ (\omega)_n & (\omega \text{ finite and not } \langle\cdot\rangle), \\ \bigsqcup_{n\geqq 0} (\omega)_n & (\omega \text{ infinite}). \end{cases}$$

Clearly $X \subseteq Bd(g)$. If $x \in Bd(g)$, then by (i) and (ii) there are $x_n \in X$ such that $x = \lim \langle i_n \circ j_n(x_n)\rangle_{n=0}^{\infty} = \lim \langle x_n\rangle_{n=0}^{\infty}$. But since X is closed, $x \in X$. $\square$

We can now characterize the closure operation which gives the maximal members of $\simeq$-equivalence classes. Let $\mathrm{Cl}(X)$ be the closure of X in the Cantor topology.

THEOREM 7. (i) *For any* $X \subseteq D$, $\mathrm{Con} \circ \mathrm{Cl}(X)$ *is the least convex set containing X which is closed in the Cantor topology.*

(ii) *For* $X \in \mathscr{F}[D]$, $X^* = \mathrm{Con} \circ \mathrm{Cl}(X)$.

(iii) *For* $X \in \mathscr{F}[D]$, $X \simeq \mathrm{Cl}(X)$.

Proof. (i) Con ∘ Cl (X) is clearly convex. Suppose $\vec{y}=\langle y_n\rangle_{n=0}^{\infty}$ is a convergent sequence in Con ∘ Cl (X). Then there are sequences $\vec{x}=\langle x_n\rangle_{n=0}^{\infty}$ and $\vec{z}=\langle z_n\rangle_{n=0}^{\infty}$ in Cl (X) such that $x_n \sqsubseteq y_n \sqsubseteq z_n$ for all n. Now $\vec{x}$ has a subsequence $\vec{x}'$ converging to $x' \in$ Cl (X). As $x_n \sqsubseteq y_n$ for all n, $x' \sqsubseteq \lim \vec{y}$. Similarly there is a z' in Cl (X) such that $\lim \vec{y} \sqsubseteq z'$. Therefore $\lim \vec{y}$ is in Con ∘ Cl (X).

So Con ∘ Cl (X) is also closed in the Cantor topology. If $Y \supseteq X$ and is convex and topologically closed, then $Y=$ Con ∘ Cl $(Y) \supseteq$ Con ∘ Cl (X).

(ii) By definition, $X^*=\{d\in D \mid \forall m \geqq 0. j_m(d)\in \mathrm{Con}(j_m(X))\}$. Therefore $d\in X^*$ iff for all $m\geqq 0$ there are $u^{(m)}$, $v^{(m)}$ in X such that $j_m(u^{(m)})\sqsubseteq j_m(d)\sqsubseteq j_m(v^{(m)})$. Since every sequence has a convergent subsequence, and $j_m(x)\sqsubseteq j_m(y)$ implies $j_n(x)\sqsubseteq j_n(y)$ for $n\leqq m$, we may assume without loss of generality that $\langle u^{(m)}\rangle_{m=0}^{\infty}$ and $\langle v^{(m)}\rangle_{m=0}^{\infty}$ are convergent to, say, u and v, respectively. Then

$$\begin{aligned} u &= \lim \langle u^{(m)}\rangle_{m=0}^{\infty} = \lim \langle i_m \circ j_m(u^{(m)})\rangle_{m=0}^{\infty} \qquad \text{(by Lemma 4(ii))} \\ &\sqsubseteq \lim \langle i_m \circ j_m(d)\rangle_{m=0}^{\infty} \\ &= d \\ &\sqsubseteq v \qquad \text{(similarly).} \end{aligned}$$

Therefore d is in Con ∘ Cl (X) and so $X^* \subseteq$ Con ∘ Cl (X).

Conversely, if x is in Cl (X), then $x=\lim \langle x_n\rangle_{n=0}^{\infty}$ where x_n is in X for all n. By the definition of limits, for all m there is an n such that $j_m(x)=j_m(x_n)\in$ Con $(j_m(X))$. So, by the definition of X^*, $x\in X^*$. Therefore Cl $(X)\subseteq X^*$. Thus $X^*=$ Con $(X^*)\supseteq$ Con ∘ Cl (X).

(iii) We have, $X \simeq X^* =$ Con ∘ Cl $(X) \simeq$ Cl (X), using part (ii) and Theorem 2(iii). □

The next theorem gives a picture of the l.u.b. operation on increasing sequences.

THEOREM 8. *Suppose* $X_0 \sqsubseteq_M X_1 \sqsubseteq_M \cdots$ *is an increasing sequence of nonempty sets closed in the Cantor topology. Then* $\bigsqcup_{n\geqq 0}$ Con $(X_n)=\{\bigsqcup_{n\geqq 0} x_n \mid$ *for all* $n\geqq 0$, $x_n \sqsubseteq x_{n+1}$ *and* $x_n\in X_n\}^*$.

Proof. Let $Y=\{\bigsqcup_{n\geqq 0} x_n \mid$ for all $n\geqq 0$, $x_n\sqsubseteq x_{n+1}$ and $x_n\in X_n\}$. If $x_m\in X_m$, then there are x_n in X_n $(n\neq m)$ such that $\langle x_n\rangle_{n=0}^{\infty}$ is an increasing sequence. Therefore there is a y in Y such that $y \sqsupseteq x_n$. So Y is nonempty and Y^* is in $\mathscr{P}[D]$. As clearly every element of Y has a lower bound in every X_m, $Y \sqsupseteq_M X_m$ for all m.

Suppose that $Z \sqsupseteq_M X_m$ for all m, for some Z in $\mathscr{P}[D]$. We show that $Y\sqsubseteq_M Z$. If $y\in Y$, then there is an increasing sequence, $\langle x_m\rangle_{m=0}^{\infty}$, such that $y=\bigsqcup_{m\geqq 0} x_m$ and $x_m\in X_m \sqsubseteq_M Z$. Therefore, for all $m\geqq 0$ there is a z_m in Z such that $z_m \sqsupseteq x_m$. So if z is the limit of a convergent subsequence of $\langle z_m\rangle_{m=0}^{\infty}$, then $z \sqsupseteq x$, and since Z is closed, z is in Z.

Conversely take z in Z. For each $m\geqq 0$ there is an x_m in X_m such that $z \sqsupseteq x_m$. We can then find for all m and $n\leqq m$, $u^{(m,n)}$ in X_n such that $u^{(m,0)}\sqsubseteq u^{(m,1)}\sqsubseteq \cdots \sqsubseteq u^{(m,m)}=x_m\sqsubseteq z$. Let $Y_m=\{u^{(n,m)} \mid n\geqq m\}\subseteq X_m$. If $\langle v_s\rangle_{s=0}^{\infty}$ is a convergent sequence with limit v, whose members are in Y_m, then there is a sequence $\langle v_s'\rangle_{s=0}^{\infty}$ in Y_{m+1} such that $v_s' \sqsupseteq v_s$ for almost all s. Then if v' is the limit of a convergent subsequence of $\langle v_s'\rangle_{s=0}^{\infty}$, we have $v' \sqsupseteq v$. So if we take v_0 in Y_0 to be the limit of a convergent sequence whose members are in Y_0, we can successively choose $v_1, v_2 \cdots$ such that $v_0 \sqsubseteq v_1 \sqsubseteq \cdots$ and v_m is the limit of a convergent sequence whose members are in $Y_m \subseteq X_m$. So as the X_m are closed in the Cantor topology, $v_m \in X_m$ for all m. But each v_m is the limit of elements less than z. Therefore $v_m \sqsubseteq z$ and $\bigsqcup_{m \geq 0} v_m$ is in Y and is less than z. Thus $Z \sqsupseteq_M Y \simeq Y^*$. □

We now have a picture of $\mathscr{P}[D]$ which is enough for our present purposes: its elements have the form Con ∘ Cl (X) for X in $\mathscr{F}[D]$; its ordering is $\sqsubseteq_M$; its finite elements have the form Con (X) where X is a finite set of finite elements of D and l.u.b.'s of sequences have the form described in the above theorem. It would be interesting to find out a good general form for the l.u.b.'s of directed sets.

Let us consider a special case, incidentally verifying some assertions made in § 2. Suppose S^ω is the domain of finite and infinite sequences of elements from a set S, with the subsequence ordering.

