# Lecture 4: Turán's theorem and Erdős-Stone theorem

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## 1 Turán theorem - 4th proof

**Theorem 1** (Turán 1941)**.** *Let $n, r$ be two integers such that $n \geq r \geq 2$. If $G$ is $K_{r+1}$-free of order $n$, then $e(G) \leq t_r(n)$. In other words, $ex(n, K_{r+1}) \leq t_r(n)$.*

In this lecture, we present an "analytic" proof of a slightly weaker version of Theorem 1. It is due to Motzkin and Straus 1965.

We will prove the following theorem, which is slightly weaker than Theorem 1.

**Theorem 2.** *For any $n \geq r \geq 2$,*

$$ex(n, K_{r+1}) \leq \left(1 - \frac{1}{r}\right)\frac{n^2}{2}.$$

Theorem 2 is indeed weaker than Theorem 1, because it is not hard to verify that

$$t_r(n) \leq \left(1 - \frac{1}{r}\right)\frac{n^2}{2}.$$

On the other hand, Theorem 2 is strong enough to determine $\pi(K_{r+1})$. The upper bound follows because

$$\begin{aligned}
\pi(K_{r+1}) &= \lim_{n\to\infty} \frac{ex(n, K_{r+1})}{\binom{n}{2}} \\
&\leq \lim_{n\to\infty} \frac{\left(1 - \frac{1}{r}\right)\frac{n^2}{2}}{\binom{n}{2}} \\
&= \lim_{n\to\infty} \left(1 - \frac{1}{r}\right)\frac{n}{n-1} \\
&= 1 - \frac{1}{r}.
\end{aligned}$$

The lower bound is the same as before, due to the fact that $T_r(n)$ is $K_{r+1}$-free.

Define the adjacency matrix $A = A(G) = (a_{ij})$ for a graph $G$ of order $n$. Let $V = \{v_1, \cdots, v_n\}$. Then $A$ is a $n$-by-$n$ $0-1$ matrix such that $a_{ij} = 1$ if and only if $v_i v_j \in E$. Thus $A$ is symmetric. We will be interested in a quadratic form $\langle A\mathbf{x}, \mathbf{x}\rangle$ where $\mathbf{x}$ denotes a vector of length $n$. This is often called the *Lagrangian* of $G$. Define

$$f_G(\mathbf{x}) := \langle A\mathbf{x}, \mathbf{x}\rangle = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_i x_j = \sum_{v_i \sim v_j} x_i x_j.$$

Note that every edge $v_i v_j \in E$, it contributes $2x_i x_j$ to $f_G(\mathbf{x})$.

Now consider the set

$$S = \{\mathbf{x} \mid \mathbf{x} \in \mathbb{R}^n,\ \sum_{i=1}^{n} x_i = 1,\ \text{and } \forall i, x_i \geq 0\}.$$

$S$ is often called the *standard simplex* of $\mathbb{R}^n$. One way to think about it is to give each vertex a weight $x_i$ and the total weight is 1. Also give a edge $v_i v_j$ the weight $x_i x_j$. Then $f_G(\mathbf{x})$ is the total weight of all edges (times 2).

Since $S$ is a closed and bounded set, and $f_G(\cdot)$ is continuous, $f_G(\cdot)$ restricted to $S$ is bounded and its maximum is achieved for some $\mathbf{x} \in S$. Define

$$f(G) := \max_{\mathbf{x} \in S} f_G(\mathbf{x}).$$

Motzkin and Straus revealed an intimate relationship between $f(G)$ and the maximum cliques of $G$. Let us first calculate $f(K_n)$.

$$\begin{aligned}
f(K_n) &= \max_{\mathbf{x} \in S} f_{K_n}(\mathbf{x}) \\
&= \max_{\mathbf{x} \in S} \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} x_i x_j \\
&= \max_{\mathbf{x} \in S} \sum_{i=1}^{n} x_i(1 - x_i) \\
&= \max_{\mathbf{x} \in S} \left( \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} x_i^2 \right) \\
&= 1 - \min_{\mathbf{x} \in S} \sum_{i=1}^{n} x_i^2 \\
&= 1 - 1/n,
\end{aligned}$$

where we used Cauchy-Schwarz inequality ($n \sum_{i=1}^{n} x_i^2 \geq (\sum_{i=1}^{n} x_i)^2 = 1$) in the last line.

