**MTH742P: Advanced Combinatorics** 10/10/2016

# Lecture 3: Turán's theorem

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## 1 Turán Graphs

Recall that Mantel's theorem says that $ex(n, K_3) = ex(n, C_3) = \lfloor n^2/4 \rfloor$. It is natural to generalize it either to $K_r$ or $C_r$ for integers $r \geq 4$. Turán gave a satisfactory answer in the first direction. However, the second direction turns out to be a lot more complicated. We will see later about $ex(n, C_4)$.

For $ex(n, K_{r+1})$, a natural guess is to take a complete $r$-partite graph. Clearly it is $K_{r+1}$-free. This is because that in a clique of size $r+1$, at least two vertices are from the same class. However there can be no edges between them. Indeed, it also maximizes the number of edges if the class-sizes are as close as possible.

**Definition 1.** *Let $r$ be a positive integer. A graph $G$ is said to be* $r$-partite *if there is a partition of $V(G)$ into $r$ "classes"*

$$V(G) = V_1 \cup V_2 \cup \cdots \cup V_r,$$

*and*

$$V_i \cap V_j = \emptyset,$$

*for any $i, j \leq r$, such that no edges of $G$ has both its endpoints in the same class.*

As usual, a complete $r$-partite graph is to add all possible edges. Namely, the edge set contains all edges between $V_i$ and $V_j$ for any $i \neq j$.

Suppose $n_1, n_2, \cdots, n_r$ are the sizes of different partitions in a $r$-partite graph $G$, where $\sum_{i=1}^{d} n_i = n$. The number of edges is

$$e(G) = \sum_{i \neq j} n_i n_j.$$

Our goal is to maximize the number of edges. Note that if there exists $i, j$ such that $n_i - n_j > 1$, then we can increase $n_i, n_j$ by replacing $n_i$ by $n_i - 1$ and $n_j$ by $n_j + 1$. This is because

$$(n_i - 1)(n_j + 1) - n_i n_j = n_i - n_j - 1 > 0.$$

On the other hand, edges between partition $i, j$ and other classes are

$$n_i \sum_{k \neq i,j} n_k + n_j \sum_{k \neq i,j} n_k = (n_i + n_j) \sum_{k \neq i,j} n_k.$$

So it does not change since we did not change $n_i + n_j$. In summary, whenever $n_i - n_j > 1$, we can do this operation and increase the number of edges. Thus in the graph with maximized number of edges, there is no $i, j$ such that $n_i - n_j > 1$. Graphs of this kind are called Turán graphs, denoted by $T_r(n)$. Let $t_r(n) = e(T_r(n))$.

**Definition 2.** *Let $n, r$ be two integers such that $n \geq r \geq 2$. Let $n = kr + s$ where $k \geq 1$ and $0 \leq s < r$. Then the Turán graph $T_r(n)$ is the complete $r$-partite graph where $s$ many classes have size $k+1$ and $r-s$ many classes have size $k$.*

**Theorem 1** (Turán 1941)**.** *Let $n, r$ be two integers such that $n \geq r \geq 2$. If $G$ is $K_{r+1}$-free of order $n$, then $e(G) \leq t_r(n)$. In other words, $ex(n, K_{r+1}) \leq t_r(n)$.*

Before proving Theorem 1, let us calculate $\pi(K_{r+1})$ using it. First,

$$t_r(n) \geq \binom{r}{2} \lfloor n/r \rfloor^2 \geq \binom{r}{2} (n/r - 1)^2,$$

and

$$\lim_{n \to n} \frac{\binom{r}{2}(n/r-1)^2}{\binom{n}{2}} = \lim_{n \to n} \frac{(n-r)^2}{n(n-1)} \cdot \frac{r-1}{r} = 1 - \frac{1}{r}.$$

Thus, $\pi(K_{r+1}) \geq 1 - \frac{1}{r}$. Similarly,

$$t_r(n) \leq \binom{r}{2} \lceil n/r \rceil^2 \leq \binom{r}{2} (n/r + 1)^2,$$

and

$$\lim_{n \to n} \frac{\binom{r}{2}(n/r+1)^2}{\binom{n}{2}} = \lim_{n \to n} \frac{(n+r)^2}{n(n-1)} \cdot \frac{r-1}{r} = 1 - \frac{1}{r}.$$

Hence $1 - \frac{1}{r} \leq \pi(K_{r+1}) \leq 1 - \frac{1}{r}$. That is, $\pi(K_{r+1}) = 1 - \frac{1}{r}$.

