MTH742P: Advanced Combinatorics 03/10/2016

Lecture 2: Forbidden Paths and Cycles

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1 Hamiltonian cycles - Dirac's theorem

Recall that in extremal graph theory, we would like to answer questions of the following sort: 'What is the maximum/minimum possible parameter C among graphs satisfying a certain property P?' In the last lecture, we see Mantel's theorem, which answers the above question with the parameter being the number of edges and the property being triangle-free. In this lecture, we will be looking at other interesting parameters and properties.

Definition 1. *Let G be a graph. A path P (or cycle C) in G is said to be* simple *if and only if all vertices of P (or C) are distinct.*

Question: What is the minimal number of edges in a graph to guarantee the existence of a cycle? In other words, what is maximal number of edges without a cycle?

Notice that a tree T of order n contains no cycle and it has $n-1$ many edges. On the other hand, a graph G of order n and $e(G) \geq n$ must contain a cycle.

Theorem 1. *A graph G of order $n \geq 3$ contains a cycle if $e(G) \geq n$.*

One key observation is that if the minimum degree of G is at least 2, then it must contain a cycle.

Definition 2. *Let G be a graph. Define $\delta(G)$ to be the minimum degree of all vertices in G:*

$$\delta(G) := \min\{d(v) : v \in V(G)\}.$$

Suppose $\delta(G) \geq 2$. We may start from an arbitrary vertex, and go to one of its neighbours. Since $\delta(G) \geq 2$ and G is finite, we can always continue this process, until we come back to a vertex that has been visited. This forms a cycle.

With the observation in hand, we can show Theorem 1 by induction. The base case is trivial. For the induction step, if G has no cycle, then it must have a vertex v of degree 1. Consider $G \backslash v$, which has $n-1$ vertices and at least $n-1$ edges. Hence by induction hypothesis $G \backslash v$ contains a cycle.

Next let us turn our attention to cycles that visit every vertex. Contradiction.

Definition 3. *Let G be a graph of order n. A* Hamiltonian cycle *is a simple cycle of order n. Also, G is said to be* Hamiltonian *if it has a Hamiltonian cycle.*

In other words, a Hamiltonian cycle visits every vertex exactly once.

Question: What is the minimal number of edges to guarantee the existence of a Hamiltonian cycle? In other words, what is maximal number of edges without a Hamiltonian cycle?

However, this question is not very interesting, as the answer if close to the maximum possible number of edges, $\binom{n}{2}$. Consider the following family of graphs G_n. Take a K_{n-1} together with an isolated vertex v. Add one edge between K_{n-1} and v. There is no Hamiltonian cycle since $d(v) = 1$ and no cycle can go through v. On the other hand,

$$e(G_n) = \binom{n-1}{2} + 1 = \binom{n}{2} - (n-2).$$

Thus, the edge "density" of this family of graphs is

$$\frac{e(G_n)}{\binom{n}{2}} = 1 - \frac{2(n-2)}{n(n-1)} \to 1$$

as $n \to \infty$. This means that even if the graph contains almost all the possible edges, it could still be non-Hamiltonian. In contrast, by Mantel's Theorem, a triangle-free graph G has density at most

$$\frac{\lfloor n^2/4 \rfloor}{\binom{n}{2}} \to \frac{1}{2} \text{ as } n \to \infty.$$

Note that the degrees of the example we constructed above are distributed very unevenly. There are $n-1$ vertices with degree at least $n-2$ and 1 vertex with degree 1. A more interesting question is that can we guarantee the existence of Hamiltonian cycles by lower bounding the minimum degree of the graph.

Theorem 2 (Dirac 1952)**.** *Let $n \geq 3$. If G is a graph of order n and $\delta(G) \geq n/2$, then G is Hamiltonian.*

Theorem 2 is actually the best possible. Consider the graph G, which is putting together two copies of $K_{n/2}$. Since G is disconnected, it is not Hamiltonian. Moreover, $\delta(G) = n/2 - 1$. Thus Theorem 2 is the best possible for even n. The case of odd n will be left as an exercise.

Proof of Theorem 2. First we claim that G is connected. Suppose otherwise. Then pick one of the smallest components of G. It must contain at most $n/2$ many vertices. Hence any vertex in this component has degree at most $n/2 - 1$. Contradiction.

Now suppose G is not Hamiltonian. Consider the simple path P of maximum possible length $\ell \leq n-1$. That is, $P = \{x_0, x_1, \cdots, x_\ell\}$ where $x_i x_{i+1} \in E$ for all $0 \leq i \leq \ell - 1$. Since P is a maximal path, the neighbours of x_0 and x_ℓ must be all inside P. Let

$$A = \Gamma(x_1), \qquad\qquad B = \{x_{i+1} : x_i \in \Gamma(x_\ell)\}.$$

Since $\delta(G) \geq n/2$, $|A|, |B| \geq n/2$. On the other hand, it is easy to see that $x_0 \notin A$ and $x_0 \notin B$. Hence $A \cup B \subseteq \{x_1, \cdots, x_\ell\}$. Thus $|A \cup B| \leq \ell \leq n-1$. It implies that $A \cap B \neq \emptyset$ (as otherwise $|A \cup B| \geq n/2 + n/2 = n$).