THEOREM 9. (i) *If $X \in \mathscr{F}[S^\omega]$, then X^* is the least convex set containing X closed under* l.u.b.'s *of increasing sequences. In particular, if every element in X is infinite, then $X^* = X$.*

(ii) *Suppose X, $Y \in \mathscr{F}[S^\omega]$ and every element of Y is infinite. Then $X \sqsubseteq Y$ iff $X \sqsubseteq_M Y$. If every element of X is also infinite, then $X \sqsubseteq Y$ iff $X = Y$.*

Proof. (i) Suppose y is a limit point of X. Then $y \in X^*$, by Theorem 7(ii). As $X \sqsubseteq_M X^*$ by Lemma 2(iv), $y \sqsupseteq x$ for some x in X. If x is infinite $y = x$. Otherwise x is finite. Now if $y \in P_e$ where $e \sqsupseteq x$, as y is a limit point, some u in X is also in P_e. Therefore e is in Con (X). Thus y is the l.u.b. of an increasing sequence of elements of Con (X). By Theorems 7(i) and 7(ii), X^* is as described. When every element of X is infinite, the least convex set containing X and closed under l.u.b.'s of increasing sequences is X.

(ii) Suppose every element of Y is infinite. If $X \sqsubseteq_M Y$, then $X \sqsubseteq Y$ by Theorem 2(iii). If $X \sqsubseteq Y$, then

$$\begin{aligned} X &\sqsubseteq_M X^* && \text{(by Lemma 2(iv))} \\ &\sqsubseteq_M Y^* && \text{(by Corollary 1(ii))} \\ &= Y && \text{(by part (i)).} \end{aligned}$$

If every element of X is infinite, then $X \sqsubseteq_M Y$ implies $X = Y$. □

We hope that this theorem increases the plausibility of the $\sqsubseteq$ ordering for sequence domains.

6. Continuity of various functions. In this section we consider various functions which are useful when defining denotational semantics. Although we cannot manipulate sets as freely as usual it is nonetheless possible to obtain reasonable analogues of many standard functions. In § 5 we showed that $\mathscr{P}[\cdot]$ sends SFP objects to SFP objects. We now extend $\mathscr{P}[\cdot]$ to maps and thereby obtain a functor from SFP to SFP.

We will feel free to use the results of previous sections without explicit reference, when their application presents no particular difficulties. The results used in this way are Theorems 2, 3, 5 (for the properties of SFP objects), Lemma 2(iv) (in the form $X \simeq X^*$ and $X \sqsubseteq_M X^*$ when $X \in \mathscr{F}[D]$), Corollary 1, Lemma 4 and Theorem 7. We also use the easily proved fact that if D is an SFP-object, U is a finite set of finite elements of D and $X \in \mathscr{F}[D]$, then $U \sqsubseteq X^*$ implies $U \sqsubseteq_M X$.

The next lemma provides a useful criterion for continuity.

LEMMA 5. *Let E and F be algebraic* ipo's. *Let f be a monotonic map from E to F. Then f is continuous iff for every x in E and every finite y in F, such that* $y \sqsubseteq f(x)$, *there is a finite u in E such that* $u \sqsubseteq x$ *and* $y \sqsubseteq f(u)$.

Proof. Suppose f is continuous, $x \in E$, $y \sqsubseteq f(x)$ and y is finite. Then, as E is algebraic and f is continuous,

$$y \sqsubseteq f(x) = \bigsqcup\{f(u) | u \sqsubseteq x \text{ and } u \text{ is finite}\}$$

and the set on the RHS is directed. As y is finite $y \sqsubseteq f(u)$ for some finite $u \sqsubseteq x$.

Conversely, suppose f is monotonic and the condition of the hypothesis holds. Suppose $X \subseteq E$ is directed. As f is monotonic, $f(\bigsqcup X) \sqsupseteq \bigsqcup f(X)$. Conversely, suppose $y \sqsubseteq f(\bigsqcup X)$ and y is finite. Then by assumption, there is a finite $u \sqsubseteq \bigsqcup X$ such that $y \sqsubseteq f(u)$. As u is finite and X is directed, $u \sqsubseteq x$ for some x in X. So $y \sqsubseteq f(u) \sqsubseteq f(x) \sqsubseteq \bigsqcup f(X)$, as f is monotonic. As F is algebraic, $f(\bigsqcup X) \sqsubseteq \bigsqcup f(X)$. □

Function extension. Suppose $f : D \to E$ where D and E are SFP objects. Define $\hat{f} : \mathscr{P}[D] \to \mathscr{P}[E]$ by:

$$\hat{f}(X) = (f(X))^* \qquad (X \in \mathscr{P}[D]).$$

We use Lemma 5 to show that $\hat{f}$ is continuous. First, suppose X, Y are in $\mathscr{P}[D]$ and $X \sqsubseteq Y$. Then $\hat{f}(X) \simeq f(X) \sqsubseteq f(Y) \simeq \hat{f}(Y)$. So f is monotonic.

Next suppose $X \in \mathscr{P}[D]$, $V \in \mathscr{P}[E]$, V is finite in $\mathscr{P}[E]$ and $V \sqsubseteq \hat{f}(X)$. Then $V = U^*$ for a finite set U of finite elements of E. So $U \sqsubseteq_M U^* = V \sqsubseteq \hat{f}(X) \simeq f(X)$. Therefore $U \sqsubseteq_M f(X)$. As X is closed in the Cantor topology, the construction in the proof of Lemma 4(iv) provides us with a $g : \Omega \to D$ such that $X = Bd(g)$ and if $\omega \in \Omega$ is finite so is $g(\omega)$.

Now set $U_\omega = \{u \in U | u \sqsubseteq f \circ g(\omega)\}$ $(\omega \in \Omega)$. Clearly if $\omega, \nu \in \Omega$ and $\omega \sqsubseteq \nu$, then $U_\omega \subseteq U_\nu \subseteq U$. Let $T = \{\omega \in \Omega | \omega = \perp \text{ or } \exists \nu \sqsupseteq \omega \cdot U_\nu \neq U_\omega\}$. Clearly every element of T is finite and T is a finitary tree under the subsequence ordering. As T has no infinite branches (U is finite), König's lemma tells us that T is finite. Let Y be the set of those elements of Ω whose predecessor is in T. Since T is a finite subtree of Ω every infinite element of Ω is greater than some element of Y. Therefore $g(Y) \sqsubseteq_M X$. Also $U \sqsubseteq_M f \circ g(Y)$. For if $u \in U$, then $u \sqsubseteq f(g(\omega))$ for some infinite ω. If $\nu \in Y$ and $\nu \sqsubseteq \omega$, then $U_\nu = U_\omega$ and so $u \sqsubseteq f \circ g(\nu)$. Conversely, if $\nu \in Y$, choose an infinite $\omega \in \Omega$ such that $\nu \sqsubseteq \omega$. Then $U_\nu = U_\omega$ and so $u \sqsubseteq f \circ g(\nu)$ for some $u \in U$, as $f \circ g(\omega) \in f(X) \sqsupseteq_M U$.

So $V = U^* \sqsubseteq_M U \sqsubseteq_M f(g(Y)) \simeq f((g(Y))^*) \simeq \hat{f}((g(Y))^*)$, $(g(Y))^*$ is a finite element of $\mathscr{P}[D]$ and $(g(Y))^* \sqsubseteq_M g(Y) \sqsubseteq_M Y$. Therefore f is continuous, by Lemma 5.

Just as in the finite case, function extension preserves all the identities and commutes with composition. Suppose $f : D \to E$ and $g : E \to F$ where D, E and F are SFP objects. Take X in $\mathscr{P}[D]$. As X is finitely generable, so is $f(X)$ and so $f(X) \simeq \hat{f}(X)$. Therefore $g(f(X)) \simeq g(\hat{f}(X)) \simeq \hat{g}(\hat{f}(X))$ (as $f(X)$ is finitely generable). Therefore as $\widehat{g \circ f}(X)$ is finitely generable, $\hat{g} \circ \hat{f}(X) = \widehat{g \circ f}(X)$.

It follows that if we define the action of $\mathscr{P}$ on morphisms $f: D \to E$ by:

$$\mathscr{P}[f] = \hat{f},$$

then $\mathscr{P}$ turns into a functor from SFP to SFP.

