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$\mathbb{T}^\omega$ as a Universal Domain

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In mathematical semantics, in the sense of Scott, the question arises of what domains of interpretation should be chosen. It has been felt by the author, and others, that lattices are the wrong choice and instead one should use complete partial orders (cpo's), which do not necessarily have the embarrassing top element. So far, however, no mathematical theory as pleasant as that developed for $P\omega$ in the paper "Data Types as Lattices" has been available. The present paper is intended to fill this gap and is a close analog of the $P\omega$ paper, replacing $P\omega$ by $\mathbb{T}^\omega$, the ω-power of the three-element truthvalue cpo, $\mathbb{T}$.

1. INTRODUCTION

The category of continuous lattices [11] has a very pleasant mathematical theory. For the purpose of giving mathematical semantics for programming languages one only considers the separable ones—those which have a countable basis for their open sets. They can be analyzed in a simple way by using the framework of $P\omega$—the lattice of subsets of ω—as presented by Scott in [12]. Various constructions can be given by easy definitions and in particular many inverse limit constructions can be easily obtained using least fixed points. One has a single language, LAMBDA, which serves to define both domains and continuous functions between them. The language LAMBDA also provides a general theoretical definiton of computability. With this mathematical foundation one can give a semantics for programming languages and use LCF-like logics [7] for investigating their properties and proving programs correct.

The present paper discusses complete partial orders (cpo's) using $\mathbb{T}^\omega$, the Cartesian product of denumerably many copies of $\mathbb{T}$, the truthvalue cpo. The treatment is more or less that of Scott in [12]. Our intention is to show how one can do without the top element, T, but retain many of the advantages of lattices. That is, we are addressing the issues raised in Sections 1.9 and 3.3 of [13]. If so desired, $\mathbb{T}^\omega$ and its attendant theory can be given and discussed within the framework of $P\omega$ itself.

With lattices difficulties are caused by the top element. Sometimes T can be interpreted as the most multivalued element, as in [12] It may also be possible to interpret it as the maximal degree of inconsistency. In most applications though, T has no real computational significance and instead is an embarrassment which causes a plethora of special decisions in definitions and cases in proofs For example, when there are four

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truthvalues there are two different possibilities for defining the conditional function:

$$\mathrm{COND}(T, x, y) = T$$

or else

$$\mathrm{COND}(T, x, y) = x \sqcup y.$$

Either one leads to a failure of an expected identity among those given in [5], say. As illustrated in [8] such difficulties can lead to the natural semantics using lattices not being "fully abstract."

To avoid the top element we use complete partial orders. Two examples are given in Fig. 1.

FIGURE 1

DEFINITION. A *complete partial order* (*cpo*) is a partial order with a bottom element ($\perp$) in which every directed subset, X, has a least upper bound (lub), $\bigsqcup X$.

Any Cartesian product of cpo's is itself a cpo under the natural componentwise ordering. So, for example, $\mathbb{T}^\omega$ is the cpo which is the Cartesian product of ω copies of $\mathbb{T}$.

It is not difficult to carry out the inverse limit construction on cpo's, as in [16], or to use retractions instead, as in [2], with the aid of a single "logical" space such as: $D \cong \mathbb{T} + (D \to D)$. However, one would rather have a simple universal space, like $P\omega$, and develop it along the lines indicated above for lattices.

Just as the category of separable continuous (or algebraic) lattices was more appropriate than that of all complete lattices so one expects to deal with special categories of cpo's. The natural cpo to try as a universal space is $\mathbb{T}^\omega$ as $P\omega = \mathbb{O}^\omega$. The appropriate category is then that of the separably continuous coherent cpo's. The definition of separable continuity is found in Section 4, but it is worth giving the definitions of coherence and the weaker property of consistent completeness here.

DEFINITIONS. Let D be a cpo. A subset, X, of D is *bounded* iff for some d in D and every x in X, $d \sqsubseteq x$; X is *pairwise consistent* iff every pair of elements of X is bounded. A cpo D is *consistently complete* iff every bounded subset has a lub; it is *coherent* iff every pairwise consistent subset has a lub.

Coherence seems to have been explicitly defined for the first time in [4]. The first cpo shown in Fig. 2 is not consistently complete. The second is, but is not coherent.

In [12] Scott recommends using the closed subsets of the retracts of $P\omega$ to avoid the top element. These turn out to be the separably continuous consistently complete cpo's rather than just the coherent ones. It is not clear to the author how much good coherence

FIGURE 2

is, but on the other hand all the usual domains are coherent and all the usual constructions, except the powerdomain constructions [9, 15] preserve coherence. One of these constructions does not even preserve consistent completeness and so it seems we are still some way off from an equally pleasant framework for handling the semantics of programs with nondeterministic features.

Section 2 of this paper introduces $\mathbb{T}^\omega$. It emphasizes a view of elements of $\mathbb{T}^\omega$ as disjoint pairs of sets of integers which provide both positive and negative information. This extends the view of elements of $P\omega$ as providing positive information. Continuous functions are "coded" as elements of $\mathbb{T}^\omega$ by extending the idea used for $P\omega$ which only provides the positive part of the coding. Section 3 presents a language, called LAMBDA, for defining elements of and functions over $\mathbb{T}^\omega$, thus fulfilling the same purposes as does Scott's language of the same name for $P\omega$. Our language is theoretically adequate as it is shown that LAMBDA definability coincides with computability for both elements and functions. We also invest some effort to show why it is even convenient to use LAMBDA to make definitions by developing some notation for truthvalues, integers, and infinite sequences of truthvalues, giving pairing and projection functions, and showing how to make recursive definitions with the least fixed-point operator, which is itself defined by the paradoxical combinator.

Section 4 discusses the category of domains which are retracts of $\mathbb{T}^\omega$. By generalizing the idea used for embedding the function space it is shown that they are, as mentioned above, the separably continuous coherent cpo's. We can then define the effectively given ones to be those given by the computable retractions. This leads to a definition of the computable elements of such domains and the computable functions between them. The result of section three shows that all such retractions, elements, and functions are LAMBDA definable. We invest some effort to show why LAMBDA is convenient for giving definitions of domains via retractions. Following Scott in [12] we give simple definitions of the retractions for some basic domains such as the integers and the truthvalues, show how constructions such as sum, product, and exponentiation are effected and how domain equations can be solved using least fixed points. Section 5 discusses the important subcategory of the algebraic domains which are the ones generally used for semantics. They are the retracts given by the partial closures which form a more inclusive class of retractions than the usual closures. Section 6 discusses the topological aspects of $\mathbb{T}^\omega$. We present an appropriate analog of the notion of injective space [11] and of the extension and embedding theorems in [12]. The idea is to adapt the work from the lattice case by using a stronger notion of subspace in the definition of injective space.

In general, whenever theorems are mechanical reworkings of theorems in [12] (or are just easy) their proofs will be omitted. It should perhaps be admitted that in the end it is more cumbersome to work with cpo's than lattices, but we still feel that the lattices have some unnatural properties. The present paper and [12] demonstrate the pleasures of two categories of separably continuous, complete cpo's. One wonders if there are other such categories and whether they would prove useful for mathematical semantics. Such categories should, in the light of present experience, be Cartesian closed, have ω-products, and be closed under retracts (or perhaps just partial closures).

2. $\mathbb{T}^\omega$ and Its Continuous Functions

There are three slightly different ways to define $\mathbb{T}^\omega$. First we have already seen it as the *Cartesian product* of ω copies of $\mathbb{T}$. So each element t is a vector:

$$\langle t^{(0)}, t^{(1)}, t^{(2)}, \ldots\rangle.$$

The ordering is that inherited from $\mathbb{T}$: $t \sqsubseteq u$ iff $(\forall i \geqslant 0,\ t^{(i)} \sqsubseteq u^{(i)})$. The vector point of view motivates many of the definitions given below.

From a second point of view, $\mathbb{T}^\omega$ is the set of *partial predicates*, $p\colon \omega \to^P \{0, 1\}$. If predicates are subsets of $\omega \times \{0, 1\}$, the ordering is the subset ordering. It turns out that no extra generality would be obtained by using partial functions, $f\colon \omega \to^P \omega$, instead. For we shall see in Section 4 that $\mathbb{N}^\omega$ is a computable retract of $\mathbb{T}^\omega$. Here $\mathbb{N}$ is the integer cpo, $\{\perp, 0, 1, \ldots\}$ with the discrete ordering: $x \sqsubseteq y$ iff $x = y$ or $x = \perp$.

However, the preferred interpretation of $\mathbb{T}^\omega$ in this paper is as the class of *pairs of disjoint sets* of integers. Thus:

$$\mathbb{T}^\omega = \{\langle u_0, u_1\rangle \mid u_0 \subseteq \omega \wedge u_1 \subseteq \omega \wedge u_0 \cap u_1 = \varnothing\}.$$

The ordering is given by:

$$\langle u_0, u_1\rangle \sqsubseteq \langle v_0, v_1\rangle \qquad \text{iff } u_0 \subseteq v_0 \quad \text{and} \quad u_1 \subseteq v_1 .$$

If $t = \langle u_0, u_1\rangle$ is in $\mathbb{T}^\omega$, we put $(t)_i = u_i$ $(i = 0, 1)$. This viewpoint is closest to that of $P\omega$ and seems to give a relatively natural development. In the case of $P\omega$, an element, u can be regarded as giving *positive* information about a set of integers; an element $\langle u, v\rangle$ of $\mathbb{T}^\omega$ gives both *positive* and *negative* information. It partially specifies a set which contains u and whose complement contains v. The maximal elements give *perfect* or *total* specifications. They are the elements, t, such that $(t)_0 \cup (t)_1 = \omega$ and also correspond to the total predicates.

Since there are so many maximal elements, $\mathbb{T}^\omega$ is not a lattice. Let us call two elements x and y of a cpo *compatible* (= *consistent* = *bounded*) if they have an upper bound and denote that by $x \uparrow y$. The contrary case is written as $x \# y$. In the case of $\mathbb{T}^\omega$, $x \# y$ iff $(x)_0 \cap (y)_1 \neq \varnothing$ or $(x)_1 \cap (y)_0 \neq \varnothing$. If $x \uparrow y$ then $\langle (x)_0 \cup (y)_0, (x)_1 \cup (y)_1\rangle$ is in $\mathbb{T}^\omega$ and is the least upper bound of x and y. Thus:

$$\forall x, y \in \mathbb{T}^\omega,\ x \uparrow y \to x \sqcup y \text{ exists.}$$

In conjunction with the fact that $\mathbb{T}^\omega$ is a cpo, we see that $\mathbb{T}^\omega$ is consistently complete. So $\mathbb{T}^\omega$ would become a complete lattice if a top element were added. In general, consistent completeness implies the existence of greatest lower bounds of nonempty sets (glb's). In the present case for any nonempty subset, X, of $\mathbb{T}^\omega$ we find:

$$\bigsqcap X = \left\langle \bigcap \{(x)_0 \mid x \in X\}, \bigcap \{(x)_1 \mid x \in X\} \right\rangle.$$

The cpo $\mathbb{T}^\omega$ is even coherent. Suppose $\{x, y, z\}$ is a pairwise consistent subset of $\mathbb{T}^\omega$. Thus $((x)_0 \cup (y)_0) \cap ((x)_1 \cup (y)_1) = \varnothing$ and so on. Then, $((x)_0 \cup (y)_0 \cup (z)_0) \cap ((x)_1 \cup (y)_1 \cup (z)_1) = \varnothing$, showing:

$$\forall x, y, z \in \mathbb{T}^\omega, x \uparrow y \wedge y \uparrow z \wedge z \uparrow x \rightarrow x \sqcup y \sqcup z \text{ exists.}$$

It then follows by induction that every finite pairwise consistent set has a lub. Since $\mathbb{T}^\omega$ is a cpo, we conclude that it is coherent. In the presence of consistent completeness, coherence can be rephrased as:

$$\forall x \in \mathbb{T}^\omega, \qquad \forall Y \subseteq \mathbb{T}^\omega, \qquad \bigsqcup Y \text{ exists} \wedge x \# \bigsqcup Y \rightarrow \exists y \in Y, \qquad x \# y. \tag{1}$$

The *finite* elements of $\mathbb{T}^\omega$ play a special role. They are the elements, b, such that $(b)_0 \cup (b)_1$ is finite. Alternatively, they are the elements which, if dominated by the lub of a directed subset, are dominated by an element of that subset. The finite elements are sometimes called *compact* or *isolated*. There are denumerably many of them. For any x in $\mathbb{T}^\omega$, $x = \bigsqcup \{b \sqsubseteq x \mid b \text{ finite}\}$ and the set on the right-hand side of the equation is directed. Thus the finite elements form a *basis* of $\mathbb{T}^\omega$, and $\mathbb{T}^\omega$ is *separably* (= *countably* = ω—) *algebraic*. We can even introduce a standard 1–1 enumeration, $b_0, b_1, \ldots$ of the finite elements of $\mathbb{T}^\omega$ by taking b_n to be the unique pair, $\langle u, v\rangle$, of sets of integers such that: $n = \sum_{i \in u} 2 \cdot 3^i + \sum_{u \in v} 3^i$. The idea here is to use ternary expansions. Note that if $k \in (b_n)_i$ then $k < n$ $(i = 0, 1)$. The relations $k \in (b_n)_i$, $b_n \uparrow b_m$, $b_n \sqsubseteq b_m$, and $b_n = b_m \sqcup b_k$ are primitive recursive in their parameters.