Suppose $x_t \in A \cap B$ for some t. Then consider the following cycle C

$$x_1 - x_t - x_{t+1} - x_{t+2} - \cdots - x_{\ell-1} - x_\ell - x_{t-1} - x_{t-2} - \cdots - x_2 - x_1.$$

A picture can be found in Figure 1. Clearly, C has length ℓ.

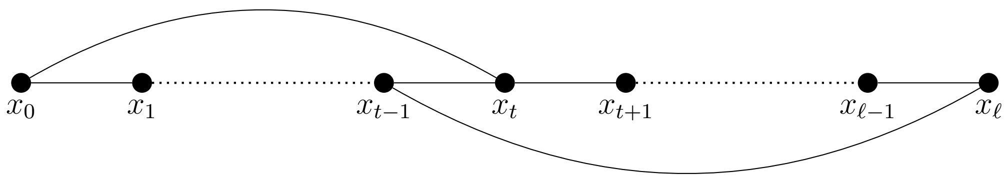

Figure 1: The path P and cycle C

If $\ell = n$, then C is a Hamiltonian cycle. Contradiction.

If $\ell < n$, we then construct a simple path P' of length $\geq \ell + 1$. As $\ell < n$, there exists at least one vertex $v \notin C$. However, G is connected. Hence there exists a simple path from v to some vertex x_r in C. Construct the path P' as follows: start from v, to x_r, and then traverse C to x_{r-1}. The length of P' is at least $\ell + 1$. It contradicts to the maximality of P. □

1.1 Forbidding a path of length k

The way we prove Dirac's theorem is useful to answer the following question.

Question: What is the maximal number of edges in a graph of order n without a simple path of length k?

Let try to guess the answer first. An easy way to avoid paths of length k is when every component has size at most k. Then to maximize the number of edges, we put all possible edges in each component. Thus our construction G is n/k many copies of cliques K_k (assuming $k \mid n$). In this case,

$$e(G) = \frac{n}{k}\binom{k}{2} = \frac{n}{k} \cdot \frac{k(k-1)}{2} = \frac{(k-1)n}{2}.$$

We will show that this is indeed the best possible.

Theorem 3. *Let G be a graph of order n and there is no path of length k in n. Then*

$$e(G) \leq \frac{(k-1)n}{2}.$$

The proof of Theorem 3 relies on the following lemma, which is a similar result to Dirac's theorem.

Lemma 4. *Let G be a connected graph of order n and $\delta(G) \geq k/2$ for some integer $k < n$. Then G contains a simple path of length k.*

Proof. Suppose that G contains no path of length k. Let $P = \{x_0, x_1, \cdots, x_\ell\}$ be a path of maximum length $\ell < k$.

Since P is a maximal path, the neighbours of x_0 and x_ℓ must be all inside P. Let

$$A = \Gamma(x_1), \qquad\qquad B = \{x_{i+1} : x_i \in \Gamma(x_\ell)\}.$$

Since $\delta(G) \geq k/2$, $|A|, |B| \geq k/2$. On the other hand, it is easy to see that $x_0 \notin A$ and $x_0 \notin B$. Hence $A \cup B \subseteq \{x_1, \cdots, x_\ell\}$. Thus $|A \cup B| \leq \ell < k$. It implies that $|A| \cap |B| \neq \emptyset$ (as otherwise $|A \cup B| \geq k/2 + k/2 = k$).

Suppose $x_t \in A \cap B$ for some t. Then consider the following cycle C

$$x_1 - x_t - x_{t+1} - x_{t+2} - \cdots - x_{\ell-1} - x_\ell - x_{t-1} - x_{t-2} - \cdots - x_2 - x_1.$$

(Recall Figure 1.) Clearly, C has length ℓ.

Since $\ell < k < n$, we then construct a simple path P' of length $\geq \ell + 1$. As $\ell < n$, there exists at least one vertex $v \notin C$. However, G is connected. Hence there exists a simple path from v to some vertex x_r in C. Construct the path P' as follows: start from v, to x_r, and then traverse C to x_{r-1}. The length of P' is at least $\ell + 1$. It contradicts to the maximality of P. $\square$

With Lemma 4 in hand, we are now ready to prove Theorem 2.

Proof of Theorem 2. If $k = 1$, then there is no possible edge in G and $e(G) = 0$.