The singleton function. Let D be an SFP object. Define $[\![\cdot]\!]: D \to \mathscr{P}[D]$ by:

$$[\![x]\!] = \{x\} \qquad (x \in D).$$

It is clear that $[\![\cdot]\!]$ is monotonic. Suppose V is a finite element of $\mathscr{P}[D]$ and $V \sqsubseteq \{x\}$ (for x in D). Then $V = U^*$ where U is a finite set of finite elements of D. So $U \sqsubseteq_M \{x\}$. Therefore x is an upper bound of U and there is a finite d less than x in $\mathscr{U}(U)$. Then $V \sqsubseteq [\![d]\!]$ and d is finite and $d \sqsubseteq x$. By Lemma 5, $[\![\cdot]\!]$ is continuous.

Union. Let D be an SFP object. Define $\uplus: \mathscr{P}[D] \times \mathscr{P}[D] \to \mathscr{P}[D]$ by:

$$X \uplus Y = \mathrm{Con}\,(X \cup Y) \qquad (X, Y \in \mathscr{P}[D]).$$

If X and Y are closed in the Cantor topology so is $X \cup Y$. Therefore $\uplus$ is well-defined.

If $X \sqsubseteq X'$ and $Y \sqsubseteq Y'$ for X, X', Y, Y' in $\mathscr{P}[D]$, then $X \cup Y \sqsubseteq_M X' \cup Y'$ as $X \sqsubseteq_M X'$ and $Y \sqsubseteq_M Y'$. Therefore $\uplus$ is monotonic.

For continuity, we use the fact that the Cartesian product of two algebraic ipo's is an algebraic ipo. If E, F are algebraic ipo's, its finite elements are those of the form $\langle d, e\rangle$ where $d \in E$, $e \in F$ and d and e are finite.

Now, suppose V, X, Y are in $\mathscr{P}[D]$, V is a finite element of $\mathscr{P}[D]$, and $V \sqsubseteq X \uplus Y$. Then $V = U^*$ where U is a finite set of finite elements of D. Then $U \sqsubseteq_M V \sqsubseteq_M X \uplus Y \sqsubseteq_M X \cup Y$.

We can then find nonempty sets U_1 and U_2 such that $U = U_1 \cup U_2$, $U_1 \sqsubseteq_M X$ and $U_2 \sqsubseteq_M Y$. But then $\langle U_1^*, U_2^*\rangle$ is a finite element of $\mathscr{P}[D] \times \mathscr{P}[D]$, by the above discussion of Cartesian product, $\langle U_1^*, U_2^*\rangle \sqsubseteq \langle X, Y\rangle$ and $V \sqsubseteq_M U = U_1 \cup U_2 \sqsubseteq_M U_1^* \uplus U_2^*$. By Lemma 5, $\uplus$ is continuous.

We may now see that any reasonable proof system for domains in SFP based on $\sqsubseteq$ may also be used to prove theorems about $\subseteq$ and $\in$ provided it has symbols and suitable axioms for $\cup$ and $[\![\cdot]\!]$. For $X \subseteq Y$ iff $X \cup Y = Y$ iff $X \uplus Y = \mathrm{Con}\,(Y) = Y$ $(X, Y \in \mathscr{P}[D])$ and $x \in X$ iff $[\![x]\!] \subseteq X$ $(x \in D, X \in \mathscr{P}[D])$. Further $\uplus$ is associative, commutative and idempotent. The notation $[\![x_1, \cdots, x_m]\!]$ will be useful—it abbreviates $[\![x_1]\!] \uplus \cdots \uplus [\![x_n]\!]$.

Big Union. Let D be an SFP object. The "big union" function, $\biguplus: \mathscr{P}[\mathscr{P}[D]] \to \mathscr{P}[D]$ is defined by:

$$\biguplus(\mathscr{X}) = \mathrm{Con}\,(\{x \in D \mid \exists X \in \mathscr{X}. x \in X\}) \qquad (\mathscr{X} \in \mathscr{P}[\mathscr{P}[D]]).$$

To see that $\biguplus$ is well-defined we must prove that if $\mathscr{X} \in \mathscr{P}[\mathscr{P}[D]]$, then $Y = \{x \in D \mid \exists X \in \mathscr{X}. x \in X\}$ is closed in the Cantor topology. Let $\langle x_n\rangle_{n=0}^{\infty}$ be a convergent sequence in Y, with limit x. Each x_n is in some X_n in $\mathscr{X}$ and, without loss of generality, we may assume that $\langle X_n\rangle_{n=0}^{\infty}$ is convergent with limit X in $\mathscr{X}$. If x is in P_e, then almost all the x_n are in P_e and so almost all the X_n are in $P_{\{\perp,e\}^*}$. Therefore X too is in $P_{\{\perp,e\}^*}$ and so some element in X is in P_e. If $\langle e_n\rangle_{n=0}^{\infty}$ is an increasing sequence of finite elements whose l.u.b. is x we can therefore find a sequence $\langle y_n\rangle_{n=0}^{\infty}$ in X such that $y_n \sqsupseteq e_n$. Taking a convergent subsequence we find an upper bound of x in X. Now suppose $x \in N_e$. Then so are almost all the x_n and so almost all the X_n are in $N_{\{e\}}$. Therefore X too is in $N_{\{e\}}$ and so some element in X is in N_e. We now obtain a lower bound of x in X. Since X is convex, x itself is in X and so is in Y, showing that, as required, Y is closed.

Next we show that $\biguplus$ is monotonic. Suppose $\mathscr{X} \sqsubseteq \mathscr{X}'$ for $\mathscr{X}, \mathscr{X}'$ in $\mathscr{P}[\mathscr{P}[D]]$. Then $\mathscr{X} \sqsubseteq_M \mathscr{X}'$. Now if $x \in \bigcup\mathscr{X}$, then $x \in X$ for some $X \in \mathscr{X}$. Therefore $X \sqsubseteq_M X'$ for some $X' \in \mathscr{X}'$. Therefore $x \sqsubseteq x'$ for some $x' \in X' \subseteq \bigcup\mathscr{X}'$. Similarly, if $x' \in \bigcup\mathscr{X}'$, then $x' \sqsupseteq x$ for some $x \in \bigcup\mathscr{X}$. Therefore $\bigcup\mathscr{X} \sqsubseteq_M \bigcup\mathscr{X}'$ and so $\biguplus\mathscr{X} \sqsubseteq_M \biguplus\mathscr{X}'$.

For continuity, suppose $\mathscr{X} \in \mathscr{P}[\mathscr{P}[D]]$, $V \in \mathscr{P}[D]$, V is finite in $\mathscr{P}[D]$, and $V \sqsubseteq \biguplus\mathscr{X}$. Then $V = U^*$ where U is a finite set of finite elements of D. So $U \sqsubseteq_M \bigcup\mathscr{X}$. Let $\mathscr{X}' = \{Y^* \mid Y \subseteq U \text{ and } \exists X \in \mathscr{X}. Y \sqsubseteq_M X\}$. Then $\mathscr{X}'$ is a finite set of finite elements of $\mathscr{P}[D]$, $\mathscr{X}' \sqsubseteq_M \mathscr{X}$ and $U = \bigcup\mathscr{X}'$. Therefore, $V \sqsubseteq \biguplus(\mathscr{X}')^*$, $(\mathscr{X}')^*$ is finite in $\mathscr{P}[\mathscr{P}[D]]$, and $(\mathscr{X}')^* \sqsubseteq \mathscr{X}$. So by Lemma 5, $\biguplus$ is continuous.

It can be shown that if $\mathscr{X}$ is any nonempty subset of $\mathscr{P}[D]$, then $\biguplus\mathscr{X}^* = (\bigcup\mathscr{X})^*$. This fact should increase the intuitive appeal of certain definitions. It can be used to prove that, $p^*q = \biguplus \circ \mathscr{P}[q_\perp] \circ p$, where * is the operation defined in § 2, which was used to give the denotational semantics of a simple nondeterministic language. The operator * is therefore continuous, as we shall see in § 7 that $\mathscr{P}$ acts continuously on functions.

Cartesian Product. Let D and E be SFP objects. We shall see below that $D \times E$ is also an SFP object.