**Theorem 3** (Motzkin and Straus 1965)**.** *Let $G$ be a graph of order $n$. Let $k$ be the maximum size of cliques in $G$. Then $f(G) = 1 - \frac{1}{k}$.*

*Proof.* Let $\mathbf{y} \in S$ be the point that achieves the maximum of $f_G(\cdot)$, and its *support* $supp(\mathbf{y}) := \{v_i \mid y_i > 0\}$ is as small as possible. Let $K = supp(\mathbf{y})$. We claim that the induced subgraph $G[K]$ is a $k$-clique.

Suppose otherwise. Then there exists $y_1, y_2 > 0$ such that $v_1 \not\sim v_2$. Assume without loss of generality that $\sum_{v_i \in \Gamma(v_1)} y_i \geq \sum_{v_i \in \Gamma(v_2)} y_i$. Then define a new vector $\mathbf{y}' \in S$ as

$$(y_1 + y_2, 0, y_3, y_4, \cdots, y_n).$$

Now we have that

$$\begin{aligned}
f_G(\mathbf{y}') - f_G(\mathbf{y}) &= \sum_{v_i \sim v_j} y'_i y'_j - \sum_{v_i \sim v_j} y_i y_j \\
&= 2(y_1 + y_2) \sum_{v_i \in \Gamma(v_1)} y_i - 2y_1 \sum_{v_i \in \Gamma(v_1)} y_i - 2y_2 \sum_{v_i \in \Gamma(v_2)} y_i \\
&= 2y_2 \left( \sum_{v_i \in \Gamma(v_1)} y_i - \sum_{v_i \in \Gamma(v_2)} y_i \right) \\
&\geq 0.
\end{aligned}$$

Intuitively, we can always move the weight of $v_2$ to $v_1$ without decreasing $f_G(\cdot)$, as long as $v_1 v_2$ is not an edge and $\sum_{v_i \in \Gamma(v_1)} y_i \geq \sum_{v_i \in \Gamma(v_2)} y_i$.

On the other hand, $\mathbf{y}$ achieves the maximum of $f_G(\cdot)$. Thus $\mathbf{y}'$ is also the maximum. However $supp(\mathbf{y}')$ is one element smaller than $K = supp(\mathbf{y})$, contradicting the minimality of $K$. Hence $G[K]$ is a $k$-clique.

As we have calculated before the theorem, since $G[K]$ is a $k$-clique,

$$f(G) = f_G(\mathbf{y}) = 1 - \frac{1}{k}. \qquad \square$$

Why does Theorem 3 imply Theorem 2? This is because we can evaluate $f_G(\cdot)$ at another vector $\mathbf{z} = (1/n, 1/n, \cdots, 1/n)$. Then

$$f_G(\mathbf{z}) = 2 \sum_{e \in E} \left( \frac{1}{n} \right)^2 = \frac{2e(G)}{n^2}.$$

Due to the definition of $f(G)$,

$$f(G) \geq f_G(\mathbf{z}).$$

Now if $G$ is $K_{r+1}$-free, the maximum clique of $G$ is at most of size $r$. A consequence is that $f(G) \leq 1 - 1/r$. Thus

$$e(G) = \frac{n^2}{2} \cdot f_G(\mathbf{z}) \leq \frac{n^2}{2} \cdot f(G) \leq \frac{n^2}{2} \cdot \left( 1 - \frac{1}{r} \right).$$

This is Theorem 2.

## 2 The Erdős-Stone theorem

Now we give a general upper bound of $ex(n, H)$ for an arbitrary $H$. It will depend on the chromatic number of $H$.