Otherwise $k \geq 2$, we do an induction on n (for each fixed integer $k \geq 2$). The base case is when $n \leq k$ and is trivial. This is because

$$e(G) \leq \binom{n}{2} = \frac{(n-1)n}{2} \leq \frac{(k-1)n}{2}.$$

For the induction step, we want to show the theorem for a graph G of order $n > k$ assuming it holds for any graph of order $< n$. If G is disconnected, then let G_0 be a component of order $n_0 > 0$ and G_1 be the rest of the graph. Clearly

$$e(G) \leq e(G_0) + e(G_1).$$

Moreover, G_0 is of order $n_0 < n$ and G_1 has $n - n_0 < n$ many vertices. By induction hypothesis,

$$e(G_0) \leq \frac{k-1}{2} \cdot n_0,$$
$$e(G_1) \leq \frac{k-1}{2} \cdot (n - n_0).$$

Combine all of the above:

$$e(G) \leq \frac{k-1}{2}(n_0 + n - n_0) = \frac{k-1}{2}n.$$

Otherwise, G is connected. If $\delta(G) \geq k/2$, then by Lemma 4 there exists a path of length k. Contradiction

Therefore $\delta(G) < k/2$. It implies that there exists a vertex $v \in V(G)$ such that

$$d(v) \leq \lceil k/2 \rceil - 1 \leq \frac{k-1}{2}.$$

Now consider the graph $G' = G \backslash v$. G' has $n-1$ many vertices and hence we can apply the induction hypothesis:

$$e(G') \leq \frac{k-1}{2}(n-1).$$

Thus,

$$e(G) = e(G') + d(v) \leq \frac{k-1}{2}(n-1) + \frac{k-1}{2} = \frac{k-1}{2}n. \qquad \square$$

We note that the edge density of graphs without a path of length k is at most

$$\frac{\frac{(k-1)n}{2}}{\binom{n}{2}} = \frac{k-1}{n-1} \to 0 \qquad \text{as } n \to \infty.$$

2 Turán numbers and Turán densities

Let us fit the examples we have seen so far into a general theory.

Definition 4. *Let F be an unlabelled graph. We say that a graph G is* F-free *if G does not contain any isomorphic copy of F as a subgraph.*

Notice that here we do mean subgraph rather than *induced* subgraph. For example, K_5 is not C_4-free because it contains a lot of cycles of length 4. However, the induced graph of K_5 on any 4 vertices is a $K_4 \neq C_4$.

Definition 5. *Let F be an unlabelled graph, and let $n \geq 2$ be an integer. Define the* Turán number *of F to be*

$$ex(n, F) := \max\{e(G) : G \text{ is an } F\text{-free graph of order } n\}.$$

Determining $ex(n, F)$ is one of the basic problems of extremal graph theory. Mantel's theorem tells us that $ex(n, K_3) = \left\lfloor \frac{n^2}{4} \right\rfloor$, and Theorem 3 shows that $ex(n, P_k) \leq \frac{(k-1)n}{2}$.

We also look at the "edge" density of F-free graphs. In particular, it is natural to consider the following limit:

$$\lim_{n\to\infty} \frac{ex(n, F)}{\binom{n}{2}}.$$

Let us first show that the limit above does exist for any graph F.

Lemma 5. *Let F be a graph. Then for any integer $n \geq 3$,*

$$\frac{ex(n,F)}{\binom{n}{2}} \leq \frac{ex(n-1,F)}{\binom{n-1}{2}}.$$

Proof. Let G be an F-free graph of order n such that $e(G) = ex(n,F)$. Let $v_0 \in V(G)$ of the minimum degree, i.e. $d(v_0) = \delta(G)$. Thus by the handshaking lemma,

$$2e(G) = \sum_{v \in V} d(v) \geq nd(v_0).$$

Let $G' = G - v$. Thus G' is an F-free graph of order $n-1$. By Definition 5,

$$e(G') \leq ex(n-1,F).$$

On the other hand,

$$e(G) = e(G') + d(v).$$

Hence

$$e(G) \leq ex(n-1,F) + \frac{2e(G)}{n}.$$

It implies that

$$ex(n,F) = e(G) \leq \frac{n}{n-2} ex(n-1,F).$$

Rearranging the terms yields

$$\frac{ex(n,F)}{ex(n-1,F)} \leq \frac{n}{n-2} = \frac{\binom{n}{2}}{\binom{n-1}{2}},$$

or equivalently,

$$\frac{ex(n,F)}{\binom{n}{2}} \leq \frac{ex(n-1,F)}{\binom{n-1}{2}}.$$

□

Lemma 5 implies that the sequence

$$\left(\frac{ex(n,F)}{\binom{n}{2}} \right)_{n=2}^{\infty}$$

is monotone non-increasing. It is also a sequence of positive real numbers. Hence its limit exists. Define

$$\pi(F) := \lim_{n \to \infty} \frac{ex(n,F)}{\binom{n}{2}}. \tag{1}$$

This limit $\pi(F)$ is also called the *Turán density* of F.

As we have seen, Mantel's theorem implies that $\pi(K_3) = \pi(C_3) = \frac{1}{2}$. Moreover, Theorem 3 implies that

$$\pi(P_k) \leq \frac{ex(n, P_k)}{\binom{n}{2}} \leq \frac{(k-1)n/2}{(n-1)n/2} = \frac{k-1}{n-1} \to 0 \text{ as } n \to \infty.$$

It implies that $0 \leq \pi(P_k) \leq 0$, and thus $\pi(P_k) = 0$. Later, we will see the Erdős-Stone theorem, which gives us precise answer of $\pi(F)$ for any F. A consequence of the Erdős-Stone theorem is that $\pi(F) = 0$ if and only if F is bipartite.