Define $\otimes : \mathscr{P}[D] \times \mathscr{P}[E] \to \mathscr{P}[D \times E]$ by:

$$\otimes(X, Y) = X \times Y \qquad (X \in \mathscr{P}[D],\ Y \in \mathscr{P}[E]).$$

We will leave most of the details to the reader. First $\otimes$ is well-defined for it is not hard to show that if $X \in \mathscr{P}[D]$ and $Y \in \mathscr{P}[E]$, then $X \times Y \in \mathscr{P}[D \times E]$. If $X, X' \in \mathscr{P}[D]$, $Y, Y' \in \mathscr{P}[E]$, $X \sqsubseteq_M X'$ and $Y \sqsubseteq_M Y'$, then $X \times Y \sqsubseteq_M X' \times Y'$. Therefore $\otimes$ is monotonic.

For continuity the essential observation is this. Suppose U is a finite set of finite elements of $D \times E$ and $U \sqsubseteq_M X \times Y$, where $X \in \mathscr{P}[D]$, $Y \in \mathscr{P}[E]$. Let $(U)_1 = \{(u)_1 \mid u \in U\}$. For each x in X choose an element $d(x)$ in $\mathscr{U}(\{u \in (U)_1 \mid u \sqsubseteq x\})$ such that $d(x) \sqsubseteq x$. Set $U_1 = \{d(x) \mid x \in X\}$. Define U_2 similarly. Then $U \sqsubseteq_M U_1 \times U_2$.

The Cartesian product function has a useful application. Suppose $f : D_1 \times \cdots \times D_n \to E$ where the D_i and E are SFP objects and we are using an iterated Cartesian product. Then we can define an extension of f to a continuous function $g : \mathscr{P}[D_1] \times \cdots \times \mathscr{P}[D_n] \to \mathscr{P}[E]$ by:

$$g(X_1, \cdots, X_n) = \tilde{f}(X_1 \otimes X_2 \otimes \cdots \otimes X_n),$$

where $\otimes$ is being used as an infix operator, associating to the left. This kind of extension allows us to define the general comprehension notation used in § 8.

Some other functions. We can define a weak analogue of $\cap$, if the SFP object D is a semi-lattice. In that case every set X has a greatest lower bound. $\sqcap X$ and $x \sqcap y$ is continuous in x and y. Then we could define $\Cap : \mathscr{P}[D] \times \mathscr{P}[D] \to \mathscr{P}[D]$ by

$$\Cap(X, Y) = \mathscr{P}[\sqcap](X \otimes Y) \qquad (X, Y \in \mathscr{P}[D]).$$

In this case $\sqcap$ itself can be regarded as a continuous function $\sqcap : \mathscr{P}[D] \to D$ defined by:

$$\sqcap(X) = \sqcap X \qquad (X \in \mathscr{P}[D]).$$

If D is a lattice, $\sqcup$ can be regarded as a continuous function $\sqcup : \mathscr{P}[D] \to D$ defined by:

$$\sqcup(X) = \sqcup X.$$

7. Solving recursive domain equations. We now consider how to solve domain equations involving $\mathscr{P}$. Enough category theory has been developed to allow a presentation along the lines of [13], [14], [20], [21]. One could also construct a universal domain along the lines of [16], and this will be discussed.

The category-theoretic approach casts $+$, $\times$, $\to$ and $\mathscr{P}$ as locally continuous, symmetric functors and looks for solutions to recursive domain equations as fixed-points of such functors. These can be found by an analogous method to that of the fixed-point theorem [17].

DEFINITION. A functor T; $(\text{IPO-P})^k \to (\text{IPO-P})((\text{SFP-P})^k \to (\text{SFP-P}))$ is *locally continuous* iff wherever $P_i \subseteq \text{Hom}(D_i, E_i)$ are directed sets of morphisms for $i = 1, k$ then:

$$T(\sqcup P_1, \cdots, \sqcup P_k) = \sqcup\{T(p_1, \cdots, p_k) | p_i \in P_i, 1 \leqq i \leqq k\},$$

the set on the right being directed.

DEFINITION. A functor $T : (\text{IPO-P})^k \to (\text{IPO-P})((\text{SFP-P})^k \to (\text{SFP-P}))$ is *symmetric* iff when p_i is in $\text{Hom}(D_i, E_i)$ for $i = 1, k$, $T(p_1^\dagger, \cdots, p_k^\dagger) = (T(p_1, \cdots, p_k))^\dagger$.

These properties are preserved under composition (of functors) and the projection and constant functors are all symmetric and locally continuous.

If $T : (\text{IPO-P})^k \to (\text{IPO-P})$ is symmetric, its restriction T_{PR} to $(\text{IPO-PR})^k$ can be considered to be in $(\text{IPO-PR})^k \to (\text{IPO-PR})$, for then, if $p_1, \cdots, p_k$ are projections:

$$\begin{aligned} T_{\text{PR}}(p_1, \cdots, p_k)^\dagger \circ T_{\text{PR}}(p_1, \cdots, p_k) &= T_{\text{PR}}(p_1^\dagger, \cdots, p_k^\dagger) \circ T(p_1, \cdots, p_k) \\ &= T_{\text{PR}}(p_1^\dagger \circ p_1, \cdots, p_k^\dagger \circ p_k) \\ &= T_{\text{PR}}(I, \cdots, I) \\ &= I. \end{aligned}$$

Similarly, $T_{\text{PR}}(p_1, \cdots, p_k) \circ T_{\text{PR}}(p_1, \cdots, p_k)^\dagger \sqsubseteq I$.

For simplicity we shall confuse T_{PR} and T when the context leaves the choice indifferent or makes clear which is intended. Similar remarks apply vis-a-vis SFP-P and SFP-PR.

Next we describe $+$, $\times$, $\to$ and $\mathscr{P}$ as functors. The first three will be locally continuous, symmetric functors from $(\text{IPO-P})^2$ to (IPO-P). We will see later that they cut down to functors from $(\text{SFP-P})^2$ to SFP-P.

We choose a separated sum for $+$. Given two ipo's D and E, $D + E$ is $\{\langle 1, d\rangle | d \in D\} \cup \{\bot\} \cup \{\langle 2, e\rangle | e \in E\}$ with the obvious ordering. If $f : D \to D'$ and

$g: E \to E'$ are continuous functions, define $f+g: D+E \to D'+E'$ by

$$f+g(x)=\begin{cases} \perp & (x=\perp), \\ \langle 1, f((x)_2)\rangle & (x \neq \perp, (x)_1 = 1), \\ \langle 2, g((x)_2)\rangle & (x \neq \perp, (x)_1 = 2). \end{cases}$$

Now if $p: D \to D'$, $q: E \to E'$ are IPO-P morphisms, we define $p+q = \langle p_1 + q_1, p_2 + q_2\rangle$. This defines + as a functor $(\text{IPO-P})^2 \to (\text{IPO-P})$.

The cartesian product of two ipo's D and E is $D \times E$ which is the usual product with the induced componentwise ordering.

If $f: D \to D'$, $g: E \to E'$ are functions, $f \times g: D \times E \to D' \times E'$ is given by: $f \times g(\langle d, e\rangle) = \langle f(d), g(e)\rangle$.

If $p: D \to D'$, $q: E \to E'$ are morphisms, $p \times q: D \times E \to D' \times E'$ is given by: $p \times q = \langle p_1 \times q_1, p_2 \times q_2\rangle$. This defines Cartesian product as a functor $(\text{IPO-P})^2 \to (\text{IPO-P})$.

Exponentiation is more interesting. Its action on objects is clear, but it is not defined in the same fashion on morphisms. Suppose $p: D \to D'$, $q: E \to E'$ are morphisms. Define $p \to q: \text{Hom}((D \to E), (D' \to E'))$ by: $(p \to q)_1(f) = q_1 \circ f \circ p_2$ $(f \in (D \to E))(p \to q)_2(g) = q_2 \circ g \circ p_1$ $(g \in (D' \to E'))$. (See Fig. 2.)

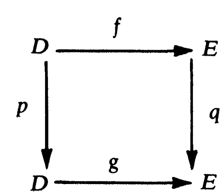

FIG. 2

This defines the exponentiation functor. The verification of symmetry and local continuity is straightforward for all three functors.

