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Lecture 10: Random Ordering and Greedy Selection

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1 Greedy recolouring

Here we are going to revisit the 2-colourability problem of k-uniform hypergraphs. Recall that $m(k)$ is the minimum number of edges such that a k-uniform hypergraph is not 2-colourable. We have derived lower and upper using the basic method:

$$2^{k-1} \leq m(k) \leq (1+o(1))\frac{e\log 2}{4}k^2 2^k.$$

We are going to improve the lower bound to

$$m(k) \geq C \cdot \left(\frac{k}{\log k}\right)^{1/2} 2^k,$$

for some constant C.

We may want to colour every vertex uniformly at random. Then we will get $m2^{1-k}$ many monochromatic edges in expectation. An improvement upon the basic method can be obtained by randomly recolouring *all* monochromatic edges after the first randomization. In a 1978 proof, Beck used this argument to show that $m(k) \geq \Omega(k^{1/3}2^k)$. In 2000, Radhakrishnan and Srinivasan improved it to $m(k) \geq \Omega((k/\log k)^{1/2}2^k)$. Their main idea is that the recolouring obeys some random ordering, and when we are about to recolour an edge, it may not be monochromatic any longer, and thus we do not need to recolour every monochromatic edge.

Cherkashin and Kozik (2015) found a simpler proof based on a random greedy colouring, which implicitly used the recolouring idea. We will present that proof here.

Theorem 1. *If there exists $p \in [0,1]$ such that $t(1-p)^k + t^2p < 1$, then a k-uniform hypergraph $H = (V,E)$ with $|E| \leq 2^{k-1}t$ is always 2-colourable.*

Proof. We may assume $m = |E| = 2^{k-1}t$ since removing edges does not destroy 2-colourability.

The random colouring process will be the following. We first assign a label x_v to every $v \in V$, where $x_v \in [0,1]$ is drawn uniformly at random. This induces a random ordering which is equivalent to a permutation at this stage. We examine vertices in an increasing order. We colour every vertex blue, unless it is the last vertex of some edge that is otherwise all blue. In this case, we colour it red. We are going to show that with positive probability, the final colouring is proper (no edge is monochromatic).

Due to our construction, no edges can be completely blue. If an edge is red, then every vertex in it must be the last vertex of some almost blue edge. Call an ordered pair (e,f)

conflicting if the last vertex of e is the first vertex of f. Thus if there is no conflicting pairs, the colouring must be proper. Note that if (e, f) is conflicting, then $e \cap f$ is unique. We proceed to show that with positive probability there is no such pairs.

To prove it, split the interval $[0, 1]$ into three subintervals, L, M, and R as follows

$$L = \left[0, \frac{1-p}{2}\right), \qquad M = \left[\frac{1-p}{2}, \frac{1+p}{2}\right), \qquad R = \left[\frac{1+p}{2}, 1\right].$$

We say $e \subset L$, M, or R if all labels of vertices of e lie in that interval. We say a conflicting pair (e, f) is type 1 if $e \in L$ or $f \in R$. The probability that there is a type 1 conflicting pair is at most the probability that some edge in E lies in L or R, either of which happens with probability $\left(\frac{1-p}{2}\right)^k$. Using the union bound, we have that

$$\begin{aligned} \Pr(\text{type 1 pair exists}) &\le m \cdot 2 \left(\frac{1-p}{2}\right)^k \\ &= 2^k t \left(\frac{1-p}{2}\right)^k = t(1-p)^k. \end{aligned}$$

We call all other conflicting pairs type 2. For each pair (e, f) with a unique intersection, we bound the probability that it's type 2. For a type 2 pair, the unique vertex $v \in e \cap f$ must have its label $x_v \in M$. This is because if $x_v \in L$, then $e \in L$ and if $x_v \in R$, then $f \in R$. Contradicting to type 2. The event $x_v \in M$ happens with probability exactly p. Moreover, all $k-1$ vertices in e must lie before x_v, which happens with probability x_v. All $k-1$ vertices in f must lie after x_v, which happens with probability $1 - x_v$. Thus

$$\Pr((e, f) \text{ is type 2}) = p x_v^{k-1}(1-x_v)^{k-1} \le p 4^{1-k},$$

where we used the fact that $x_v(1 - x_v) \le 1/4$. The number of ordered pair of edges is at most $m \cdot m = t^2 4^{k-1}$. Thus, by a union bound,

$$\Pr(\text{type 2 pair exists}) \le t^2 4^{k-1} \cdot p 4^{1-k} = t^2 p.$$

By assumption, we have that

$$\begin{aligned} \Pr(\text{conflict pair exists}) &\le \Pr(\text{type 1 pair exists}) + \Pr(\text{type 2 pair exists}) \\ &\le t(1-p)^k + t^2 p < 1. \end{aligned}$$