We now turn to the continuous functions and consider arbitrary cpo's, D and E. A function $f\colon D \rightarrow E$ is *continuous* iff $f(\bigsqcup X) = \bigsqcup \{f(x) \mid x \in X\}$ for all directed subsets, X, of D. When D and E are cpo's, $D \rightarrow E$ will always be the cpo of all continuous functions from D to E with the pointwise ordering where: $f \sqsubseteq g$ iff $\forall x \in D$, $f(x) \sqsubseteq g(x)$ for all f, g in $D \rightarrow E$. Any continuous function is monotonic. A function of several arguments is continuous iff it is continuous in each of its arguments separately. The collection of continuous functions is closed under substitution. Now we consider these ideas for $\mathbb{T}^\omega$. The next result is stated for functions of one variable but it is easy to give the analogous characterizations in the case of several variables.

THEOREM 1 (The Continuous Function Characterization Theorem). (1) *A function, f, from $\mathbb{T}^\omega$ to $\mathbb{T}^\omega$ is continuous iff for all x in $\mathbb{T}^\omega$,*

$$f(x) = \bigsqcup \{f(b_n) \mid b_n \sqsubseteq x\}.$$

(2) *A function, f, from* $\mathbb{T}^\omega$ *to* $\mathbb{T}^\omega$ *is continuous iff for all x in* $\mathbb{T}^\omega$ *and all* b_m, $b_m \sqsubseteq f(x)$ *iff* $b_m \sqsubseteq f(b_n)$ *for some* b_n *such that* $b_n \sqsubseteq x$.

Proof. (1) Suppose $f: \mathbb{T}^\omega \to \mathbb{T}^\omega$. Then, as $\{b_n \mid b_n \sqsubseteq x\}$ is directed with lub x, $f(x) = \bigsqcup \{f(b_n) \mid b_n \sqsubseteq x\}$ by the definition of continuity.

Suppose the equation holds for all x in $\mathbb{T}^\omega$ and let X be a directed subset of $\mathbb{T}^\omega$. Then:

$$f\left(\bigsqcup X\right) = \bigsqcup \left\{f(b_n) \mid b_n \sqsubseteq \bigsqcup X\right\}$$

$$= \bigsqcup \{f(b_n) \mid \exists x \in X, b_n \sqsubseteq x\} \qquad \text{(as } X \text{ is directed)}$$

$$= \bigsqcup_{x \in X} \bigsqcup \{f(b_n) \mid b_n \sqsubseteq x\}$$

$$= \bigsqcup_{x \in X} f(x) \qquad \text{(by the equation).}$$

(2) Omitted. ∎

For example, we can see that the binary functions, $\vee$ and $\wedge$, and the unary one, $\sim$, defined by:

$$\sim\langle u, v\rangle \underset{\text{def}}{=} \langle v, u\rangle,$$

$$\langle u, v\rangle \vee \langle u', v'\rangle \underset{\text{def}}{=} \langle u \cup u', v \cap v'\rangle,$$

$$\langle u, v\rangle \wedge \langle u', v'\rangle \underset{\text{def}}{=} \langle u \cap u', v \cup v'\rangle,$$

are continuous. As operators on partial predicates, $\vee$ and $\wedge$ are parallel OR and AND.

The next example is motivated by consideration of $\mathbb{T}^\omega$ as a Cartesian power. The continuous functions $tt * \cdot$ and $ff * \cdot$ are defined by:

$$tt * \langle u, v\rangle \underset{\text{def}}{=} \langle\{0\} \cup \{n + 1 \mid n \in u\}, \{n + 1 \mid n \in v\}\rangle,$$

$$ff * \langle u, v\rangle \underset{\text{def}}{=} \langle\{n + 1 \mid n \in u\}, \{0\} \cup \{n + 1 \mid n \in v\}\rangle.$$

Note that $tt * (\langle t^{(0)}, t^{(1)}, \ldots\rangle) = \langle tt, t^{(0)}, t^{(1)}, \ldots\rangle$ and the corresponding equation holds for $ff * \cdot$.

The cpo $\mathbb{T}^\omega \to \mathbb{T}^\omega$ is itself consistently complete, coherent, and ω-algebraic. For consistent completeness, suppose f, g are elements of the cpo and $f \uparrow g$, then for all x in $\mathbb{T}^\omega$, $f(x) \uparrow g(x)$ and $f \sqcup g$ can be given by:

$$f \sqcup g(x) = f(x) \sqcup g(x).$$

If F is any set of continuous functions then for any x in $\mathbb{T}^\omega$:

$$\bigsqcup F(x) = \bigsqcup \{f(x) \mid f \in F\}. \qquad (2)$$

If f and g are continuous functions then $f \sqcap g$ can be given by:

$$f \sqcap g(x) = f(x) \sqcap g(x).$$

This equation is proved by using the fact that the binary greatest lower bound function, $\sqcap$, is continuous. The equation analogous to (2) does not hold. We leave the reader to extablish coherence. The continuity of $\sqcap$ makes it easy to prove that the *parallel conditional* function, COND, is continuous where for x, y, z in $\mathbb{T}^\omega$:

$$\begin{aligned} \mathrm{COND}(x, y, z) &= y && (0 \in (x)_0), \\ &= z && (0 \in (x)_1), \\ &= y \sqcap z && (\text{otherwise}). \end{aligned}$$

The parallelism resides in the fact that $\mathrm{COND}(\bot, y, z)$ is $y \sqcap z$ rather than the bottom element. For then we have $\mathrm{COND}(\bot, y, y) = \mathrm{COND}(tt, y, \bot) = \mathrm{COND}(ff, \bot, y) = y$ where we are anticipating the identification of tt with $\langle\{0\}, \varnothing\rangle$ and ff with $\langle \varnothing, \{0\}\rangle$ in Section 3.

Theorem 1 shows that a continuous function f is determined by the triples of numbers, m, n, i such that $m \in (f(b_n))_i$. This suggests the definition of a *subbasis* of the function space. We will use a standard enumeration, (n, m) of all pairs of integers: $(n, m) = \frac{1}{2}(n + m)(n + m + 1) + m$. Note that $n \leqslant (n, m)$ and $m \leqslant (n, m)$ with equality holding only in the cases (0,0) and (1.0). Now the functions f_k are defined by

$$\begin{aligned} f_{(n,2m+i)}(x) &= \langle\{m\}, \varnothing\rangle && (\text{if } i = 0 \text{ and } x \sqsupseteq b_n), \\ &= \langle \varnothing, \{m\}\rangle && (\text{if } i = 1 \text{ and } x \sqsupseteq b_n), \\ &= \bot && (\text{otherwise}). \end{aligned}$$

LEMMA 1. (1) *Eeach f_k is continuous and finite.*

(2) $f_{(n,2m+i)} \sqsubseteq f_{(n',2m'+i')}$ *iff* $b_n \sqsupseteq b_{n'}$ *and* $m = m'$ *and* $i = i'$.

(3) $f_{(n,2m+i)} \# f_{(n',2m'+i')}$ *iff* $b_n \uparrow b_{n'}$ *and* $m = m'$ *and* $i \neq i'$.

(4) *Each f_k is join prime, in the sense that if f and g are contniuous functions and $f_k \sqsubseteq f \sqcup g$ then $f_k \sqsubseteq f$ or $f_k \sqsubseteq g$.*

(5) *The f_k's form a subbasis in that the class of lubs of finite sets of them is a basis.*

Proof. The proofs of (1)–(4) are omitted; for (5) let f be a continuous function. We have to prove that:

$$f = \bigsqcup \{f_k \mid f_k \sqsubseteq f\}.$$

To do this we take a continuous function, g, such that $f_k \sqsubseteq g$ for all k and show that $f \sqsubseteq g$. Suppose $m \in (f(x))_i$ where x is in $\mathbb{T}^\omega$ and i is 0 or 1. Then for some n, $b_n \sqsubseteq x$ and $m \in (f(b)_n)_i$ by Theorem 1. Then $f_k \sqsubseteq f$ where $k = (n, 2m + i)$ and so $f_k \sqsubseteq g$

and $m \in (g(b_n))_i$. Therefore m is also in $(g(x))_i$ and so $f(x) \sqsubseteq g(x)$, which finishes the proof. ∎

It is now possible to explain simply how to embed the function space in $\mathbb{T}^\omega$. The positive half is given by the function Graph: $(\mathbb{T}^\omega \to \mathbb{T}^\omega) \to P\omega$ where for every $f: \mathbb{T}^\omega \to \mathbb{T}^\omega$ we have:

$$\text{Graph}(f) = \{k \mid f \sqsupseteq f_k\}.$$

The other half is the information about the complement of Graph(f) which can be obtained continuously from f. Define Pred: $(\mathbb{T}^\omega \to \mathbb{T}^\omega) \to \mathbb{T}^\omega$ by:

$$\text{Pred}(f) = \langle \text{Graph}(f), \{k \mid f \# f_k\}\rangle.$$

Since one cannot have both $f \sqsupseteq f_k$ and $f \# f_k$, Pred is well defined. Continuity is established by using the finiteness of the f_k and the coherence of the function space in the form (1).

In order to find an inverse function for Pred, we need to choose characteristic segments, Seg(k) of each Pred(f_k). So for each nonnegative integer k, we put:

$$\text{Seg}(k) = \langle \{k\}, \{n \leqslant k \mid f_n \# f_k\}\rangle.$$

Then it turns out that $\text{Seg}(k) \sqsubseteq \text{Pred}(f_k)$ and $f_k \uparrow f_1$ iff Seg $(k) \uparrow \text{Seg}(1)$. The inverse of Pred is Fun: $\mathbb{T}^\omega \to (\mathbb{T}^\omega \to \mathbb{T}^\omega)$ where for each x in $\mathbb{T}^\omega$:

$$\text{Fun}(x) = \bigsqcup \{f_k \mid \text{Seg}(k) \sqsubseteq x\}.$$

The existence of the lub on the right follows from the coherence of the function space and the above remarks on the Seg(k). The continuity of Fun follows from the finiteness of the Seg(k).

THEOREM 2 (The Predicate Theorem). *Suppose* $f: \mathbb{T}^\omega \to \mathbb{T}^\omega$. *Then* $\text{Fun}(\text{Pred}(f)) = f$.