We have already seen that $\mathcal{P}$ can be taken as a functor from SFP to SFP. It induces a corresponding functor, which we also call $\mathcal{P}$, on SFP-P. Its action on objects is the same as $\mathcal{P}$. Let p be an SFP-P morphism. We define

$$\mathcal{P}[p] = \langle \mathcal{P}[p_1], \mathcal{P}[p_2]\rangle.$$

$\mathcal{P}$ is clearly symmetric. For continuity, we show that function extension preserves limits, too. This is clearly equivalent to showing $\text{EXT}: (D \to E) \to (\mathcal{P}[D] \to \mathcal{P}[E])$ where $\text{EXT}(f) = \hat{f}$ for $f: D \to E$. Now it is not hard to see that EXT is monotonic.

To prove continuity it is only necessary to take a directed set $F \subseteq (D \to E)$, a set X in $\mathcal{P}[D]$ and a map $g: D \to \mathbb{O}$ and prove that $g(\text{EXT}(\bigsqcup F)(X)) \sqsubseteq_M g(\bigsqcup_{f \in F} \text{EXT}(f)(X))$, since the other half follows from the monotonicity of EXT.

Now,

$$\text{LHS} = \hat{g}(((\widehat{\bigsqcup F})(X)))$$
$$= \{g \circ (\bigsqcup F)(x) \mid x \in X\}, \quad \text{and}$$

and

$$\text{RHS} = \hat{g}(\bigsqcup_{f \in F} (\hat{f}(X)))$$
$$= \bigsqcup_{f \in F} \hat{g}(\hat{f}(X))$$
$$= \bigsqcup_{f \in F} \{g \circ f(x) \mid x \in X\}.$$

If $\top \in \text{LHS}$, then $\top = g \circ (\bigsqcup F)(x)$ for some x in X. Therefore $\top = g \circ f(x)$ for some $f \in F$ and so $\top \in \text{RHS}$.

If $\perp \in \text{RHS}$, then $\forall f \in F. \exists x \in X. g \circ f(x) = \perp$. So $\forall n \geqq 0$, $f \in F. \exists x \in X. g \circ f \circ i_n \circ j_n(x) = \perp$. But $\{g \circ f \circ i_n \circ j_n \mid f \in F\}$ is both finite and directed. Therefore $\forall n \geqq 0. \exists x^{(n)} \in X. \forall f \in F. g \circ f \circ i_n \circ j_n(x^{(n)}) = \perp$. Without loss of generality we may assume that $\langle x^{(n)} \rangle_{n=0}^{\infty}$ converges to, say, x. Then $g \circ f(x) = \perp$ for all f in F and so $\perp \in \text{LHS}$. This finishes the proof of continuity of EXT.

We have now succeeded in exposing $+$, $\times$, $\rightarrow$ and $\mathscr{P}$ as examples of symmetric, locally continuous functors. Solving domain equations can be viewed as finding fixed points of such functors. For example suppose we want to find a domain R of resumptions which satisfies

$$R \cong S_\perp \rightarrow \mathscr{P}[S_\perp + (S_\perp \times R)].$$

Define a functor, $T: (\text{SFP-P}) \rightarrow (\text{SFP-P})$ by

$$T(R) = S_\perp \rightarrow \mathscr{P}[S_\perp + (S_\perp \times R)]$$

on objects and similarly on morphisms.

Then we want to find a fixed point of T. We need a more global form of continuity.

Notation. Suppose $\mathscr{D}^i = \langle D^i_m, p^i_{mn} \rangle$ is a directed sequence when $1 \leqq i \leqq k$, and $T: (\text{IPO-P})^k \rightarrow (\text{IPO-P})((\text{SFP-P})^k \rightarrow (\text{SFP-P}))$. Then we define $T(\mathscr{D}^1, \cdots, \mathscr{D}^k)$ to be the directed sequence $\langle T(D^1_m, \cdots, D^k_m), T(p^1_{mn}, \cdots, p^k_{mn}) \rangle$.

THEOREM 10 (Global Continuity). *Suppose $T: (\text{IPO-P})^k \rightarrow (\text{IPO-P})$ is symmetric and locally continuous. Let $\mathscr{D}^1, \cdots, \mathscr{D}^k$ be directed sequences in* IPO-PR. *Then $T(\lim_\rightarrow \mathscr{D}^1, \cdots, \lim_\rightarrow \mathscr{D}^k) \cong \lim_\rightarrow T(\mathscr{D}^1, \cdots, \mathscr{D}^k)$ $(k > 0)$. The analogous result holds for* SFP.

Proof. Let $\langle r^i_m \rangle$ be a universal cone from $\mathscr{D}^i$ to $\lim_\rightarrow \mathscr{D}^i$ $(1 \leqq i \leqq k)$. Then $\langle r_m \rangle = \langle T(r^1_m, \cdots, r^k_m) \rangle$ is a cone from $T(\mathscr{D}^1, \cdots, \mathscr{D}^k)$ to $T(\lim_\rightarrow \mathscr{D}^1, \cdots, \lim_\rightarrow \mathscr{D}^k)$. This is easily checked from the functor laws. To show that it is universal, we use the criterion of Lemma 1 and show that $\bigsqcup_{n\geqq 0} r_n \circ r_n^{\dagger} = I$:

$$\begin{aligned}
&\bigsqcup_{n\geqq 0} T(r_n^1, \cdots, r_n^k) \circ T(r_n^1, \cdots, r_n^k)^{\dagger} \\
&\qquad = \bigsqcup_{n\geqq 0} T(r_n^1 \circ r_n^{1\dagger}, \cdots, r_n^k \circ r_n^{k\dagger}) \\
&\qquad = T(\bigsqcup_{n\geqq 0} r_n^1 \circ r_n^{1\dagger}, \cdots, \bigsqcup_{n\geqq 0} r_n^k \circ r_n^{k\dagger}) \qquad \text{(by local continuity)} \\
&\qquad = T(I, \cdots, I) \qquad \text{(by Lemma 1 as the } \langle r_m^i \rangle \text{ are universal)} \\
&\qquad = I.
\end{aligned}$$

The proof for SFP is similar. □

COROLLARY 2. *Suppose* $T:(\text{IPO-P})^k \to (\text{IPO-P})$ *is symmetric and locally continuous. If* $T(D_1, \cdots, D_k)$ *is an* SFP *object whenever* $D_1, \cdots, D_k$ *are finite, then* T *cuts down to a symmetric and locally continuous functor* $T_{\text{SP}}:(\text{SFP-P})^k \to (\text{SFP-P})$.

Proof. We need only show that if $D_1, \cdots, D_n$ are SFP objects, so is $T(D_1, \cdots, D_n)$. There are directed sequences, $\mathscr{D}^i$, of finite ipo's with limit D_i. Then $T(D_1, \cdots, D_n) = \lim_{\to} T(\mathscr{D}^1, \cdots, \mathscr{D}^k)$, by Theorem 10. Hence by Theorem 5(ii) and the hypothesis, $T(D_1, \cdots, D_n)$ is an SFP object. □

It follows that $+$, $\times$ and $\to$ cut down to symmetric, locally continuous functors from $(\text{SFP-P})^2$ to (SFP-P). This could have been proved directly, but in the case of $\to$ the details are rather tedious.

The next corollary enables us to solve recursive domain equations. It is partially analogous to the fixed-point theorem of [17], but does not give any "leastness" information. This, and much else, can be found in [21].

COROLLARY 3. *Let* $T:(\text{SFP-P}) \to (\text{SFP-P})$ *be symmetric and locally continuous. Then there is an* SFP *object* D *such that* $T(D) \cong D$.

Proof. Let $\mathscr{D} = \langle T^m(\mathbb{1}), p_{mn} \rangle$ where $\mathbb{1}$ is the one-point ipo. $p_{mn} = T^{n-1}(p_{\mathbb{1}}) \circ \cdots \circ T^m(p_{\mathbb{1}})$ $(m \leqq n)$, where $p_{\mathbb{1}}$ is the unique projection $p: \mathbb{1} \to T(\mathbb{1})$. Let $D = \lim_{\to} \mathscr{D}$. Then

$$\begin{aligned}
T(D) &\cong T(\lim_{\to} \mathscr{D}) \qquad \text{(by Theorem 10)} \\
&\cong \lim_{\to} \mathscr{D} \qquad \text{(since } T(\mathscr{D}) \text{ is obtained from } \mathscr{D} \text{ by dropping the first domain in } \mathscr{D}).
\end{aligned}$$

It is well-known that dropping a term of a sequence in this way does not affect the limit. □

When we want to solve simultaneous equations such as

$$D \cong N_{\perp} + \mathscr{P}[E] + D,$$

$$E \cong (N_{\perp} \to D) + E,$$

a slight extension of Theorem 8(i) to functors $T: (\text{SFP-P})^n \to (\text{SFP-P})^n$ is necessary. For example, in the case at hand one considers $T: (\text{SFP-P})^2 \to (\text{SFP-P})^2$ defined by $T(\langle D, E\rangle) = \langle N_\perp + \mathcal{P}[E] + D, (N_\perp \to D) + E\rangle$, etc.