**Definition 1.** *Let $G$ be a graph. The* chromatic number $\chi(G)$ *is the minimum $q \in \mathbb{N}$ such that $G$ is $q$-partite.*

The name "chromatic" is because of its relationship to proper colourings of graphs. A colouring is proper if no two adjacent vertices have the same colour.

**Definition 2.** *Let $G$ be a graph. A $q$-*colouring *$\sigma$ is a mapping from $V$ to $[q]$ such that if $uv \in E$, then $\sigma(u) \neq \sigma(v)$. If a $q$-colouring exists, then $G$ is said to be $q$-*colourable.

Another way of defining chromatic number is that $\chi(G)$ is the minimum $q \in \mathbb{N}$ such that $G$ is $q$-colourable.

Clearly if $H$ is a subgraph of $G$, then $\chi(G) \geq \chi(H)$. Note that Turán's graph $T_r(n)$ is $r$-colourable but not $r-1$-colourable. Thus $\chi(T_r(n)) = r$. Indeed, if $\chi(H) = r+1$, then $H$ cannot be a subgraph of $T_r(n)$. Hence, $ex(n, H) \geq t_r(n) \geq \left(1 - \frac{1}{r}\right)\binom{n}{2}$.

The Erdős-Stone theorem says that a similar upper bound holds as well! It is also called the "fundamental theorem of extremal graph theory".

**Theorem 4.** *Let $r \in \mathbb{N}$ and $\varepsilon > 0$. Then there exists $n_1 \in \mathbb{N}$ such that the following holds. For any graph $H$ with $\chi(H) = r+1$ and $n \geq n_1$, we have that*

$$ex(n, H) \leq \left(1 - \frac{1}{r} + \varepsilon\right)\binom{n}{2}.$$

An immediate consequence is that we can calculate the Turán density of any graph.

**Corollary 5.** *For any graph $H$ with at least one edge,*

$$\pi(H) = 1 - \frac{1}{\chi(H) - 1}.$$

*Proof of Corollary 5.* Since $H$ has at least one edge, $r = \chi(H) - 1 \geq 1$. For any $\varepsilon > 0$, by Theorem 4,

$$\pi(H) = \lim_{n\to\infty} \frac{ex(n, H)}{\binom{n}{2}} \leq 1 - \frac{1}{r} + \varepsilon.$$

Since this holds for all $\varepsilon > 0$, we must have that $\pi(H) \leq 1 - 1/r$.

On the other hand, as we discussed earlier, $ex(n, H) \geq t_r(n)$ for any $n \geq 2$. Hence,

$$\pi(H) \geq \lim_{n\to\infty} \frac{t_r(n)}{\binom{n}{2}} = 1 - \frac{1}{r}.$$

Therefore $\pi(H) = 1 - 1/r$ as required. □

Note that $\chi(K_{r+1}) = r+1$ and by Theorem 1, $\pi(K_{r+1}) = 1 - 1/r$. Hence Corollary 5 says that any graph $H$ with $\chi(H) = r+1$ has the same Turán density as $K_{r+1}$.

Let's turn to the proof of Theorem 4.

### 2.1 Proof of the Erdős-Stone theorem

Let $K_r^p$ be the complete $r$-partite graph with $p$ vertices in each class. In other words, $K_r^p = T_r(pr)$, the Turán graph with $pr$ many vertices. It is easy to see that $\chi(K_r^p) = r$.

For a graph $H$ with $\chi(H) = r+1$. Let $p = |V(H)|$. Then $H$ is a subgraph of $K_{r+1}^p$. Hence we only need to prove Theorem 4 for $K_{r+1}^p$. (Note that the claim of Theorem 4 does not depend on the number of vertices in $H$.)

The key observation will be the following lemma, which roughly states that if $G$ is $K_{r+1}^p$-free, then we can find a vertex of small enough degree.