Proof. Using the definition of Pred and the fact that $\text{Seg}(k) \sqsubseteq \text{Pred}(f_k)$ we see that $\text{Seg}(k) \sqsubseteq \text{Pred}(f)$ iff $f_k \sqsubseteq f\ (k \geqslant 0)$. Therefore,

$$\begin{aligned}\text{Fun}(\text{Pred}(f)) &= \bigsqcup \{f_k \mid \text{Seg}(k) \sqsubseteq \text{Pred}(f)\} = \bigsqcup \{f_k \mid f_k \sqsubseteq f\} \\ &= f \qquad \text{(by Lemma 1.5).} \ ∎\end{aligned}$$

In general it is not the case that $\text{Pred}(\text{Fun}(x)) \sqsupseteq x$. It will be seen in Section 5 why this is expected of any similar embedding. It is left to the reader to discover when $\text{Pred}(\text{Fun}(x)) = x$.

It is interesting that if we had defined Pred by:

$$\text{Pred}(f) = \langle \text{Graph}(f), \varnothing\rangle$$

or

$$\text{Pred}(f) = \langle \{(n, m) \mid m \in (f(b_n))_0\}, \{(n, m) \mid m \in (f(b_n))_1\}\rangle$$

then it would have had no continuous inverse, Fun. For suppose there was such an inverse. In the first case if we choose continuous functions f and g # such that $f \# g$ we then find the contradiction that $f = \text{Fun}(\text{Pred}(f)) \uparrow \text{Fun}(\text{Pred}(g)) = g$ as $\text{Pred}(f) \uparrow \text{Pred}(g)$. The second case is less straightforward. Put $g = f_{(0,0)}$ and $h_r = f_{(r,1)}$ for $r \geqslant 0$. Then $g \# h_r$ for all $r \geqslant 0$. Now $\text{Pred}(g) = \langle\{(n, 0) \mid n \geqslant 0\}, \varnothing\rangle$. So we can choose a k such that $g = \text{Fun}(\langle\{(n, 0) \mid 0 \leqslant n \leqslant k\}, \varnothing\rangle)$ as g is finite, by Lemma 1, $g = \text{Fun}(\text{Pred}(g))$ and Fun is continuous. Put $g_k = \langle\{(n, 0) \mid 0 \leqslant n \leqslant k\}, \varnothing\rangle$. Then $g_k \# \text{Pred}(h_r) = \langle\varnothing, \{(n, 0) \mid b_n \sqsupseteq b_r\}\rangle$ as otherwise we would have $g = \text{Fun}(g_k) \uparrow h_r$. But we can choose a number r so that $b_n \sqsupseteq b_r$ implies that $n > k$. For example, let X be a maximal consistent subset of $\{b_i \mid 0 \leqslant i \leqslant k\}$ and choose r so that $b_r \sqsupseteq \bigsqcup X$ but $b_r \neq \bigsqcup X$. Now we find that $g_k = \langle\{(n, 0) \mid 0 \leqslant n \leqslant k\}, \varnothing\rangle \uparrow \langle\varnothing, \{(n, 0) \mid b_n \sqsupseteq b_r\}\rangle$ which is the required contradiction.

Theorem 2 can be extended to the case of several variables. First define $\text{Abs}_k : ((\mathbb{T}^\omega)^{k+1} \to \mathbb{T}^\omega) \to (\mathbb{T}^\omega \to ((\mathbb{T}^\omega)^k \to \mathbb{T}^\omega))$ for each $k > 0$, where for each f in $(\mathbb{T}^\omega)^{k+1} \to \mathbb{T}^\omega$, and $x, x_1, \ldots, x_k$ in $\mathbb{T}^\omega$:

$$\text{Abs}_k(f)(x)(x_1, \ldots, x_k) = f(x, x_1, \ldots, x_k).$$

Then define $\text{Pred}_k : ((\mathbb{T}^\omega)^k \to \mathbb{T}^\omega) \to \mathbb{T}^\omega$ and $\text{Fun}_k : \mathbb{T}^\omega \to ((\mathbb{T}^\omega)^k \to \mathbb{T}^\omega)$ for $k > 0$ by induction on k:

$$\begin{aligned} \text{Pred}_1 &= \text{Pred}, \qquad \text{Fun}_1 = \text{Fun}, \\ \text{Pred}_{k+1}(f) &= \text{Pred}(\text{Pred}_k \circ (\text{Abs}_k(f))), \\ \text{Fun}_{k+1}(x)(x_1, \ldots, x_{k+1}) &= \text{Fun}_k(\text{Fun}(x)(x_1))(x_2, \ldots, x_{k+1}). \end{aligned} \tag{3}$$

It is straighforward to use induction on k to prove that the functions Pred_k and Fun_k are well defined and continuous and to show that when f is a continuous function of k arguments:

$$\text{Fun}_k(\text{Pred}_k(f)) = f.$$

Finally we note the well-known fixed-point property of continuous functions (which is true for any cpo, not just $\mathbb{T}^\omega$ or $P\omega$):

THEOREM 3 (The Fixed-Point Theorem). *Suppose* $f: \mathbb{T}^\omega \to \mathbb{T}^\omega$. *Then f has a least fixed point*, $\text{Fix}(f) = \bigsqcup\{f^n(\bot) \mid n \geqslant 0\}$. *Furthermore, as a map*, $\text{Fix}: (\mathbb{T}^\omega \to \mathbb{T}^\omega) \to \mathbb{T}^\omega$ *is itself continuous.*

3. Computability and Definability

We now present a language, LAMBDA, which serves the same functions as the language of the same name given by Scott in [12]. Although apparently a little weak, it serves to define all the computable elements of $\mathbb{T}^\omega$. Its syntax and semantics are given in Table I. The syntax is on the left and the semantics is on the right. There would, of course, be no difficulty in giving a formal definition along the usual lines. Note that there are three unary functions, $tt * \cdot, ff * \cdot$, and $TL(\cdot)$, a *conditional expression*, *if x then y else z*, one binary function, $\cdot(\cdot)$, for *application*, and one variable-binding operator, $\lambda x \cdot \tau$, for *functional abstraction.*

TABLE I

The Language LAMBDA

$$tt * x = \langle \{0\} \cup \{n+1 \mid n \in (x)_0\}, \{n+1 \mid n \in (x)_1\}\rangle$$

$$ff * x = \langle \{n+1 \mid n \in (x)_0\}, \{0\} \cup \{n+1 \mid n \in (x)_1\}\rangle$$

$$TL(x) = \langle \{n \mid n+1 \in (x)_0\}, \{n \mid n+1 \in (x)_1\}\rangle$$

$$(\textit{if } x \textit{ then } y \textit{ else } z) = \mathrm{COND}(x, y, z)$$

$$x(y) = \mathrm{Fun}(x)(y)$$

$$\lambda x \cdot \tau = \langle (n, 2m+i) \mid m \in ([b_n/x]\tau)_i \text{ and } i = 0 \text{ or } 1\},$$
$$\{(n, 2m+i) \mid \exists n' \cdot b_{n'} \uparrow b_n \wedge m \in ([b_{n'}/x]\tau)_{1-i} \text{ and } i = 0 \text{ or } 1\}\rangle$$

Definitions. An element, x, of $\mathbb{T}^\omega$ is (*LAMBDA*) *definable* iff there is a closed LAMBDA term τ such that x is the value of τ. (Closed terms are those possessing no free variables.) A function, $f: (\mathbb{T}^\omega)^k \to \mathbb{T}^\omega$ $(k > 0)$ is (*LAMBDA*) *definable* iff there is a LAMBDA term τ with free variables $x_1, ..., x_k$ (say) such that, for all $x_1, ..., x_k$ in $\mathbb{T}^\omega$, $f(x_1, ..., x_k) = \tau$. In other words, τ defines f if $f = \lambda x_1, ..., x_k : \mathbb{T}^\omega \cdot \tau$, using the logical typed λ-calculus notation.

Theorem 4 (The Continuity Theorem). *All LAMBDA terms define continuous functions.*

Proof. By structural induction on terms. ▮

The definition of functional abstraction was chosen in accordance with the ideas in Section 2 and it is straightforward to show that for all terms τ:

$$\lambda x_1 \cdots \lambda x_k \cdot \tau = \mathrm{Pred}_k(\lambda x_1, ..., x_k : \mathbb{T}^\omega \cdot \tau). \tag{4}$$

Theorem 5 (The λ-Calculus Theorem). *The laws* (α), (β), (ξ), (ξ^*), *and* (μ) *displayed in Table II all hold.*

Proof. The proof uses the case $k = 1$ of (4). ▮

On the other hand (η) fails. For by the case $k = 1$ of (4), $\lambda y \cdot x(y) = \mathrm{Pred} \circ \mathrm{Fun}(x)$ which, in general, is not x. In fact for a suitable choice of x, $x \# \lambda y \cdot x(y)$.

Continuous functions of several variables can be reduced to functions of one variable and LAMBDA can be reduced to application and a few primitives.

TABLE II

Some λ-Calculus Laws

(α)	$\lambda x \cdot \tau = \lambda y \cdot [y/x]\tau$	(y not free in τ)
(β)	$(\lambda x \cdot \tau)(y) = [y/x]\tau$	
(ξ)	$\lambda x \cdot \tau = \lambda x \cdot \sigma$	iff $\tau = \sigma$ for all x in $\mathbb{T}^\omega$
(ξ^*)	$\lambda x \cdot \tau \sqsubseteq \lambda x \cdot \sigma$	iff $\tau \sqsubseteq \sigma$ for all x in $\mathbb{T}^\omega$
(η)	$y = \lambda x \cdot y(x)$	
(μ)	$x \sqsubseteq x'$ and $y \sqsubseteq y'$ implies $x(y) \sqsubseteq x'(y')$	

THEOREM 6 (The Reduction Theorem). *Suppose* $f: (\mathbb{T}^\omega)^k \to \mathbb{T}^\omega$ $(k > 0)$ *and* u *is in* $\mathbb{T}^\omega$. *Then* $f(x_1, ..., x_k) = u(x_1) \cdots (x_k)$ *holds for all* $x_1, ..., x_k$ *in* $\mathbb{T}^\omega$ *iff* $\lambda x_1 \cdots \lambda x_k \cdot u(x_1) \cdots (x_k) = \mathrm{Pred}_k(f)$.

Proof. Sufficiency is proved by induction on k, using (β), Theorem 2, and (3). Necessity follows from (4) by taking τ to be the term $u(x_1) \cdots (x_k)$. ∎

From now on we will identify continuous functions over $\mathbb{T}^\omega$ of k arguments, f, with their image, $\mathrm{Pred}_k(f)$ in $\mathbb{T}^\omega$. Theorem 6 shows that this identification respects application and using Lemma 1, one can prove that Pred_k commutes with the lub operation. The identification respects composition and we have, for example, $\mathrm{Pred}(f \circ g) = \mathrm{Pred}(f) \circ \mathrm{Pred}(g)$ where the composition operator on the right is $\lambda f \cdot \lambda g \cdot \lambda x \cdot f(g(x))$. This is proved using Theorem 6. However, in general $\mathrm{Pred}_k(f \sqcap g)$ is *not* $\mathrm{Pred}_k(f) \sqcap \mathrm{Pred}_k(g)$, although we do have for all sets, F, of continuous functions over $\mathbb{T}^\omega$ of k arguments and $x_1, ..., x_k$ in $\mathbb{T}^\omega$ that:

$$\Big(\bigsqcap_{f\in F} \mathrm{Pred}_k(f)\Big)(x_1) \cdots (x_k) = \Big(\bigsqcap F\Big)(x_1, ..., x_k).$$

Finally, the identification respects definability for if τ defines f then, by (4), $\lambda x_1 \cdots \lambda x_k \cdot \tau$ defines $\mathrm{Pred}_k(f)$. Since functions are being identified with predicates we call our model the *partial predicate model* to contrast with the graph model [12].

THEOREM 7 (The Combinator Theorem). *The LAMBDA-definable elements (functions) can be generated by iterated application from the six constants* CONST, CONSF, TL, COND, K, *and* S (*and variables*) *where* CONST $=_{\mathrm{def}} \lambda x \cdot tt * x$, CONSF $=_{\mathrm{def}} \lambda x \cdot ff * x$, $TL =_{\mathrm{def}} \lambda x \cdot TL(x)$, COND $=_{\mathrm{def}} \lambda x \cdot \lambda y \cdot \lambda z$ *if* x *then* y *else* z, $K =_{\mathrm{def}} \lambda x \cdot \lambda y \cdot x$ *and* $S =_{\mathrm{def}} \lambda x \cdot \lambda y \cdot \lambda z \cdot x(z)(y(z))$.