An alternative approach to solving recursive equations is to follow the universal domain idea [16] and solve just one equation, $\mathcal{U} \cong N_\perp + (\mathcal{U} + \mathcal{U}) + (\mathcal{U} \times \mathcal{U}) + (\mathcal{U} \to \mathcal{U}) + \mathcal{P}[\mathcal{U}]$. Then we can represent subdomains by retractions. These are continuous functions $f: \mathcal{U} \to \mathcal{U}$ such that $f^2 = f$; f *represents* the ipo $\text{dom}(f) = \{d \in \mathcal{U} \mid f(d) = d\}$. As usual the retractions can be considered as elements of $\mathcal{U}$ and there are continuous functions $\oplus, \otimes$ and $\ominus$ such that if f and g are retractions, then so are $f \oplus g$, $f \otimes g$ and $f \ominus g$ and we have $\text{dom}(f \oplus g) \cong \text{dom}(f) + \text{dom}(g)$, $\text{dom}(f \otimes g) \cong \text{dom}(f) \times \text{dom}(g)$ and $\text{dom}(f \ominus g) \cong \text{dom}(f) \to \text{dom}(g)$.

There is also a suitable function $\circledP$ for $\mathcal{P}$ defined by:

$$\circledP(f) = \varphi_{\mathcal{P}} \circ \mathcal{P}[f] \circ \psi_{\mathcal{P}}.$$

Here $\langle \varphi_{\mathcal{P}}, \psi_{\mathcal{P}}\rangle$ is the evident projection of $\mathcal{U}$ onto $\mathcal{P}[\mathcal{U}]$, EXT is the function extension defined above and we have followed the usual practice and identified $N_\perp$, $\mathcal{U} + \mathcal{U}$, $\mathcal{U} \times \mathcal{U}$, $\mathcal{U} \to \mathcal{U}$ and $\mathcal{P}[\mathcal{U}]$ as subdomains of $\mathcal{U}$. One then finds solutions to the recursion equations by solving the corresponding equations using the fixed point theorem. For example, to solve $P \cong V \to \mathcal{P}[L \times V \times P]$, where V and L are represented by r and l, respectively, one defines a continuous function f on $\mathcal{U}$ by $f = \lambda p \in \mathcal{U}.(r \ominus (\circledP(l \otimes v \otimes p)))$. Its least fixed point, p, is a retraction and represents a domain satisfying the equation. Simultaneous equations can also be solved in this way.

We will not pause to spell out the details of this approach. It should be noted that in general $\text{dom}(f)$ is *not* algebraic and hence not an SFP object. Rather it is, presumably, a continuous ipo in an appropriate sense. Thus this approach should lead to stronger results as it allows the possibility of a powerdomain construction on certain continuous objects. Extension of our direct approach, linking algebraic ipo's to continuous ones should also give these results.

Questions relating to $\mathcal{P}(\omega)$ also arise. By Theorem 1.6 of [17], we can embed any SFP object in $\mathcal{P}(\omega)$, as described in § 5, and this gives rise to a lattice with intermediate points as remarked at the beginning of this section. But these intermediate points seem to clutter up the domain and we do not know any simple continuous function analogous to $\circledP$ defined above. Of course we do know some such continuous function since $\mathcal{U}$ will be embeddable in $\mathcal{P}(\omega)$ and $\mathcal{P}(\omega)$ is embeddable in $\mathcal{U}$—indeed it can be embedded in $\mathcal{P}[N_\perp]$. But, rather than tagging $\mathcal{P}(\omega)$ on at the end, as it were, what is wanted is a simple development of $\mathcal{P}[\,\cdot\,]$ in the context of $\mathcal{P}(\omega)$ or a similar "simple" structure. In Scott's words, we want an analytic, not a synthetic development. However, we have at least developed the powerdomain construction enough to apply it to give some semantics as promised in the introduction.

8. Applications. We conclude by first giving the semantics for our illustrative language with simple parallelism and then giving an oracle-free semantics for Milner's multiprocessing language.

The programs of the first language are described by the grammar:

$$\pi ::= (\nu := \tau)|(\pi_1;\ \pi_2)|(\pi_1 \textit{ or } \pi_2)|(\textit{if } \nu \textit{ then } \pi_1 \textit{ else } \pi_2)$$
$$|(\textit{while } \nu \textit{ do } \pi)|(\pi_1 \textit{ par } \pi_2),$$

where ν and τ are as described in § 2.

The semantic domain of resumptions R is constructed, as described in the previous section, to satisfy the equation

$$R \cong S_\perp \to \mathscr{P}[S_\perp + (S_\perp \times R)].$$

To avoid being pedantic, we will identify R with the RHS and regard $S_\perp$, $(S_\perp \times R)$ as subsets of $S_\perp + (S_\perp \times R)$. The discriminator function $(:S_\perp): S_\perp + (S_\perp \times R) \to \mathbb{T}$ is defined by:

$$(x : S_\perp) = \begin{cases} \perp & (x = \perp), \\ \textit{true} & (x \in S_\perp), \\ \textit{false} & (x \in (S_\perp \times R)). \end{cases} \qquad (x \in S_\perp + (S_\perp \times R))$$

Suppose $---x---$ and $\sim\sim y \sim\sim$ are expressions possibly having occurrences of variables x and y ranging over D and E, respectively, such that $\sim\sim y \sim\sim$ is continuous in y and so defines a continuous function $f: E \to \mathscr{P}[D]$ and $---x---$ is continuous in x and so defines a continuous function $g: D \to F$, where D, E, F are SFP objects. Then the expression,

$$[\![\, ---x--- \,|\, x \in \sim\sim y \sim\sim \,]\!]$$

is taken as defining $p(e)$ where the function $p: E \to \mathscr{P}[F]$ is $\mathscr{P}[g] \circ f$ when y has value e. With this notation we can easily describe various helpful operations on resumptions. It can be extended to several elements on the right of the $|$.

Choice. $r?r' = \lambda\sigma \in S_\perp . r(\sigma) \uplus r'(\sigma)$, $(r, r' \in R)$.

Sequence. $r^*r' = \lambda\sigma \in S_\perp . [\![\mathrm{COND}\,(x : S_\perp, \langle x, r'\rangle, \langle (x)_1, ((x)_2^* r')\rangle) | x \in r(\sigma)]\!]$.

Parallelism.

$$r /\!\!/ r' = \lambda\sigma \in S_\perp [\![\mathrm{COND}\,(x : S_\perp, \langle x, r'\rangle, \langle (x)_1, r' /\!\!/ (x)_2\rangle) | x \in r(\sigma)]\!]$$
$$\uplus [\![\mathrm{COND}\,(x : S_\perp, \langle x, r\rangle, \langle (x)_1, (x)_2 /\!\!/ r\rangle) | x \in r'(\sigma)]\!].$$

These recursive definitions are to be taken as shorthand versions of explicit definitions using the least fixed-point operation. The choice combinator is associative, commutative and idempotent; the sequence combinator is associative and the parallelism combinator is associative and commutative, but not idempotent in general.

We have used a slightly different conditional combinator than in § 2, namely, the combinator $\mathrm{COND}: \mathbb{T} \times D \times D \to E$ defined by

$$\mathrm{COND}\,(t, x, y) = \begin{cases} \perp & (t = \perp), \\ x & (t = \textit{true}), \\ y & (t = \textit{false}), \end{cases} \qquad (t \in \mathbb{T},\ x, y \in D).$$

Strictly speaking we should write COND_D, but both here and later D will be understood from the context.