Turán's Theorem is a classical result and there have been many beautiful proofs for it. The nice structure is ideal for all kinds of induction proofs. We will give several different ones, each of which entertains a different trick. First let us show it in Turán's original way.

*1st proof of Theorem 1.* We do an induction on $n$. The base case is when $n = r$. In this case any graph $G$ of order $n$ cannot have subgraph $K_{r+1}$. So $e(G)$ is maximized at $K_n = K_r$, which is the same as $T_r(r)$.

For the induction step, let $G$ be a $K_{r+1}$-free graph of order $n$ and $G$ maximizes the number of edges. First we argue that $G$ has a $K_r$. For each non-edge $(v_0, v_1) \notin E$, there must exist $r+1$ many vertices $S = \{v_0, v_1, \cdots, v_r\}$ such that the induced subgraph by $S$ is $K_{r+1}$ minus the edge $(v_0, v_1)$, since otherwise we can add this edge to $G$ and contradict the maximality. Then take one vertex of this non-edge, say $v_1$, together with all of the rest $v_2, \cdots, v_r$. This set $v_1, \cdots, v_r$ is a $r$-clique. Call this $r$-clique $A$.

Let $B = G\backslash A$. Then $B$ is $K_{r+1}$-free and of order $n-r$. Apply the induction hypothesis,

$$e(B) \leq t_r(n-r).$$

We also know that $e(A) = \binom{r}{2}$. Thus the only piece missing is a good bound on $e(A, B)$. As usual, let us consider how many neighbours in $A$ can a vertex $v$ in $B$ have. This number cannot be $r$, since if so then $A$ and $v$ together form a $K_{r+1}$. Hence it is at most $r-1$. It implies that

$$e(A, B) \leq (r-1)\,|B| = (r-1)(n-r).$$

Put everything together,

$$\begin{aligned} e(G) &= e(A) + e(B) + e(A, B) \\ &\leq \binom{r}{2} + t_r(n-r) + (r-1)(n-r) \\ &= t_r(n). \end{aligned}$$

The last line is because

$$t_r(n) - t_r(n-r) = \binom{r}{2} + (r-1)(n-r).$$

(Imagine $T_r(n)$ then remove one vertex from each class.) □

Next we present a proof by Erdős. It does not only prove the theorem but also pins down the extremal graphs.

**Theorem 2** (Erdős 1970)**.** *Let $G$ be a graph of order $n$ and $K_{r+1}$-free. Then there is a $r$-partite graph $H$ on the same vertex set such that*

$$d_G(v) \leq d_H(v)$$

*for every $v \in V$.*

Note that $G$ is not necessarily $r$-partite even if $G$ is $K_{r+1}$-free. Theorem 1 follows from Theorem 2 because $T_r(n)$ maximizes the number of edges among all *$r$-partite* graphs.

*Proof of Theorem 2.* We do an induction on $r$. The base case of $r = 1$ is trivial: $K_2$-free means the graph is empty.

For the induction step, let $r \geq 2$ and assume that the theorem holds for any integer $< r$. Let $v$ be the vertex of maximum degree $\Delta(G)$. Consider the induced subgraph of $\Gamma(v)$, the neighbourhood of $v$. Call it $G_1$; that is, $G_1 = G[\Gamma(v)]$. Then $G_1$ is $K_r$-free, as otherwise $\Gamma(v) \cup v$ is a $K_{r+1}$. By the induction hypothesis on $G_1$, there exists a $(r-1)$-partite graph $H_1$ such that $V(G_1) = V(H_1)$, and $d_{G_1}(v) \leq d_{H_1}(v)$ for every $v \in V(G_1)$.