It is even possible to show that *TL* and COND are definable from CONST, CONSF, K, and S, but we find it more convenient to use the broader base. Another well-known combinator is the paradoxical combinator $Y =_{\mathrm{def}} \lambda x \cdot (\lambda y \cdot x(y(y)))(\lambda y \cdot x(y(y)))$, which turns out to give *least* fixed points.

THEOREM 8 (The First Recursion Theorem). *For any continuous function, $f: \mathbb{T}^\omega \to \mathbb{T}^\omega$, $Y(f) = \mathrm{Fix}(f)$, the least fixed point of f.*

Proof. Let $a = \mathrm{Fix}(f)$ and $d = \lambda y \cdot f(y(y))$.

Then $Y(f) = d(d) = f(d(d)) = f(Y(f))$. Therefore $Y(f)$ is a fixed point of f and so as a is the least one, $a \sqsubseteq Y(f)$.

For the converse we show by induction on l that if $b_l \sqsubseteq d$ then $b_l(b_l) \sqsubseteq a$. The case where $l = 0$ is trivial. For $l > 0$ suppose $k \in (b_l(b_l))_i$.

Then for some $b_n \sqsubseteq b_l$, $k \in (b_l(b_n))_i$ and $(n, 2k + i) \in (b_l)_0$. Therefore $l > n$ and as $b_l \sqsubseteq d$, $k \in (d(b_n))_i$. Now we have:

$$\begin{aligned} d(b_n) &= f(b_n(b_n)) && \text{(by the definition of } d\text{)},\\ &\sqsubseteq f(a) && \text{(by induction hypothesis)},\\ &= a && \text{(as } a \text{ is a fixed point of } f\text{)}. \end{aligned}$$

So $k \in (a)_i$ and we see that $b_l(b_l) \sqsubseteq a$, as desired. ∎

Theorem 8 concerns the pure λ-calculus aspects of LAMBDA as the term Y does not contain any occurrences of $tt * \cdot$, $ff * \cdot$, $TL(\cdot)$, or the conditional. More information can be obtained by following the ideas of Hyland and Wadsworth [3, 17]. As is now customary, we work with the $\lambda\Omega$-calculus which is obtained from the λ-calculus by adding a constant, Ω, which is to denote $\perp$.

DEFINITIONS. A $\lambda\Omega$-term, σ, is in *$\beta\Omega$-normal form* iff it contains no subterms of any of the forms $(\lambda x \cdot \sigma)(\tau)$, $\lambda x \cdot \Omega$, or $\Omega(\sigma)$. We use $=_\beta$ for β-equivalence and, following [3], we put, for any $\lambda\Omega$-term, σ:

$$\omega(\sigma) =_{\mathrm{def}} \{\sigma' \mid \sigma' \text{ is a } \beta\Omega\text{-normal form obtained from some } N, \text{ where } N =_\beta M, \text{ by replacing subterms of } N \text{ by } \Omega\}.$$

THEOREM 9 (The Approximant Theorem). *For all values assigned to the free variables of a $\lambda\Omega$-term, σ, the set of values of members of $\omega(\sigma)$ is directed and has as lub the value of σ.*

Proof. The proof uses indexed terms, following exactly the lines of Theorem 2.5 of [3] or Theorem 5.2 of [17]. ∎

This strengthens Theorem 8 as $\omega(Y) = \{\lambda f \cdot f^n(\Omega) \mid n \geqslant 0\}$. We further conjecture that for two $\lambda\Omega$-terms σ and τ, $\sigma \sqsubseteq \tau$ for all values of their free variables iff $\omega(\sigma) \subseteq \omega(\tau)$. The proof would involve further investigation of the way the η-rule fails in the model. There should also be a similar characterization of when $\sigma \uparrow \tau$ for all values of their free variables.

Next we shall see how to use LAMBDA to give simple explicit definitions for some interesting functions and elements of $\mathbb{T}^\omega$. The bottom element is the least fixed point of the identity and so:

$$\perp = Y(\lambda x \cdot x) = \langle \perp, \perp, \ldots \rangle.$$

The truthvalues are:

$$tt = tt * \perp = \langle tt, \perp, \perp, \ldots \rangle$$

and

$$ff = ff * \perp = \langle ff, \perp, \perp, \ldots \rangle .$$

So we are identifying $\mathbb{T}$ with the first component of $\mathbb{T}^\omega$, in harmony with the definition of the conditional in Section 2. Arbitrary elements can be turned into truthvalues by the function BOOL, where for any x in $\mathbb{T}^\omega$:

$$\mathrm{BOOL}(x) = \textit{if } x \textit{ then } tt \textit{ else } ff.$$

Note that Bool just projects $\mathbb{T}^\omega$ onto its first component. Sometimes we need the *strict* (= *sequential*) conditional, SCOND where for x, y, z in $\mathbb{T}^\omega$:

$$\mathrm{SCOND}(x, y, z) = \textit{if } x \textit{ then } (\textit{if } x \textit{ then } y \textit{ else } \perp) \textit{ else } z.$$

This function differs from COND in that $\mathrm{SCOND}(\perp, y, z)$ is $\perp$ rather than $y \sqcap z$. It is also useful to have a function, $\cdot * \cdot$, which generalizes $tt * \cdot$ and $ff * \cdot$ where for x, y in $\mathbb{T}^\omega$:

$$x * y \underset{\text{def}}{=} (\textit{if } x \textit{ then } tt * y \textit{ else } ff * y) = \langle x^{(0)}, y^{(0)}, y^{(1)}, \ldots \rangle .$$

Note that we are using infix notation for $*$. Any missing brackets should be associated to the right. Whenever they help the eye, we will prefer infix and other notational devices to the standard form for function application. The function, $*$, helps to define finite vectors of truthvalues. For example,

$$\langle ff, ff, ff, tt \rangle \underset{\text{def}}{=} ff * ff * ff * tt.$$

This idea allows one to represent integers by:

$$\bar{n} \underset{\text{def}}{=} \mathrm{CONSF}^n(tt) = \langle ff, \ldots, ff, tt \rangle$$

where there are n ff's. The test for zero and successor and predecessor functions is easily defined:

$$\mathrm{ZERO} \underset{\text{def}}{=} \mathrm{BOOL},$$

$$x \dot{-} 1 \underset{\text{def}}{=} \mathrm{TL}(x),$$

and

$$x + 1 \underset{\text{def}}{=} \mathrm{SCOND}(x)(ff * x)(ff * x).$$

Note that we only care that they give the right results when applied to numbers and send $\perp$ to $\perp$. Actually every partial recursive function $f(m, n, \ldots)$ is definable in the sense that there is a term τ such that if $f(m, n, \ldots) = l$ then $\tau(\bar{m}, \bar{n}, \ldots) = \bar{l}$ and if $f(m, n, \ldots)$ is not defined then $\tau(\bar{m}, \bar{n}, \ldots) = \perp$, and $\tau(\ldots, \perp, \ldots) = \perp$. First of all the successor function, the constantly zero function, and the projections are all definable and the class of functions definable in this sense is closed under composition. Next primitive recursion is handled using the least fixed-point operator Y as in [12]. Finally we treat minimalization by example: Suppose f is a binary primitive recursive function and we want to define $g(m) = \mu n \cdot f(m, n) = 0$. If $\hat{f}$ defines f, then $\hat{g}$ defines g where:

$$\hat{g} \underset{\text{def}}{=} Y(\lambda h \cdot \lambda m \cdot \lambda n \cdot \text{SCOND}(\text{ZERO}(\hat{f}(m, n)))\, n(h(m, n + 1)))(\bar{0}).$$

This is almost all we will need to show that LAMBDA definability and computability coincide.

We can now make the infinite vector notation, $\langle x^{(0)}, x^{(1)},\ldots\rangle$, official in LAMBDA. First for x, n in $\mathbb{T}^{\omega}$ we define,

$$x^{(n)} = (\textit{if } \text{Zero}(n) \textit{ then } \text{BOOL}(x) \textit{ else } (TL(x))^{(n \dot{-} 1)}). \tag{5}$$

Notice the recursive style of the definition. It is intended that $\lambda x \cdot \lambda n \cdot x^{(n)}$ is the least function satisfying (5). In other words, it is the fuction f where:

$$f = Y(\lambda f \cdot \lambda x \cdot \lambda n \cdot \textit{if } \text{Zero}(n) \textit{ then } \text{BOOL}(x) \textit{ else } f(TL(x), n \dot{-} 1))).$$

This method of definition will also be used below. Now if we recursively define $\langle x\rangle$ by:

$$\langle x\rangle = x(\bar{0}) * \langle \lambda n \cdot x(n + 1)\rangle,$$

we see that $x = \langle \lambda n \cdot x^{(n)}\rangle$ as expected. Also, $\langle x\rangle^{(\bar{n})} = \text{BOOL}(x(\bar{n}))$ for x in $\mathbb{T}^{\omega}$ and $n \geqslant 0$. The vector notation makes it easy to give definitions coordinate by coordinate. For example, if we put $\text{BOR}(x, y) = (\textit{if } x \textit{ then } tt \textit{ else } y)$, we have:

$$x \vee y = \langle \lambda n \cdot \text{BOR}(x^{(n)}, y^{(n)})\rangle,$$

and $\wedge$ and $\sim$ can be defined similarly.

Pairs of arbitrary elements of $\mathbb{T}^{\omega}$ are dealt with rather differently from pairs of truth-values. For x, y in $\mathbb{T}^{\omega}$ we put:

$$[x, y] = x * y * [TL(x), TL(y)],$$

and also,

$$\pi_0(x) = x * \pi_0(TL(TL(x))) \qquad \text{and} \qquad \pi_1(x) = \pi_0(TL(x)).$$

Note that $[x, y] = \langle x^{(0)}, y^{(0)}, x^{(1)}, y^{(1)},\ldots\rangle$. The point of the definition is that $[\cdot, \cdot]$ is an isomorphism of $\mathbb{T}^{\omega} \times \mathbb{T}^{\omega}$ and $\mathbb{T}^{\omega}$ as is demonstrated by the equations $\pi_0([x, y]) = x$, $\pi_1([x, y]) = y$, and $[\pi_0(x), \pi_1(x)] = x$ for x, y in $\mathbb{T}^{\omega}$. Similar methods can be used to show that $(\mathbb{T}^{\omega})^n \cong \mathbb{T}^{\omega}$ where n is any nonnegative integer, or even ω.

Having seen something of the scope of LAMBDA definability, it is now time to connect up definability and computability.

DEFINITIONS. An element, x, of $\mathbb{T}^\omega$ is *computable* iff $\{n \mid b_n \sqsubseteq x\}$ is recursively enumerable. A function, f, of k arguments, is *computable* iff the relation $m \in (f(b_{n_1}, ..., b_{n_k}))_i$ is recursively enumerable in $m, n_1, ..., n_k, i$.

Note that an element x is computable iff $(x)_0$ and $(x)_1$ are a disjoint pair of re sets iff x corresponds to a partial recursive predicate. The computable functions are closed under composition and the operation of fixing an argument to be a computable value. Also the computable elements are closed under the application of computable functions to computable elements. Our definitions of computability agree with those in [1, 14], for example, and we claim it is also intuitively reasonable.

THEOREM 10 (The Definability Theorem). (1) *An element of* $\mathbb{T}^\omega$ *is computable iff it is* LAMBDA *definable.*

(2) *A function (of k arguments) is computable iff it is* LAMBDA *definable.*

Proof. In one direction, it follows from Theorem 7 that every LAMBDA-definable function and element is computable. For one can check that the six constants are computable and that Fun is computable, and so every element (function) generated by iterated application from the constants (and variables) is computable.

Conversely, suppose an element x of $\mathbb{T}^\omega$ is computable. Then it corresponds to a partial recursive predicate which can be defined, in the sense stated above, by a term τ. But then x is LAMBDA definable, as $x = \langle \tau \rangle$.