With the aid of the function $\mathcal{V}$ described in § 2 we can define the denotational semantics, $\mathfrak{R}$: Statements $\to R$ by structural induction on Statements:

$$\mathfrak{R}[\![(x_i := \tau)]\!] = \lambda\sigma \in S_\perp.\mathrm{COND}\,(EQ(\sigma, \sigma), [\![\langle x_1, \cdots, x_{i-1}, \mathcal{V}[\![\tau]\!](\sigma), x_{i+1}, \cdots, x_n\rangle]\!], \perp),$$

$$\mathfrak{R}[\![(\pi_1; \pi_2)]\!] = \mathfrak{R}[\![\pi_1]\!]^* \mathfrak{R}[\![\pi_2]\!],$$

$$\mathfrak{R}[\![(\pi_1 \; or \; \pi_2)]\!] = \mathfrak{R}[\![\pi_1]\!] \,?\, \mathfrak{R}[\![\pi_2]\!],$$

$$\mathfrak{R}[\![(if\; x_i \; then \; \pi_1 \; else \; \pi_2)]\!] = \lambda\sigma \in S_\perp.\mathrm{COND}\,(EQ((\sigma)_i, 0), \mathfrak{R}[\![\pi_1]\!](\sigma), \mathfrak{R}[\![\pi_2]\!](\sigma)),$$

$$\mathfrak{R}[\![(while\; x_i \; do \; \pi)]\!] = Y(\lambda r \in R, \lambda\sigma \in S_\perp.\mathrm{COND}\,(EQ((\sigma)_i, 0), \mathfrak{R}[\![\pi]\!]^* r(\sigma), [\![\sigma]\!])),$$

$$\mathfrak{R}[\![(\pi_1 \; par \; \pi_2)]\!] = \mathfrak{R}[\![\pi_1]\!] /\!/ \mathfrak{R}[\![\pi_2]\!].$$

The predicate $EQ : D^2 \to \mathbb{T}$ used above is defined for discrete domains D by:

$$EQ(x, y) = \begin{cases} \perp & (x = \perp \text{ or } y = \perp), \\ true & (x = y \neq \perp), \\ false & (x \neq \perp, y \neq \perp, x \neq y), \end{cases} \qquad (x, y \in D).$$

Various ad hoc possibilities are available to deal with the fairness problem. For example one could define a parallelism combinator $/\!/_m$ which is like $/\!/$ except that it does not run a branch for more than m elementary operations.

This combinator is commutative but not associative; its use is equivalent to a local use of a nondeterministic oracle. It will, presumably, not give rise to a fully abstract semantics. One can also look at other cases, such as buffers [8] which need not even give rise to nondeterminism. Perhaps one could achieve some workable combination of nondeterminism, oracles and special cases, but we feel that some new insight will be needed.

Let us conclude with a semantics for Milner's language. We give only an abbreviated account here, following the general pattern laid down in the papers [10], [11]. Some inessential variations have been made for consistency's sake.

The language has *identifiers* with metavariable x and a class of *expressions* with metavariable ε. The expressions are given by:

$$\varepsilon ::= x \mid \varepsilon_1(\varepsilon_2) \mid (\varepsilon_1; \varepsilon_2) \mid (\pi x.\varepsilon) \mid (let\; rec\; x\; be\; \varepsilon_1\; in\; \varepsilon_2) \mid (let\; slave\; x\; be\; \varepsilon_1\; in\; \varepsilon_2) \mid (\varepsilon_1\; or\; \varepsilon_2) \mid (\varepsilon_1\; par\; \varepsilon_2) \mid (if\; \varepsilon_0\; then\; \varepsilon_1\; else\; \varepsilon_2) \mid (while\; \varepsilon_1\; do\; \varepsilon_2) \mid (\varepsilon_1\; renew\; \varepsilon_2)$$

The semantic domains (SFP objects) comprise basic values, B, which is not specified further, addresses L (a discrete ipo), $\mathbb{T}$, nondeterministic processes P and pairs of values W. The fundamental domain equations are:

$$V \cong B + L + \mathbb{T} + P + W,$$

$$W \cong V \times V,$$

$$P \cong V \to \mathcal{P}[L \times V \times P].$$

The solution is obtained as described above; the equation for V should be thought of as employing a generalized (five-way) separated sum rather than an iterated binary separated sum. We have five injection functions of B, L, etc., into V. These are all called "*in V*", and used in a postfix fashion. There are also five postfixed discriminator functions $:B$, $:L$, etc., defined similarly to the function $:S$ used above.

Finally there are five postfixed projection functions from V onto $B, L \cdots$ named $|B, |L \cdots$ where, for example,

$$(v|B) = \begin{cases} b, & v = (b\ in\ V), \\ \perp & \text{(otherwise)}. \end{cases}$$

We have also the Cartesian pairing and tripling functions $\langle -, - \rangle$ and $\langle -, -, - \rangle$ and projection functions $(-)_1, (-)_2, (-)_3$; for convenience the Cartesian product in the equation for P is to be regarded as employing the generalized product.

In L there are two distinct addresses, ζ and ν; the latter is intended to address a process for generating a sequence of distinct addresses, also distinct from ζ and ν. In B there is a special value whose injection into V is denoted by "!".

One useful combinator is $K: D \to (E \to D)$, defined by $Kxy = y$. Here D and E are arbitrary domains which should really appear as suffices. Another is the least fixed-point combinator $Y: (D \to D) \to D$. Two more specialized ones are ID, in P, and QUOTE in $V \to P$ defined by:

$$\text{ID} = \lambda v \in V. [\![\langle \iota, v, \perp \rangle]\!],$$

$$\text{QUOTE} = \lambda v \in V. K(\text{ID}\ v).$$

Now we need various functions on processes.

Conditional. DECIDE in $P \to P \to P$ is defined by:

$$\text{DECIDE}\ pq = \lambda v \in V.\text{COND}\ ((v|\mathbb{T}), p!, q!).$$

Extension. EXTEND in $P \to (P \to P) \to P$ is defined recursively by:

$$\text{EXTEND}\ pf = \lambda v \in V. \biguplus [\![\text{COND}\ (EQ(t_1, \iota), ft_3t_2, [\![\langle t_1, t_2, \text{EXTEND}\ t_3 f\rangle]\!] | t \in pv]\!].$$

Serial Composition. $*$ in $P \to P \to P$ is defined by:

$$p^*q = \text{EXTEND}\ p(Kq).$$

Choice. ? in $P \to P \to P$ is defined by

$$p\ ?\ q = \lambda v \in V. pv \uplus qv.$$

Parallel Composition. $\|$ in $P \to P \to P$ is defined recursively by:

$$\begin{aligned}(p\|q) = \lambda v \in V. \biguplus [\![&\text{COND}\ (EQ(s_1, \zeta), (q^* \lambda u \in V.\text{ID}(\langle s_2, u\rangle\ in\ V)) \\ &(v|W)_2, \text{COND}\ (EQ(t_1, \iota), (p^* \lambda u \in V.\text{ID}\ (\langle u, t_2\rangle\ in\ V))(v|W)_1, \\ &[\![\langle s_1, s_2, \lambda u \in V.(s_3\|q)(\langle u, (v|W)_2\rangle\ in\ V)\rangle, \\ &\langle t_1, t_2, \lambda u \in V.(p\|t_3)\langle (v|W)_1, u\rangle\ in\ V\rangle\rangle]\!])) \\ &| s \in p(v|W)_1, t \in q(v|W)_2]\!].\end{aligned}$$

Here we have used the extension of the $[\![\ |\ ]\!]$ notation mentioned above to several variables.