Then we construct the graph $H$ as follows. Take $H_1$ and the rest of vertices as the vertex set. That is, $V(H) = V(H_1) \cup (V(G)\backslash V(H_1)) = V(G)$. Keep all edges in $H_1$. Moreover, for every $u \in V(G)\backslash V(H_1)$, add all edges between $u$ and every vertex in $V(H_1)$. Thus we have that $d_H(u) = |V(H_1)| = |V(G_1)|$ for every $u \in V(G)\backslash V(H_1)$.

On the other hand, we know that $|V(G_1)| = d_G(v) = \Delta(G) \geq d_G(u)$ for any $u \in V(G)$. Hence $d_H(u) = \Delta(G) \geq d_G(u)$. This holds for all $u \in V(G)\backslash V(H_1)$.

For any $u \in V(H_1)$, due to the construction of $H$,

$$\begin{aligned} d_H(u) &= d_{H_1}(u) + |V(G)\backslash V(H_1)| \\ &\geq d_{G_1}(u) + |V(G)\backslash V(G_1)| \\ &\geq d_G(u). \end{aligned}$$

The last step holds because in the original graph $G$, $u$ has $d_{G_1}(u)$ many neighbours in $V(G_1)$, and at most $|V(G)\backslash V(G_1)|$ many neighbours outside of $V(G_1)$.

We still have to verify that $H$ is $r$-partite. It is because $H_1$ is $(r-1)$-partite. Keep these $r-1$ classes of $H_1$, and define a new class comprising of $V(G)\backslash V(H_1)$. By construction, there is no edge between any two vertices of $V(G)\backslash V(H_1)$. □

The next proof has a similar flavor as Erdős's proof, but it uses an interesting trick of "vertex duplication".

*3rd proof of Theorem 1.* Let $G$ be a $K_{r+1}$-free graph with maximum number of edges. We prove the following claim, which implies the theorem.

**Claim 3.** *If $uv \in E$, then for any vertex $w$, either $uw \in E$ or $vw \in E$.*

*Proof of Claim 3.* Suppose the contrary; that is $uw \notin E$ and $vw \notin E$. There are two cases.

1. $d(w) < d(u)$ or $d(w) < d(v)$. Without loss of generality, assume that $d(w) < d(u)$. Duplicate $u$ to create a new vertex $u'$ such that $\Gamma(u') = \Gamma(u)$. If there is $K_{r+1}$ in the new graph, then it must contain $u'$. Thus it is still an clique if we replace $u'$ by $u$. This contradicts to $G$ being $K_{r+1}$-free. Hence the new graph is still $K_{r+1}$-free.

   Then we remove $w$. Call this graph $G'$. Clearly $G'$ is $K_{r+1}$-free. However, the number of edges is

$$\begin{aligned} e(G') &= e(G) + d(u') - d(w) \\ &= e(G) + d(u) - d(w) > e(G). \end{aligned}$$

   This contradicts to $G$ having maximum number of edges.

2. $d(w) \geq d(u)$ and $d(w) \geq d(v)$. Duplicate $w$ twice and remove $u$ and $v$. By exactly the same reasoning as above, the new graph is $K_{r+1}$-free, while the number of edges is

$$e(G) + d(w) + d(w) - (d(u) + d(v) - 1) > e(G).$$

   Again it contradicts to $G$ having maximum number of edges. □

Due to Claim 3, we know that if $uw \notin E$, $vw \notin E$, then $uv \notin E$. For an arbitrary vertex $v \in V$, there is no edge between vertices in $\Gamma^+(v) := \{v\} \cup \Gamma(v)$. Moreover, any other vertex $u \notin \Gamma^+(v)$ is adjacent to every vertex in $\Gamma^+(v)$. Thus, $G$ has to be a complete $k$-partite graph with $k \leq r$. If $k < r$, then we can view it as a complete $r$-partite graph with the last $r - k$ many classes empty. As we have discussed prior to the first proof, Turán's graph $T_r(n)$ maximizes the number of edges among all complete $r$-partite graphs. Thus $G$ has to be $T_r(n)$. $\square$