In the case of computable functions of one variable, f, the definition of Pred shows that Pred(f) is computable. Then it is defined by some term, τ, say and f itself is defined by $\tau(x)$. In the case of several variables it is easier to proceed indirectly and we indicate the idea by taking the case of a computable function, f, of two variables. Since f, π_0, and π_1 are computable, $g(x) =_{\text{def}} f(\pi_0(x), \pi_1(x))$ is a computable function of one variable which therefore can be defined by a term τ, say. But then f itself is defined by $\tau([x, y])$. ∎

Thus the computable elements of $\mathbb{T}^\omega$ form a model for LAMBDA which obeys (α), (β), (ξ^*), and (μ). Indeed any class containing the definable elements and closed under application forms a model obeying these laws. In the model of the computable elements, the maximal ones correspond to the recursive sets and those elements which are not dominated by any maximal elements correspond to the recursively inseparable sets.

There seems to be no difficulty in working out the theory analogous to that of Section 3 in [12] on enumeration and degrees. This task is left to the interested reader. He might note that the reducibility relationship is just that one obtained via the recursive operators defined in Section 9.8 of [10] except that one uses partial predicates rather than functions which makes no difference to the degree structure. However, the degrees *are* different to the enumeration degrees, as is shown in [10].

4. RETRACTIONS

The retractions on $\mathbb{T}^\omega$ are a convenient means whereby complete partial orders other than $\mathbb{T}^\omega$ are considered.

DEFINITION. An element, a, of $\mathbb{T}^\omega$ is a *retraction* iff $a \circ a = a$. Its associated cpo is $\mathrm{Dom}(a) =_{\mathrm{def}} \langle\{x \mid a(x) = x\}, \sqsubseteq\rangle$ where $\sqsubseteq$ is inherited from $\mathbb{T}^\omega$.

If a is a retraction, the retract $\mathrm{Dom}(a)$ is a cpo with least element $a(\bot)$ and its lub operation is that of $\mathbb{T}^\omega$, when restricted to directed sets. We will write $x{:}\, a$ to mean x is a member of $\mathrm{Dom}(a)$ and $\lambda x{:}\, a \cdot \tau$ for $\lambda x \cdot [a(x)/x]\, \tau$.

Many basic cpo's are, to within isomorphism, given by retractions. For example, $\mathbb{T}$ is given by:

$$\mathrm{BOOL} = \lambda x \cdot (\mathit{if}\ x\ \mathit{then}\ tt\ \mathit{else}\ ff).$$

The cpo, $\mathbb{N}$, is given by the retraction INT, where:

$$\mathrm{INT} \underset{\mathrm{def}}{=} \lambda x \cdot \mathit{if}\ \mathrm{ZERO}(x)\ \mathit{then}\ \bar{0}\ \mathit{else}\ \mathrm{INT}(x \dot{-} 1) + 1.$$

Note that $\mathrm{Dom(INT)} = \{\bot, \bar{0}, \bar{1}, \ldots\}$ and so the inherited order is correct.

The domain of binary sequences with the subsequence ordering is given by:

$$\mathrm{BINSEQ} \underset{\mathrm{def}}{=} \lambda x \cdot \mathit{if}\ \mathrm{NE}(x)\ \mathit{then}\ \bot\ \mathit{else}\ x^{(0)} * \mathrm{BINSEQ}(\mathrm{TL}(x)).$$

Here NE is just $\lambda x \cdot \mathrm{SCOND}(x)(ff)(ff)$. The idea is that finite binary sequences are represented by finite vectors of truthvalues, such as $ff * tt * tt * ff$. Infinite ones are represented by maximal elements of $\mathbb{T}^\omega$.

The domain $P\omega$ is given by:

$$PW \underset{\mathrm{def}}{=} \lambda x \cdot \langle \lambda n \cdot \mathit{if}\ x^{(n)}\ \mathit{then}\ tt\ \mathit{else}\ \bot\rangle.$$

The idea here is that subsets of ω are represented by vectors with no ff components. Since $P\omega$ is a retract of $\mathbb{T}^\omega$ all the lattices of [12] are retracts of $\mathbb{T}^\omega$. Of course $\mathbb{T}^\omega$ is not a retract of $P\omega$ as retracts of lattices are lattices.

As mentioned in Section 1 one way of characterizing the retracts is to identify them as the coherent separably continuous cpo's. We now give the required definitions.

DEFINITIONS. Let D be a cpo and let d and e be elements of D. Then $d \ll e$ iff for every directed subset, X, of D, $e \sqsubseteq \bigsqcup X$ implies $d \sqsubseteq x$ for some x in X. Read $d \ll e$ as "d is way below e." A subset B of D is a *basis* for D iff for every element, d, $B_d =_{\mathrm{def}} \{b \in B \mid b \ll d\}$ is directed and has lub d. The cpo D is *separably* (= *countably* = ω—) *continuous* iff it has a denumerable basis.

This definition of the ω-continuous cpo's agrees with those given in [11, 14]. Here are the elementary properties of $\ll$.

LEMMA 2. *Let D be an ω-continuous cpo. Let x, y, z be in D.*

(1) $x \ll y \to x \sqsubseteq y$.

(2) $x \sqsubseteq y \leftrightarrow \forall d \in D\ (d \ll x \to d \ll y)$.

(3) $x \sqsubseteq y \wedge y \ll z \to x \ll z$.

(4) $x \ll y \leftrightarrow \exists d \in D\ (x \ll d \wedge d \ll y)$.

(5) $x \ll z \wedge y \ll z \wedge x \sqcup y$ *exists* $\to x \sqcup y \ll z$.

In the case of $\mathbb{T}^\omega$, $x \ll y$ iff $x = b_n \sqsubseteq y$ for some n. So $\mathbb{T}^\omega$ is separably continuous with basis $\{b_n \mid n \geq 0\}$. Thus $\mathbb{T}^\omega$ is a coherent separably continuous cpo. For the purposes of this paper such cpo's will be called *domains.*

THEOREM 11 (Characterization Theorem for Domains). *A cpo is a domain iff it is, to within isomorphism, a retract of $\mathbb{T}^\omega$.*

Proof. Let a be a retraction. Suppose X is a pairwise consistent subset of Dom(a). Then it is also pairwise consistent in $\mathbb{T}^\omega$ and so its lub, $\bigsqcup X$, in $\mathbb{T}^\omega$, exists. But then its lub in Dom(a) is $a(\bigsqcup X)$ and so Dom(a) is coherent.

Now set $B =_{\text{def}} \{a(b_n) \mid n \geq 0\}$. If x is in Dom(a) then $x = a(x) = \bigsqcup \{a(b_n) \mid b_n \sqsubseteq x\}$. The set on the right is directed and is a subset of $B_x = \{d \in B \mid d \ll x$ holds in Dom(a)$\}$. For suppose $Y \subseteq$ Dom(a) is directed and $x \sqsubseteq \bigsqcup Y$. Then if $b_n \sqsubseteq x$, $b_n \sqsubseteq y$ for some y in Y and so $a(b_n) \sqsubseteq a(y) = y$. So $\{a(b_n) \mid b_n \sqsubseteq x\}$ is a directed subset of B_x with lub x. This implies that B_x is directed with lub x. As B is countable, we have shown Dom(a) to be a domain.

Conversely suppose D is a domain and let e_0, e_1,... be an enumeration of a countable basis of D. Define $\varphi: D \to \mathbb{T}^\omega$ by:

$$\varphi(d) \underset{\text{def}}{=} \langle \{n \mid d \gg e_n\}, \{n \mid d \# e_n\} \rangle.$$

It is straightforward to check continuity. The proof uses consistent completeness.

For each t in $\mathbb{T}^\omega$, let $X_t =_{\text{def}} \{e_i \mid i \in (t)_0 \wedge \forall j \leq i,\ e_j \# e_i \to j \in (t)_1\}$. First X_t is pairwise consistent. For suppose e_i and $e_{i'}$ are in X_t. We can suppose that $i \in (t)_0$ and if $j \leq i$ and $e_j \# e_i$ then $j \in (t)_1$ and similarly for i'. If $i' \leq i$ then $e_{i'} \uparrow e_i$ as otherwise $i' \in (t)_1$ which contradicts $i' \in (t)_0$. Similarly if $i \leq i'$, $e_{i'} \uparrow e_i$ and so X is pairwise consistent. One can therefore define $\psi: \mathbb{T}^\omega \to D$ by:

$$\psi(t) = \bigsqcup_D X_t\,.$$

This is a good definition as D is coherent. The function ψ is continuous, the important point being that the universal quantifier in the definition of X_t is restricted to a finite set.

Suppose $x \in D$. We show that $e_i \ll x$ iff $e_i \in X_{\varphi(d)}$. Now, if $e_i \ll x$ then $i \in (\varphi(d))_0$ and if $j \leq i$ and $e_j \# e_i$ then $e_j \# x$ and so $j \in (\varphi(d))_1$. Thus $e_i \in X_{\varphi(d)}$. Conversely, if $e_i \in X_{\varphi(d)}$ then for some i', $e_i = e_{i'}$ and $i' \in (\varphi(d))_0$ and so $e_i = e_{i'} \ll x$. Therefore, $\psi \circ \varphi(x) = \bigsqcup \{e_i \mid e_i \ll x\} = x$. It follows that D is isomorphic to $\varphi(D)$, the image of D under φ and so that $\varphi \circ \psi$ is a retraction with domain $\varphi(D)$. ∎

Next we consider operations on retracts which correspond to domain constructions in the style of [12].

DEFINITION. For retractions a and b we write $a \circ\!\leqslant b$ for $a = a \circ b = b \circ a$.

Although we have not defined retractions on arbitrary cpo's, it should be clear that $a \circ\!\leqslant b$ implies that Dom(a) is a retract of Dom(b). Now $\circ\!\leqslant$ is a partial order and if a and b are commuting retractions, $a \circ b$ is their glb wrt $\circ\!\leqslant$. If a sequence of retractions is increasing with respect to $\circ\!\leqslant$ and $\sqsubseteq$, its lub with respect to $\sqsubseteq$ is a retraction which is also its lub with respect to $\circ\!\leqslant$. The element, $\perp$, is the least strict retraction. (A function a, is *strict* iff $a(\perp) = \perp$.) The element $(\lambda x \cdot x)$ is the largest retraction. For any retraction a there is a strict retraction, a', representing the same domain to within isomorphism:

$$a'(x) = \langle (a(x))_0 - (a(\perp))_0 \,, (a(x))_1 - (a(\perp))_1 \rangle.$$

Note that forming a' is not even a monotonic function of a.

The three main operators on retractions are $\circ\!\!\rightarrow$, $\otimes$, and $\oplus$:

$$a \circ\!\!\rightarrow b = \lambda f \cdot b \circ f \circ a,$$

$$a \otimes b = \lambda x \cdot [a \circ \pi_0(x), b \circ \pi_1(x)],$$

$$a \oplus b = \lambda x \cdot \mathrm{SCOND}(x)(\mathrm{CONST} \circ a \circ TL(x))(\mathrm{CONSF} \circ b \circ TL(x)).$$

One calculates that if a, a', b, b' are any elements of $\mathbb{T}^\omega$:

$$(a \circ\!\!\rightarrow b) \circ (a' \circ\!\!\rightarrow b') = (a' \circ a) \circ\!\!\rightarrow (b \circ b'),$$

$$(a \otimes b) \circ (a' \otimes b') = (a \circ a') \otimes (b \circ b'),$$

$$(a \oplus b) \circ (a' \oplus b') = (a \circ a') \oplus (b \circ b').$$

THEOREM 12 (The Function Space Theorem). *Let a, b, a', b', c be retractions. Then:*

(1) *$a \circ\!\!\rightarrow b$ is a retraction and is strict if b is,*

(2) *f: $a \circ\!\!\rightarrow b$ iff $f = \lambda x$: $a \cdot f(x)$ and $\forall x$: $a \cdot f(x)$: b,*

(3) *if $a \circ\!\leqslant a'$ and $b \circ\!\leqslant b'$ then $a \circ\!\!\rightarrow b \circ\!\leqslant a' \circ\!\!\rightarrow b'$,*

(4) *if f: $a \circ\!\!\rightarrow b$ and f': $a' \circ\!\!\rightarrow b'$ then $f \circ\!\!\rightarrow f'$: $(b \circ\!\!\rightarrow a') \circ\!\!\rightarrow (a \circ\!\!\rightarrow b')$,*

(5) *if f: $a \circ\!\!\rightarrow b$ and f': $b \circ\!\!\rightarrow c$ then $f' \circ f$: $a \circ\!\!\rightarrow c$.*

Part 2 tells us that Dom($a \circ\!\!\rightarrow b$) $\cong$ Dom(a) $\rightarrow$ Dom(b) and also indicates the isomorphism. Indeed one has a category with objects the strict retractions and morphisms the functions f: $a \circ\!\!\rightarrow b$. As can be seen from Theorems 11 and 12 this category is equivalent to the category of domains and continuous functions. The operator $\circ\!\!\rightarrow$ on the former category is a functor which induces the exponentiation functor on the second category.