Thus, with positive probability, there is a random ordering such that no conflict pair exists, and the resulting colouring is proper. □

We are still left to find an explicit expression of the condition of Theorem 1. Once again, we use the bound $1-p \le e^{-p}$ and thus want to bound $te^{-pk} + t^2 p$. The condition in Theorem 1 is typical in that the two terms come from type 1 error and type 2 error, respectively. A common strategy is to make sure $te^{-pk} < 1/2$ and $t^2 p < 1/2$. From the second one we have that $p < \frac{1}{2t^2}$. Plugging it back in the first and taking the log, we want to have

$$\log 2t - \frac{k}{2t^2} < 0,$$

which implies that

$$2t^2 \log 2t < k.$$

Thus $t = \left(\frac{k}{\log k}\right)^{1/2}$ satisfies the above for sufficiently large k. In fact, if we do a bit of analysis, we will arrive at a tighter $t = \sqrt{2}\left(\frac{k}{\log k}\right)^{1/2}$, but our rough bound is asymptotically tight already.

Corollary 2. $m(k) \geq \Omega\left(\left(\frac{k}{\log k}\right)^{1/2} 2^{k-1}\right)$.

Note that the procedure in the proof above can also be formulated as a recoloring. Indeed, after selecting the random labels x_v, one can color all vertices v with $x_v \in L \cup M$ blue and all vertices u with $x_u \in R$ red. We can then go over the vertices of M according to the order of their labels, and recolor any blue vertex which is the last one in a blue edge red.

2 Yet another proof of Turán's theorem

Recall that $T_r(n)$ is the Turán graph with n vertices and r classes, and denote its number of edges by $t_r(n)$. Let $\kappa(G)$ denote the size of the maximum clique of G. Then Turán's theorem states that $ex(n, K_{r+1}) \leq t_r(n)$, or equivalently:

Theorem 3 (Turán's Theorem)**.** *Let $n \geq r \geq 2$. Let $G = (V, E)$ be a graph of n vertices and m edges. If $\kappa(G) \leq r$, then $m \leq t_r(n)$.*

Recall that we use $\alpha(G)$ to denote the size of the maximum independent set of G. An equivalent formulation of Turán's theorem is the following.

Theorem 4. *Let $n \geq r \geq 2$. Let $G = (V, E)$ be a graph of n vertices and m edges. If $m < \binom{n}{2} - t_r(n)$, then $\alpha(G) > r$.*

The reason is that the complement of an independent set is a clique. Let $G^c = (V, E^c)$ be the graph on the same set of vertices as G, but $(u, v) \in E^c$ if and only if $(u, v) \notin E$. Then we see that $\alpha(G) = \kappa(G^c)$ and $|E| + |E^c| = \binom{n}{2}$. It follows that Theorem 3 and Theorem 4 are equivalent:

$$\begin{aligned}
& \text{If } \kappa(G) \leq r, \text{ then } |E| \leq t_r(n). \\
\Leftrightarrow\quad & \text{If } |E| > t_r(n), \text{ then } \kappa(G) > r. \\
\Leftrightarrow\quad & \text{If } |E^c| < \binom{n}{2} - t_r(n), \text{ then } \alpha(G^c) > r.
\end{aligned}$$

Last time, we have shown that $\alpha(G) \geq \frac{n^2}{4m}$ if $m \geq \frac{n}{2}$. In order to prove Theorem 4, we first show a stronger bound on $\alpha(G)$. It is due to Caro (1979) and Wei (1981).

Lemma 5.

$$\alpha(G) \geq \sum_{v \in V} \frac{1}{d(v)+1}.$$

Proof. We choose a uniformly at random permutation of all vertices in V. This gives us a random ordering $<$. Choose an independent set I greedily in an increasing order. To be more specific, we first put the smallest vertex into I and remove all its neighbours. At every subsequent step, we put the smallest vertex of the current set of vertices into I, and then remove all its neighbours. Clearly I is an independent set by choice.

For the convenience of analysis later, we will in fact use a subset of the one generated by the process above. Let

$$I := \{v \in V \mid (u,v) \in E \Rightarrow v < u\}.$$

For a vertex v, it is put into I if and only if it is ordered smallest among all its neighbours. (For the greedy procedure, there is another possibility, which is difficult to analyze. The other possibility is that smaller neighbours are eliminated by some earlier vertices.) Let X_v be the indicator variable of the event that v is chosen. Thus,

$$\mathbb{E}\, X_v = \Pr(v \in I) \geq \frac{d(v)!}{(d(v)+1)!} = \frac{1}{d(v)+1}.$$

Hence, by the linearity of expectations

$$\mathbb{E}\,|I| = \mathbb{E}\left(\sum_{v \in V} X_v\right) = \sum_{v \in V} \mathbb{E}\, X_v = \sum_{v \in V} \frac{1}{d(v)+1}.$$

There must exist an ordering such that the greedily chosen independent set has size $\geq \sum_{v \in V} \frac{1}{d(v)+1}$. □

Proof of Theorem 4. Note that $f(x) = \frac{1}{x+1}$ is a convex function (consider the second derivative). It implies that

$$\sum_{v \in V} \frac{1}{d(v)+1} = \sum_{v \in V} f(d(v))$$

is minimized if $d(v)$'s differ by at most 1. (To see this, one may show that $f(x_1) + f(x_2) > f(x_1 - 1) + f(x_2 + 1)$ if $x_1 - x_2 \geq 2$.)