Renewal. : in $P \to P \to P$ defined by:

$$p : q = p^* \lambda v \in V.[\![\langle \iota, v, q\rangle]\!]$$

Binding. BIND in $L \to P \to P \to P$ defined recursively by:

$$\text{BIND}\ \alpha pq = \lambda v \in V. \biguplus [\![\text{COND}\ (EQ(t_1, \alpha), \text{EXTEND}\ q\ (\text{BIND}\ \alpha t_3)t_2, [\![\langle t_1, t_2, \text{BIND}\ \alpha t_3 q\rangle]\!] \mid t \in pv]\!].$$

The domain of environments is Env = (Identifiers → P). It is ranged over by r. Although Identifiers is a set, Env is an SFP object if given the pointwise ordering. We denote by $r[p/x]$ the environment r' differing from r only in that $r[\![x]\!] = p$. The semantic function, $\mathscr{E}$: Expressions → (Env → P) can now be defined by structural induction on expressions by:

$$\mathscr{E}[\![x]\!]r = r[\![x]\!],$$

$$\mathscr{E}[\![\varepsilon_1(\varepsilon_2)]\!]r = (\mathscr{E}[\![\varepsilon_1]\!]r^*(\lambda v \in V.\mathscr{E}[\![\varepsilon_2]\!]r^*\ \text{COND}\ (v : L, \lambda u \in V.[\![\langle v|L, u, ID\rangle]\!], v|P)))!,$$

$$\mathscr{E}[\![(\varepsilon_1; \varepsilon_2)]\!]r = \mathscr{E}[\![\varepsilon_1]\!]r^*\mathscr{E}[\![\varepsilon_2]\!]r,$$

$$\mathscr{E}[\![(\pi x.\varepsilon)]\!]r = \text{QUOTE}\ ((\lambda v \in V.\mathscr{E}[\![\varepsilon]\!]r[(\text{QUOTE}\ v)/x]!)\ \textit{in}\ V),$$

$$\mathscr{E}[\![(\textit{let rec x be}\ \varepsilon_1\ \textit{in}\ \varepsilon_2)]\!]r = \mathscr{E}[\![\varepsilon_2]\!](Y(\lambda r' \in \text{ENV}.r[\mathscr{E}[\![\varepsilon_1]\!]r'/x])),$$

$$\mathscr{E}[\![(\textit{let slave x be}\ \varepsilon_1\ \textit{in}\ \varepsilon_2)]\!]r = \mathscr{E}[\![\varepsilon_1]\!]r^*(\lambda u \in V.[\![\langle \nu, !, (\lambda v \in V.\text{BIND}\ (v|L) (\mathscr{E}[\![\varepsilon_2]\!]r[(\text{QUOTE}\ v)/x])(u|P)!)\rangle]\!]).$$

$$\mathscr{E}[\![(\varepsilon_1\ \textit{or}\ \varepsilon_2)]\!]r = \mathscr{E}[\![\varepsilon_1]\!]r ? \mathscr{E}[\![\varepsilon_2]\!]r,$$

$$\mathscr{E}[\![(\varepsilon_1\ \textit{par}\ \varepsilon_2)]\!]r = \mathscr{E}[\![\varepsilon_1]\!]r /\!/ \mathscr{E}[\![\varepsilon_2]\!]r,$$

$$\mathscr{E}[\![(\textit{if}\ \varepsilon_0\ \textit{then}\ \varepsilon_1\ \textit{else}\ \varepsilon_2)]\!]r = \mathscr{E}[\![\varepsilon_0]\!]r^*\ \text{DECIDE}\ (\mathscr{E}[\![\varepsilon_1]\!]r)(\mathscr{E}[\![\varepsilon_2]\!]r),$$

$$\mathscr{E}[\![(\textit{while}\ \varepsilon_1\ \textit{do}\ \varepsilon_2)]\!]r = Y(\lambda p \in P.\mathscr{E}[\![\varepsilon_1]\!]r^*\ \text{DECIDE}\ (\mathscr{E}[\![\varepsilon_2]\!]p^*r)\text{ID}),$$

$$\mathscr{E}[\![(\varepsilon_1\ \textit{renew}\ \varepsilon_2)]\!]r = \mathscr{E}[\![\varepsilon_2]\!]r^*(\lambda v \in V.(\mathscr{E}[\![\varepsilon_1]\!]r : (v|P))!).$$

Now the denotational semantics, $\mathscr{D}$: Expressions → (Env → P) is obtained by binding in an address-generating process GEN as mentioned above. Thus:

$$\mathscr{D}[\![\varepsilon]\!] = \lambda v \in \text{Env}.\ (\text{BIND}\ \nu(\mathscr{E}[\![\varepsilon]\!]r_0)\ \text{GEN}),$$

where r_0 is a suitable standard environment; $\mathscr{D}[\![\varepsilon]\!]$ will be a process which does not interrogate any addresses, provided none of the processes assigned by r_0 do.

In [11], Milner considered a different binding combinator which involved a domain $Q \cong Q \to P$. It is straightforward to adapt that to the present context.

Acknowledgments. I would like to thank Robin Milner for his part in the development of this work. He contributed not only stimulation, but also general philosophy and concrete ideas. I have also profited from some conversations with Gilles Kahn, and from the many helpful suggestions of the referees. The research was carried out under the direction of Dr. R. M. Burstall.

REFERENCES

[1] J. W. DE BAKKER, *Recursive Procedures*, Mathematical Centre Tracts 24, Math. Centre, Amsterdam, 1971.

[2] J. W. DE BAKKER AND W. P. DE ROEVER, *A calculus for recursive program schemes*, Automata, Languages and Programming, M. Nivat, ed., North-Holland/American Elsevier; New York, 1972.

[3] G. BIRKHOFF, *What can lattices do for you? Trends in Lattice Theory*, J. C. Abbott, ed., Van Nostrand, New York, 1970.

[4] N. BOURBAKI, *Elements of Mathematics: General Topology*, part 1, Addison-Wesley, Reading, Mass., 1966.

[5] J. M. CADIOU AND J. J. LEVY, *Mechanizable proofs about parallel processes*, Proc. 14th Annual Symposium on Switching and Automata Theory, 1973, p. 34.

[6] P. M. COHN, *Universal Algebra*, Harper and Row, New York, 1965.

[7] P. HITCHCOCK AND D. M. R. PARK, *Induction rules and termination proofs*, Automata, Languages and Programming, M. Nivat, ed., North-Holland/American Elsevier, New York, 1972.

[8] G. KAHN, *The semantics of a simple language for parallel programming*, Proceedings of IFIP Congress '74, Stockholm, Sweden, North-Holland, Amsterdam, 1974.

[9] Z. MANNA, *The correctness of non-deterministic programs*, Artificial Intelligence, 1 (1970), pp. 1–26.

[10] R. MILNER, *An approach to the semantics of parallel programs*, Proceedings of the Convegno di Informatica Teorica, Instituto di Elaborazione delle Informazione, Pisa, 1973.

[11] ———, *Processes: A mathematical model of computing agents*, Logic Colloquium, '73, North-Holland, Amsterdam, 1973.

[12] P. MOSSES, *The mathematical semantics of Algol 60*, Programming Research Group Technical Monograph PRG-12, University of Oxford, 1974.

[13] G. D. PLOTKIN, *LCF considered as a programming language*, Theoret. Computer Sci., 1977, to appear.

[14] J. C. REYNOLDS, *Notes on a lattice-theoretic approach to the theory of computation*, Dept. Systems and Information Science, Syracuse University, Syracuse, N.Y., 1972.

[15] ———, *On the relation between direct and continuation semantics*, Automata, Languages and Programming—2nd Colloquium, Univ. of Saarbrücken, Lecture Notes, in Computer Science, vol. 14, Springer-Verlag, Berlin, 1974.

[16] D. SCOTT, *Continuous lattices*, Toposes, Algebraic Geometry and Logic, F. W. Lawvere, ed., Springer-Verlag Lecture Notes, vol. 274, Springer-Verlag, Berlin, 1970.

[17] ———, *Data types as lattices*, Logic Conference, Kiel 1974, A. Dold and B. Eckmann, eds., Springer-Verlag Lecture Notes, vol. 499, Springer-Verlag, Berlin, 1975.

[18] D. SCOTT AND C. STRACHEY, *Towards a mathematical semantics for computer languages*, Proceedings of the Symposium in Computers and Automata, Microwave Research Institute Symposia Series, vol. 21, 1971.

[19] C. P. WADSWORTH, *The relation between computational and denotational properties for Scott's D_∞-models of the lambda calculus*, this Journal, 5 (1976), pp. 488–521.

[20] M. WAND, *On the recursive specification of data types*, Category Theory Applied to Computation and Control, Lecture Notes in Computer Science, vol. 25, Springer-Verlag, Berlin, 1974.

[21] ———, *Fixed-point constructions in order-enriched categories*, Tech. Rep. 23, Computer Science Dept., Indiana University, Bloomington, Indiana, 1975.