THEOREM 13 (The Product Theorem). *Let a, b, a', b' be retractions. Then:*

(1) *$a \otimes b$ is a retraction which is strict if a and b are,*

(2) *d: $a \otimes b$ iff $\pi_0(d)$: a and $\pi_1(d)$: b,*

(3) *if $a \circ\!\!\leqslant a'$ and $b \circ\!\!\leqslant b'$ then $a \otimes b \circ\!\!\leqslant a' \otimes b'$,*

(4) *if f: $a \circ\!\!\rightarrow b$ and f': $a' \circ\!\!\rightarrow b'$ then $f \otimes f'$: $a \otimes a' \circ\!\!\rightarrow b \otimes b'$.*

We see from part 2 that $\mathrm{Dom}(a \otimes b) \cong \mathrm{Dom}(a) \times \mathrm{Dom}(b)$, and, as in Section 4 of [12], one sees that $\otimes$ is the product functor and the category of strict retractions is Cartesian closed. The same holds for the category of domains.

THEOREM 14 (The Sum Theorem). *Suppose a, b, a', b' are retractions. Then:*

(1) *$a \oplus b$ is a strict retraction,*

(2) *d: $a \oplus b$ iff $d = \bot$ or $d = \mathrm{CONST} \circ TL(d)$ and $TL(d)$: a or $d = \mathrm{CONSF} \circ TL(d)$ and $TL(d)$: b,*

(3) *if $a \circ\!\!\leqslant a'$ and $b \circ\!\!\leqslant b'$ then $a \oplus b \circ\!\!\leqslant a' \oplus b'$,*

(4) *if f: $a \circ\!\!\rightarrow b$ and f': $a' \circ\!\!\rightarrow b'$ then $f \oplus f'$: $a \oplus a' \circ\!\!\rightarrow b \oplus b'$.*

Thus $\mathrm{Dom}(a \oplus b) \cong \mathrm{Dom}(a) + \mathrm{Dom}(b)$ the separated sum of $\mathrm{Dom}(a)$ and $\mathrm{Dom}(b)$. There are various injection and projection functions such as in Section 4 of [12]. There is no difficulty in extending $\oplus$ and $\otimes$ to more factors, even denumerably many. In particular, we see that $\mathbb{N}^\omega$ is a computable retract of $\mathbb{T}^\omega$ which is why there was no loss of generality in our considering partial predicates ($\mathbb{T}^\omega$) rather than partial functions ($\mathbb{T}^\omega$). If one used strict retractions one also has amalgamated sums and products [2, 6].

THEOREM 15 (The Limit Theorem). *Suppose F is a function in $\mathbb{T}^\omega$ which maps retractions to retractions and set $c = Y(F)$. Then c is a retraction. Suppose too that F preserves strictness and is monotonic with respect to $\circ\!\!\leqslant$. Then c is strict and if d is a strict retraction such that $Fd \circ\!\!\leqslant d$ then $c \circ\!\!\leqslant d$. Finally $\mathscr{D} =_{\mathrm{def}} \langle D_m, \varphi_{mn}\rangle$ is a directed ω-sequence in the category of domains and embeddings with colimit, $\underrightarrow{\lim}\, \mathscr{D} = \mathrm{Dom}(c)$. Here $D_m = \mathrm{Dom}(F^m(\bot))$ and for $m \leqslant n$, $\varphi_{mn} = \lambda d{:}\, D_m \cdot d$.*

Proof. First, suppose F sends retractions to retractions. Then each $F^m(\bot)$ is a retraction and so too, therefore, is c as $c \circ c = \bigsqcup F^m(\bot) \circ F^n(\bot) = \bigsqcup_m F^m(\bot) \circ F^m(\bot) = c$. Next, suppose F preserves strictness and is monotonic with respect to $\circ\!\!\leqslant$. Then each $F^m(\bot)$ is strict and so too, therefore, is c. Also $\langle F^m(\bot)\rangle_{n=0}^{\infty}$ is an increasing sequence with respect to both the orderings, $\sqsubseteq$ and $\circ\!\!\leqslant$. Therefore, by a remark made above, c is the lub of this sequence with respect to both orderings. Now if d is a strict retraction such that $Fd \circ\!\!\leqslant d$ then by induction on m, one sees that $F^m(\bot) \circ\!\!\leqslant d$ for all m. Therefore $c \circ\!\!\leqslant d$.

We only sketch the proof of the last part. Let D and E be cpo's. A continuous map, f: $D \rightarrow E$ is an *embedding* iff there is a, necessarily unique, continuous map, f^R: $E \rightarrow D$ such that $f^R \circ f = I_D$, the identity on D and $f \circ f^R \sqsubseteq I_E$. The map f^R is called the *right adjoint* of f. The category, Dom^E, of domains and embeddings is a subcategory of the category of domains and continuous functions. To say that $\mathscr{D}$ is a directed sequence in Dom^E means each φ_{mn} is an embedding, $\varphi_{mn} = I_{D_m}$, and $\varphi_{mn} \circ \varphi_{lm} = \varphi_{ln}$ when

$0 \leqslant l \leqslant m \leqslant n$. These facts are easily checked and it turns out that $\varphi^R_{mn} = \lambda d: D_n \cdot F^m(\perp)(d)$.

Now Dom^E is a full subcategory of the category CPO^E considered in [16] where it was shown that in that category, $\varinjlim \mathscr{D}$ exists and is $D_\infty =_{\mathrm{def}} \{\langle x_m \rangle_{m=0} \mid x_m \in D_m \varphi^R_{m(m+1)}(x_{m+1}) = x_m\}$ with the componentwise ordering. But we can define a map θ: Dom$(c) \to D$ by, $\theta(d) = \langle F^m(\perp)(d)\rangle$ and check that θ is an isomorphism. Therefore D_∞ is an object in Dom^E and so it is also $\varinjlim \mathscr{D}$ in Dom^E which concludes the proof, as Dom$(c) \simeq D$. ∎

Thus certain colimits can be defined, using Y, and then analyzed using categorical notions. It should be remarked that Dom^E is actually closed under colimits of directed ω-sequences. With the aid of these last few theorems and the basic retractions such as BOOL, INT, and so on one can solve many recursive domain equations just as in [12]. The domains obtained in this way are effectively given in the sense that they are given by computable retractions. The computable elements of such domains are those elements which are computable as elements of $\mathbb{T}^\omega$ itself, according to the definition given in Section 3. The effective maps between such domains are then the computable elements of the appropriate function spaces found by Theorem 12. Following the ideas of Smyth [14] one can give an independent definition of the effectively given domains by using his idea of an effectively given basis. Then one obtains an effective version of Theorem 11.

In his paper on $P\omega$, Scott recommended dealing with cpo's by using a suitable top-cutting operator. It does not seem too convenient to use arbitrary pairs $\langle a, t\rangle$ of retractions and top-cutting operators to represent domains, where $\langle a, t\rangle$ represents $\{u: a \mid t(u) = \perp\}$. The difficulty is that it does not seem to be possible to define an exponentiation operator. However we can use a *fixed* t with the aid of our considerations on $\mathbb{T}^\omega$. Define $\langle TW, TOP\rangle$ by putting for u in $P\omega$:

$$TOP(u) = \{n \mid \exists m \geqslant 0 \cdot \{2m, 2m+1\} \subseteq u\},$$
$$TW(u) = u \cup TOP(u).$$

Then $\langle TW, TOP\rangle$ represents $\mathbb{T}^\omega$ and we temporarily identify each element x of $\mathbb{T}^\omega$ with $\{2m + i \mid m \in (x)_i \ (i = 0, 1)\}$. Clearly, computable elements are identified with computable elements.

Define the function Ap: $(P\omega)^2 \to P\omega$ by:

$$\begin{aligned} Ap(u, v) &= \mathrm{Fun}(u)(v) \qquad && (u, v \in \mathbb{T}^\omega) \\ &= T && (u \notin \mathbb{T}^\omega \text{ or } v \notin \mathbb{T}^\omega). \end{aligned}$$

The function Ap is clearly computable and so as K, S, CONST, and CONSF are computable, using Theorem 7, we can associate to each term τ of the language LAMBDA for $\mathbb{T}^\omega$ a term τ^* of the language LAMBDA for $P\omega$ which has the same free variables, $x_1, \ldots, x_k$ say, and which has the same value as τ when $x_1, \ldots, x_k$ take values in $\mathbb{T}^\omega$.

Now if a is a retraction in $\mathbb{T}^\omega$ then $\bar{a} =_{\mathrm{def}} \lambda u \cdot Ap(a)(u)$ is a retraction in $P\omega$ and, indeed, $\langle \bar{a}, TOP\rangle$ represents Dom(a). If a is defined by τ then $\bar{a}$ is clearly defined by $\lambda u \cdot AP(\tau^*)(u)$ where AP defines Ap. So we can conveniently use the results of this section to define the retractions we want in $P\omega$. It is also possible to find suitable operators $\overline{\circ\!\!\rightarrow}$ and so on First, one defines a computable function Ret: $P\omega \rightarrow P\omega$ such that if a is a retraction in $\mathbb{T}^\omega$ then $\text{Ret}(\bar{a}) = a$. Then $\overline{\circ\!\!\rightarrow}$ is defined by:

$$u \mathbin{\overline{\circ\!\!\rightarrow}} v \underset{\text{def}}{=} \text{Ap}(\text{Ap}(\text{Ap}(\circ\!\!\rightarrow, \text{Ret}(u)), \text{Ret}(v))),$$

which ensures that $\bar{a} \mathbin{\overline{\circ\!\!\rightarrow}} \bar{b} = \overline{a \circ\!\!\rightarrow b}$. In this way one can work rather directly in $P\omega$.

5. Partial Closures

Most domains which arise in mathematical semantics are, in fact, algebraic. These domains are given by a special kind of retraction—the partial closures.

Definitions. Let D be a cpo. An element, d, of the cpo is *finite* (= *isolated* = *compact*) iff $d \ll d$; D is *separably* (= *countably* = ω—) *algebraic* iff it has a countable basis of finite elements.

The lub of any two finite elements is finite. Any basis of an ω-algebraic cpo contains all the finite elements. So an ω-algebraic cpo has essentially one basis, the set of its finite elements. We have seen above that $\mathbb{T}^\omega$ and its function space are ω-algebraic.

In the case of $P\omega$, ω-algebraic lattices arose from closures, which are just retractions dominating the identity. In the case of $\mathbb{T}^\omega$ the identity is the only closure. (For suppose a is a closure but $a(x) \neq x$. As $a(x) \sqsupseteq x$ there is an element $y \sqsupseteq x$ such that $y \# a(x)$. But $a(y) \sqsupseteq y$ and $a(y) \sqsupseteq a(x)$. So $y \uparrow a(x)$, which is a contradiction.) We ask instead that a is increasing on enough elements to ensure that Dom(a) is algebraic.

Definition. Let a be a retraction and put $F(a) =_{\text{def}} \{b_n \mid b_n \sqsubseteq a(b_n)\}$. Then a is a *partial closure* iff $a(x) = \bigsqcup \{a(b_n) \mid b_n \sqsubseteq a(x) \wedge b_n \in F(a)\}$, for all x in $\mathbb{T}^\omega$.