For a fixed $\sum_{i=1}^n d_i = 2m$, let $d_1 \leq d_2 \leq \cdots \leq d_n$ be the unique sequence of degrees (up to permutations) such that $|d_i - d_j| \leq 1$. Thus, by Lemma 5,

$$\alpha(G) = \sum_{v \in V} \frac{1}{d_G(v)+1} \geq \sum_{i=1}^{n} \frac{1}{d_i + 1}.$$

On the other hand, consider the complement of Turán's graph, $T_r(n)^c$. Denote it by H and $d_{H,1} \le d_{H,2} \le \cdots d_{H,n}$ be its degree sequence. It is composed by r isolated cliques of size $\lfloor n/r \rfloor$ and $\lceil n/r \rceil$. The number of edges is $\binom{n}{2} - t_r(n) > m$, and the degrees differ by at most 1. Thus, $d_i \le d_{H,i}$ for any $i \in [n]$, and the inequality is strict for at least one i. It implies that

$$\sum_{i=1}^{n} \frac{1}{d_i + 1} > \sum_{i=1}^{n} \frac{1}{d_{H,i} + 1}.$$

Moreover, notice that for each clique of H, it contributes exactly 1 to $\sum_{i=1}^{n} \frac{1}{d_{H,i}+1}$. Hence,

$$\sum_{i=1}^{n} \frac{1}{d_{H,i} + 1} = r.$$

Combining everything together, we have that

$$\begin{aligned} \alpha(G) = \sum_{v \in V} \frac{1}{d_G(v) + 1} &\ge \sum_{i=1}^{n} \frac{1}{d_i + 1} \\ &> \sum_{i=1}^{n} \frac{1}{d_{H,i} + 1} = r. \end{aligned}$$

This finishes the proof. □

3 Lovász Local Lemma

Usually, when we use the probabilistic method, the desired event holds with very high probability. On the other hand, suppose we have n mutually independent events, each of which holds with probability p, then the conjunction of all of them holds with probability p^n. This is exponentially small but strictly positive.

In most applications, the desired event cannot be decomposed into n mutually independent ones. The Lovász Local Lemma provides a way to deal with dependencies. It was first found by Erdős and Lovász in 1975 and is extremely powerful, especially if the dependencies are rare.

Let A_1, A_2, $\ldots$, A_n be events in an arbitrary probability space. A directed graph $D = (V, E)$ on the set of vertices $V = [n]$ is called *the dependency digraph* for the events $A_1, \ldots, A_n$ if for each $i \in [n]$, the event A_i is mutually independent of all events $\Gamma(i) := \{A_j : (i,j) \notin E\}$.

Lemma 6 (The Local Lemma)**.** *Suppose $D = (V, E)$ is a dependency digraph for events $A_1, \cdots, A_n$ and there are real numbers $x_1, \cdots, x_n$ such that $0 \le x_i < 1$ and for all $i \in [n]$*

$$\Pr(A_i) \le x_i \prod_{(i,j) \in E} (1 - x_j). \tag{1}$$

Then

$$\Pr\left(\bigwedge_{i=1}^{n} \overline{A_i}\right) \geq \prod_{i=1}^{n}(1-x_i) > 0.$$

One should think A_i's as "bad" events, and what we are after is some "perfect" object that avoids all undesired events. In particular, when the condition of Lemma 6 holds, with positive probability, no event A_i holds and thus a perfect object exists.

The symmetric case of Lemma 6 is often useful.

Corollary 7. *Suppose D is the dependency digraph and $|\Gamma(i)| \leq d$ for any $i \in [n]$. If*

$$ep(d+1) \leq 1,$$

then

$$\Pr\left(\bigwedge_{i=1}^{n} \overline{A_i}\right) > 0.$$

Proof. If $d = 0$ then the corollary is trivial. Otherwise, take $x_i = \frac{1}{d+1} < 1$. The corollary follows trivially since for any $d \geq 1$, $\left(1 - \frac{1}{d+1}\right)^d > \frac{1}{e}$. □

As shown by Shearer in 1985, the constant e in Corollary 7 is the best possible. However, due to the algorithmic approach of Moser and Tardos (2011), the condition can be slightly improved to $epd \leq 1$.