Note that if b_n and b_m are in $F(a)$ then so is $b_n \sqcup b_m$, if it exists. A *projection* is a retraction which is less than the identity. Every projection, p, is a partial closure, as $p(b_n)$ is in $F(p)$ for all n. In particular the functions BOOL, INT, BINSEQ, and PW are all projections. Note that the corresponding domains are all ω-algebraic. This is explained by the next theorem, which also gives some justification for the rather ugly definition of a partial closure.

Theorem 16 (The Partial Closure Theorem). *Let a be a retraction. The following are equivalent*:

(1) *The retraction a is a partial closure.*

(2) *There is a retraction c and a projection p such that* Dom(a) = Dom(c), *and* $c \circ p = c \sqsupseteq p$.

(3) *The domain,* Dom(a), *is ω-algebraic.*

Proof. (3) $\Rightarrow$ (1) Suppose Dom(a) is ω-algebraic and let F be the set of its finite elements. We show that a is partial closure. If $d \in F$ then $d = \bigsqcup \{a(b_n) \mid b_n \sqsubseteq d\}$. As d is finite and the set on the right is directed, $d = a(b_n)$ for some b_n such that $b_n \sqsubseteq d$. So $b_n \in F(a)$. It follows that as Dom(a) is ω-algebraic, for any x in $\mathbb{T}^\omega$, $a(x) = \bigsqcup \{d \sqsubseteq a(x) \mid d \in F\} = \bigsqcup \{a(b_n) \mid b_n \sqsubseteq a(x) \wedge b_n \in F(a)\}$. That is, a is a partial closure.

(1) $\Rightarrow$ (2) Now suppose that a is a partical closure. Define p by $p(d) = \bigsqcup \{b_n \sqsubseteq d \mid b_n \in F(a)\}$. Then p is a projection and $p \sqsubseteq a$, as if $b_n \sqsubseteq d$ and $b_n \in F(a)$ then $b_n \sqsubseteq a(b_n) \sqsubseteq a(d)$. Let $c = a \circ p$. Then $c \circ c = a \circ p \circ a \circ p \sqsupseteq a \circ p \circ p \circ p = c$ and also $c \circ c = a \circ p \circ a \circ p \sqsubseteq a \circ a \circ p = c$. Thus c is a retraction. Also $c \circ p = a \circ p \circ p = c$ and $c = a \circ p \sqsupseteq p \circ p = p$. It remains to show that Dom(a) = Dom(c). As $c = a \circ p$, Dom(c) $\subseteq$ Dom(a). Suppose d: a. Then $d = a(d) = \bigsqcup \{a(b_n) \mid b_n \sqsubseteq a(d) \wedge b_n \in F(a)\} = a(\bigsqcup \{b_n \mid b_n \sqsubseteq d \wedge b_n \in F(a)\}) = a \circ p(d)$. So Dom($a$) $\subseteq$ Dom(c), as required.

(2) $\Rightarrow$ (3) Let $B = \{c(b_n) \mid n \geqslant 0\}$. We show that B is a basis of finite elements for Dom(c). First suppose X is a directed subset of Dom(c) and $c(b_n) \sqsubseteq \bigsqcup X$. Then $p(b_n) \sqsubseteq \bigsqcup X$, as $p \sqsubseteq c$ and so for some x in X, $p(b_n) \sqsubseteq x$ as $p(b_n)$ is finite in $\mathbb{T}^\omega$. Therefore $c(b_n) = c \circ p(b_n) \sqsubseteq c(x) = x$. So each $c(b_n)$ in B is finite in Dom(c). But if x: c then $x = c(x) = \bigsqcup \{c(b_n) \mid b_n \sqsubseteq x\}$. So $\{c(b_n) \mid b_n \sqsubseteq x\}$ is a directed subset of $B_x = \{c(b_n) \mid c(b_n) \ll x$ holds in Dom(a)$\}$ and also has lub x. Thus B is a basis for Dom(a) = Dom(c). ∎

As the function space, $\mathbb{T}^\omega \to \mathbb{T}^\omega$ is ω-algebraic, Fun is a partial closure. The retraction, c, and the projection, p, mentioned in part (2) of Theorem 16, can be taken to be Fun and Funpart, respectively, where Funpart is given by Funpart(x) = $\bigsqcup \{\text{Seg}(k) \mid \text{Seg}(k) \sqsubseteq x\}$.

Corollary 1 (Characterization Theorem for Algebraic Domains). *A cpo D is an ω-algebraic domain iff it is given by a partial closure.*

Proof. If D is ω-algebraic then it is an ω-continuous domain. Therefore by Theorem 11 it is given by a retraction. As it is ω-algebraic, Theorem 16 implies that the retraction is a partial closure. The converse is immediate from Theorem 16. ∎

Theorem 17 (Exponentiation, Sum, and Product Theorems for Partial Closures). *Let a and b be partial closures. Then so are $a \circ\!\!\rightarrow b$, $a \otimes b$, and $a \oplus b$.*

Proof. It is known (see any of [4, 8, 14]), that if D and C are consistently complete ω-algebraic cpo's then so is $D \to C$. So if a and b are partial closures, Dom(a) and Dom(b) are consistently complete ω-algebraic domains and so too, therefore, is Dom($a \circ\!\!\rightarrow b$) $\cong$ Dom(a) $\to$ Dom(b), as coherence follows from the fact that $a \circ\!\!\rightarrow b$ is a retraction and Theorem 11. Therefore by Theorem 16, $a \circ\!\!\rightarrow b$ is a partial closure. A similar proof works for $\otimes$ and $\oplus$. Alternatively one can obtain a short proof by using condition (2) of Theorem 16. ∎

The section concludes with the description of a universe function, V, for the partial closures. First we need some notation. Put

$$[m; n] \underset{\text{def}}{=} \bigsqcup \{\text{Seg}((m, 2r + i)) \mid r \in (b_n)_i, i = 0 \text{ or } 1\} \qquad \text{for all } m \geqslant 0 \text{ and } n \geqslant 0.$$

Then $[m; n]$ behaves as a step function in that $[m; n](x) = b_n$ if $x \sqsupseteq b_m$ but otherwise is $\bot$. Also we have $[m; n] \sqsubseteq f$ iff $b_n \sqsubseteq f(b_m)$, for any function f. Now we can put:

$$\mathrm{Pcpart}(a) \underset{\mathrm{def}}{=} \bigsqcup \{[1; m] \sqsubseteq a \mid \exists n \cdot b_m \sqsubseteq b_n \wedge [m; n] \sqsubseteq a\}.$$

Note that Pcpart is a computable projection dominated by Fnpart. Also:

$$\mathrm{Pcpart}(a) \circ \mathrm{Pcpart}(a)(x) \sqsupseteq \mathrm{Pcpart}(a)(x). \tag{6}$$

For suppose $[l; m] \sqsubseteq a$, $[m; n] \sqsubseteq a$, $b_m \sqsubseteq b_n$ and $b_l \sqsubseteq x$. Then $b_m \sqsubseteq \mathrm{Pcpart}(a)(x)$ and $[m; m] \sqsubseteq \mathrm{Pcpart}(a)$ as $[m; m] \sqsubseteq [m; n]$. Therefore $b_m \sqsubseteq \mathrm{Pcpart}(a) \circ \mathrm{Pcpart}(a)(x)$. Now define Q by putting for a in $\mathbb{T}^\omega$:

$$Q(a) \underset{\mathrm{def}}{=} \lambda x \cdot \mathrm{Pcpart}(a)(x).$$

By (6), $Q(a) \circ Q(a) \sqsupseteq Q(a)$ and so $(Q(a))^n$ is increasing with n and we may define the computable function V by putting for any a in $\mathbb{T}^\omega$:

$$V(a) = \bigsqcup_{n \geqslant 1} (Q(a))^n.$$

THEOREM 18 (Universe Theorem for ω-Algebraic Domains). *The function V is a partial closure operation and its fixed points comprise the set of all partial closure operations.*

Proof. First we show that $V(a) = a$ iff a is a partial closure operation. Suppose a is a partial closure operation. Then for any x in $\mathbb{T}^\omega$:

$$\begin{aligned} a(x) &= \bigsqcup \{a(b_l) \mid b_l \sqsubseteq x\} \\ &= \bigsqcup \{a(b_m) \mid b_l \sqsubseteq x \wedge b_m \sqsubseteq a(b_l) \wedge b_m \in F(a)\} \\ &\qquad\qquad\qquad \text{(by the definition of a partial closure)} \\ &= \bigsqcup \{b_n \mid b_l \sqsubseteq x \wedge b_m \sqsubseteq a(b_l) \wedge b_m \in F(b_m) \wedge b_m \sqsubseteq b_n \sqsubseteq a(b_m)\} \\ &= \mathrm{Pcpart}(a)(x) \qquad \text{(by the definition of Pcpart)}. \end{aligned}$$

So, as a is a function, $\mathrm{Pcpart}(a) = a$; thus $Q(a) = a$ and as a is a retraction, $V(a) = a$.

Conversely, suppose $V(a) = a$. Then $a^2 = V(a) \circ V(a) = \bigsqcup_{n \geqslant 1} (Q(a))^{2n} = V(a) = a$. So a is a retraction. As $Q(a) \circ V(a) = V(a)$, $a = V(a) = Q(a) \circ V(a) = Q(a) \circ a$. Therefore, if x is in $\mathbb{T}^\omega$:

$$\begin{aligned} a(x) &= Q(a)(a(x)) \\ &= \bigsqcup \{b_m \mid \exists l, \exists n, b_l \sqsubseteq a(x) \wedge [l; m] \sqsubseteq a \wedge [m; n] \sqsubseteq a \wedge b_m \sqsubseteq b_n\} \\ &\qquad\qquad\qquad \text{(by the definition of Pcpart)} \\ &= \bigsqcup \{b_m \mid b_m \sqsubseteq a(x) \wedge b_m \in F(a)\} \qquad \text{(as } a \text{ is a retraction)} \\ &= \bigsqcup \{a(b_m) \mid b_m \sqsubseteq a(x) \wedge b_m \in F(a)\}. \end{aligned}$$

The last line follows because if $b_n \sqsubseteq a(b_m)$, $b_m \sqsubseteq a(x)$, and $b_m \in F(a)$ then $(b_m \sqcup b_n) \in F(a)$ and $(b_m \sqcup b_n) \sqsubseteq a(x)$. So we see that a is indeed a partial closure.

Finally, we must show that V itself is a partial closure operation. We have already seen that Pcpart is a projection. Also $V \sqsupseteq$ Pcpart as $Q \sqsupseteq$ Pcpart. Now, $(Q \circ \text{Pcpart})(a) = \lambda x \cdot \text{Pcpart}^2(a)(x) = Q(a)$, as Pcpart is a retraction. Therefore, $V \circ \text{Pcpart} = V$. It follows that $V \circ V \sqsupseteq V$ and it only remains to show that $V \circ V \sqsubseteq V$. If $k \geqslant 1$ we have:

$$\begin{aligned} Q((Q(a))^k) &= \lambda x \cdot \text{Pcpart}((Q(a))^k)(x) \\ &\sqsubseteq \lambda x \cdot \text{Funpart}((Q(a))^k)(x) \qquad (\text{as Pcpart} \sqsubseteq \text{Funpart}) \\ &= Q(a)^k \qquad (\text{as this is a function}). \end{aligned}$$

Therefore $V^2(a) \sqsubseteq \bigsqcup_{k,n \geqslant 1} Q(a)^{k+n} = V(a)$, concluding the proof. ▮

In the light of this theorem, we can rewrite Theorem 17 as: $(\lambda a: V \cdot \lambda b: V \cdot a\,op\,b)$: $V \circ\!\!\rightarrow (V \circ\!\!\rightarrow V)$, where op is $\circ\!\!\rightarrow$, $\otimes$, or $\oplus$. We also have: $(\lambda f: V \circ\!\!\rightarrow V \cdot Y(f))$: $(V \circ\!\!\rightarrow V) \circ\!\!\rightarrow V$, as V is a retraction. This shows that the direct limits considered in Theorem 15 are ω-algebraic if their components are. As it happens, one can show that if $\mathscr{D}$ is any directed ω-sequence of ω-algebraic domains in Dom^E then $\varinjlim \mathscr{D}$ is ω-algebraic.

6. Topological Considerations

The domain $\mathbb{T}$ has a very natural T_0 topology which is closely associated with the partial order on $\mathbb{T}$. As basis one takes the empty set together with $\{ff\}$, $\{tt\}$, and $\mathbb{T}$. Then $\mathbb{T}^\omega$ inherits the product topology which has a basis consisting of the empty set together with the sets B_n where for $n \geqslant 0$:

$$B_n = \{x \in \mathbb{T}^\omega \mid x \sqsupseteq b_n\}.$$

It can be shown without difficulty that the continuous functions from $\mathbb{T}^\omega$ to $\mathbb{T}^\omega$ are just those functions which are continuous with respect to this T_0 topology. One can then go on to consider the various kinds of equational sets as in Section 6 of [12]. It is straightforward to formulate and prove analogs of the theorems there. We just quote one.

Theorem 19 (The $\mathscr{B}_\delta$ Theorem). *The sets that are countable intersections of Boolean compbinations of open sets of $\mathbb{T}^\omega$ are precisely those of the form: $\{x \mid f(x) = g(x)\}$ where f and g are continuous functions from $\mathbb{T}^\omega$ to $\mathbb{T}^\omega$.*

This theorem indicates the possible scope of equational theories for LAMBDA. In the rest of the section we give a topological equivalent of the concept of a domain and show that the resulting category is isomorphic to the category of domains and continuous maps. Along the way, we present analogs of the extension and embedding theorems in Section 1 of [12], and an analog of the notion of injective space of [11]. The development of the material largely follows that of the latter paper. The main difference is that we use a stronger notion than that of a topological subspace.

DEFINITION. Suppose X is a subspace of a topological space Y. It is an *isochordal* subspace iff whenever U_1, U_2 are disjoint open subsets of X then there are disjoint open subsets V_1, V_2 of Y such that $U_i = V_i \cap X$ $(i = 1, 2)$.

For example, Cantor Space is, to within isomorphism, an isochardal subspace of $\mathbb{T}^\omega$. Under the evident embedding $\mathbb{T}^\omega$ is not an isochordal subspace of $P\omega$. It should be remarked that the definition of an isochordal subspace was chosen just so that the crucial Lemma 3 (see below) would hold. Indeed Lemma 3 can be taken as an alternative definition.

THEOREM 20 (Embedding Theorem). *Suppose X is a separable T_0 space. Then there is an embedding of X into $\mathbb{T}^\omega$ as an isochordal subspace of $\mathbb{T}^\omega$.*

Proof. Let $U_0, U_1 \cdots$ be an enumeration of the countable basis of X. Define $\epsilon: X \to \mathbb{T}^\omega$ by putting, for x in X:

$$\epsilon(x) = \langle \{n \mid x \in U_n\}, \{m \mid \exists n \cdot x \in U_n \wedge U_n \cap U_m = \varnothing\}\rangle.$$

The function ϵ is certainly well defined and continuous. The T_0 hypothesis ensures that ϵ is 1–1.

If U is an open set of X then:

$$\begin{aligned}\epsilon(U) &= \{\epsilon(x) \mid x \in U\}\\ &= \epsilon(X) \cap \{t \in \mathbb{T}^\omega \mid \exists i,\ U_i \subseteq U \wedge i \in (t)_0\}.\end{aligned}$$

Therefore $\epsilon(U)$ is an open subset of $\epsilon(X)$.

Finally let U^1, U^2 be disjoint open sets of X. If $x \in U^1$ and $U_j \subseteq U^2$ then $j \in (\epsilon(x))_1$. So:

$$\begin{aligned}\epsilon(U^1) &= \epsilon(X) \cap \{t \in \mathbb{T}^\omega \mid \exists i,\ U_i \subseteq U^1 \wedge i \in (t)_0 \wedge (\forall j \leqslant i,\ U_j \subseteq U^2 \to j \in (t)_1)\}\\ &= \epsilon(X) \cap V^1, \qquad \text{say.}\end{aligned}$$

The definition of V^1 shows it is open as the universal quantifier is restricted to a finite set. Define V^2 similarly but with U^2 and U^1 interchanged. To see that V_1 and V_2 are disjoint suppose, for the sake of contradiction, that t is an element of $V^1 \cap V^2$. Let i be the smallest number such that $U_{i_1} \subseteq U^1$ and $i_1 \in (t)_0$ and $(\forall j \leqslant i_1,\ U_j \subseteq U^2 \to j \in (t)_1)$. Define i_2 similarly. If $i_2 \leqslant i$ then $i_2 \in (t)_1$, which contradicts $i_2 \in (t)_0$. Similarly we have a contradiction if $i_1 \leqslant i_2$, which concludes the proof. ∎

DEFINITION. Let D be a topological space. It is *injective over isochordal subspaces* iff whenever X is an isochordal subspace of a space Y and $f: X \to D$ is topologically continuous then f can be extended to a topologically continuous map $\bar{f}: Y \to D$.

THEOREM 21 (Extension Theorem). *$\mathbb{T}^\omega$ is injective over isochordal subspaces.*

This is an immediate corollary of two lemmas.

LEMMA 3. *$\mathbb{T}$ is injective over isochordal subspaces.*

Proof. Let X be an isochordal subspace of Y and suppose $f: X \to \mathbb{T}$ is topologically continuous. Set $U_1 = f^{-1}(tt)$ and $U_2 = f^{-1}(ff)$. By hypothesis there are disjoint open sets V_1, V_2 of Y such that $U_i = X \cap V_i$. Define $\bar{f}: Y \to \mathbb{T}$ by:

$$\begin{aligned} \bar{f}(y) &= tt && (y \in V_1), \\ &= ff && (y \in V_2), \\ &= \bot && (\text{otherwise}). \end{aligned}$$

Then $\bar{f}$ is the required extension. ▌

LEMMA 4. *Any Cartesian product of spaces which are injective over isochordal subspaces is itself injective over isochordal subspaces.*

Proof. Obvious. ▌

Now we consider topological retractions of $\mathbb{T}^\omega$. They are clearly just the retractions defined in Section 4 since the two notions of continuity coincide. If a is a retraction its associated topological retract, Inj(a), is $\{a(x) \mid x \in \mathbb{T}^\omega\}$ equipped with the subspace topology. The retract inherits the T_0-property from $\mathbb{T}^\omega$. If U is an open set of Inj(a) then $U = f^{-1}(U) \cap \text{Inj}(a)$. So if U_1, U_2 are disjoint open subsets of Inj(a) then $f^{-1}(U_1)$, $f^{-1}(U_2)$ are disjoint open subsets of $\mathbb{T}^\omega$ containing U_1 and U_2, respectively. Thus Inj(a) is always an isochordal subspace of $\mathbb{T}^\omega$.

THEOREM 22 (Characterization Theorem for Injective Spaces). *A topological space D is separable and injective over isochordal subspaces iff it is a retract of $\mathbb{T}^\omega$ to within isomorphism.*

Proof. One half is like the proof of Proposition 1.4 of [11] and the other half is like that of Corollary 1.6 of that paper. ▌

COROLLARY 2 (The Isomorphism Theorem). *The category with objects those separable spaces which are injective over isochordal subspaces and with morphims the topologically continuous maps is isomorphic to the category of domains and (order-) continuous maps.*

Proof. Theorem 22 shows that the first category is equivalent to the full subcategory whose objects are the spaces, Inj(a). The work in Section 4 shows that the second category is equivalent to the full subcategory whose objects are the domains, Dom(a). We describe an isomorphism, F, of the latter two categories. The equivalences then show how to build an isomorphism of the former two.

For objects we put $F(\text{Inj}(a)) = \text{Dom}(a)$. As the retraction a only determines the carrier of the structures this is a well-defined bijection of the structures. For a morphism

f: Inj(a) $\to$ Inj(b) we put $F(f) = f$. To see that this is well-defined let ι_a and ι_b be the inclusion maps from $a(\mathbb{T}^\omega)$ and $b(\mathbb{T}^\omega)$, respectively, to $\mathbb{T}^\omega$. Then $f = b \circ (\iota_b \circ f \circ a) \circ \iota_a$. As a and ι_b are topologically continuous so is $\iota_b \circ f \circ a$ which is therefore order continuous, as the two notions of continuity coincide for $\mathbb{T}^\omega$. As b and ι_a are order continuous, the equation shows that f is. Thus F sends topologically continuous maps to order continuous maps. Similarly if f: Dom(a) $\to$ Dom(b) then f is topologically continuous. This is enough to show that F is an isomorphism. ▮

The isomorphism shows that the domain Dom(a) is determined by the topology Inj(a) and vice versa. We will now make this dependence explicit. For each retraction, a, we define a topology by requiring that a set U is open just when:

(i) $\forall x, y \in \mathrm{Dom}(a),\ x \in U \wedge x \sqsubseteq y \to y \in U$;

(ii) $\forall X \subseteq \mathrm{Dom}(a),\ X$ directed $\wedge \bigsqcup X \in U \to \exists x \in X,\ x \in U$.

Then the topology obtained is that of Inj(a). First consider the case when a is the identity. Each B_n obeys (i) and (ii). Suppose U does and $y \in U$. Then by (ii) and the algebraicity of $\mathbb{T}^\omega$ there is a b_n such that $b_n \sqsubseteq y$ and $b_n \in U$. Thus by (i), $B_n \subseteq U$ and so $U = \bigsqcup \{B_n \mid B_n \subseteq U\}$ which shows the topologies are identical.

In the general case suppose U is open in Inj(a). Then $U = \mathrm{Inj}(a) \cap a^{-1}(U)$ and $a^{-1}(U)$ is open. This equation makes it easy to check conditions (i) and (ii). Conversely suppose U obeys conditions (i) and (ii) and consider $a^{-1}(U)$. If $x, y \in \mathbb{T}^\omega$ and $x \sqsubseteq y$ and $x \in a^{-1}(U)$ then $a(x) \sqsubseteq a(y)$ and $a(x) \in U$. By (i), $a(y) \in U$ and so $y \in a^{-1}(U)$. Similarly we can check condition (ii) for $a^{-1}(U)$ using the continuity of a. Thus $a^{-1}(U)$ is open in $\mathbb{T}^\omega$, by the first case examined, and so $U = \mathrm{Inj}(a) \cap a^{-1}(U)$ is open in Inj(a). Thus the two topologies are always the same.

Conversely the order can be derived from the topology. Consider a space Inj(a). Define $\sqsubseteq'$ by:

$$x \sqsubseteq' y \underset{\text{def}}{\equiv} \forall U \text{ open in Inj}(a),\quad x \in U \to y \in U.$$

The same style of proof shows that $\sqsubseteq'$ is just $\sqsubseteq$.

We conclude this section with a word on the countability hypothesis which is used throughout this paper in what seems to be a more essential way than in [12]. For example, in the proof of Theorem 20 the openness of V^1 depended on the restriction of the universal quantifier to a finite set. For that reason there is no obvious generalization of the theorem to the case of a space with basis of cardinality $\kappa > \omega$ and $\mathbb{T}^\kappa$. Again in the proof of Theorem 11 the continuity of ψ depended on the restriction of the universal quanitfier in the definition of X_t to a finite range. Again in Section 2 our ability to define *finite* characteristic segments, Seg(k) depended on the countability of the subbasis of the function space. We therefore conjecture that if $\kappa > \omega$ then $(\mathbb{T}^\kappa \to \mathbb{T}^\kappa)$ is *not* a retract of $\mathbb{T}^\kappa$. This would give a counterexample to the natural generalization of Theorem 11 and, as Lemma 4 says each $\mathbb{T}^\kappa$ is injective over isochordal subspaces, it would also give a counterexample to the natural generalization of Theorem 22.

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