**Lemma 6.** *Let $r, p \in \mathbb{N}$ and $\varepsilon > 0$. There exists $n_0 \in \mathbb{N}$ such that if $n \geq n_0$ and $G$ is a graph of order $n$ such that*

$$\delta(G) \geq \left(1 - \frac{1}{r} + \varepsilon\right) n,$$

*then $G$ contains a copy of $K_{r+1}^p$ as a subgraph.*

We will prove Lemma 6 in the next section.

The next lemma enables us to find a subgraph of high minimum degree in any graph with suitably many edges.

**Lemma 7.** *For all $c, \eta > 0$, $n > 8/\eta$, if $G$ is a graph on $n$ vertices with $e(G) \geq (c+\eta)\binom{n}{2}$, then $G$ has a subgraph $G'$ with $n' \geq \frac{1}{2}\sqrt{\eta}n$ vertices such that $\delta(G') \geq cn'$.*

*Proof.* Suppose otherwise for the sake of a contradiction. Then we can construct a sequence of graphs

$$G = G_n, G_{n-1}, G_{n-2}, \cdots, G_t$$

with $t = \left\lceil \frac{1}{2}\sqrt{\eta}n \right\rceil$, such that $G_i$ is a graph with $i$ vertices and $G_{i-1}$ is obtained from $G_i$ by removing a vertex of degree less than $ci$.

Then we have that

$$\begin{aligned}
e(G_t) &> e(G) - \sum_{i=t+1}^{n} ci \\
&\geq (c+\eta)\binom{n}{2} - c\left(\binom{n+1}{2} - \binom{t+1}{2}\right) \\
&= \eta\binom{n}{2} - c\left(\binom{n+1}{2} - \binom{n}{2}\right) + c\binom{\left\lceil \frac{1}{2}\sqrt{\eta}n \right\rceil + 1}{2} && \text{As } t = \left\lceil \frac{1}{2}\sqrt{\eta}n \right\rceil \\
&\geq \eta\binom{n}{2} - cn + c\frac{\eta n^2}{8} \\
&\geq \eta\binom{n}{2} && \text{As } n > 8/\eta \\
&\geq \frac{\sqrt{\eta} \cdot \sqrt{\eta}(n-1)}{2} \geq \binom{t}{2}.
\end{aligned}$$

But this is impossible, since $G_t$ has $t$ vertices, and the maximum number of edges is $\binom{t}{2}$. $\square$

Theorem 4 follows from Lemma 6 and Lemma 7.

*Proof of Theorem 4.* Let $H$ be a graph with $\chi(H) = r+1$ and $p = |V(H)|$. Then $H$ is a subgraph of $K_{r+1}^p$.

Suppose $G$ is a graph with $n \geq n_1$ vertices (we will choose $n_1$ later) and $e(G) \geq \left(1 - \frac{1}{r} + \varepsilon\right)\binom{n}{2}$. By Lemma 7 with $c = 1 - \frac{1}{r} + \frac{\varepsilon}{2}$ and $\eta = \frac{\varepsilon}{2}$, we can find a subgraph $G'$ of $G$ with $n' \geq \frac{1}{2\sqrt{2}}\sqrt{\varepsilon}n$ vertices such that $\delta(G') \geq \left(1 - \frac{1}{r} + \frac{\varepsilon}{2}\right)n'$.

Now we can apply Lemma 6 to $G'$ by choosing $\frac{1}{2\sqrt{2}}\sqrt{\varepsilon}n_1 \geq n_0$. Therefore $G'$ contains $K_{r+1}^p$ as a subgraph and so does $G$. $\square$

## 2.2 Proof of Lemma 6

Finally we prove Lemma 6 to complete the proof. The overall proof strategy is the following:

We do an induction on $r$. The induction hypothesis allows us to find a copy of $K_t^q$, where $q$ is a suitably chosen integer. ($q$ will be much larger than $p$, and doing this only bumps $n_0$.) Suppose for contradiction that $G$ does not contain a $K_{r+1}^p$. Then we can use the minimum degree condition to give a lower bound on the number of edges from $U$ to $\overline{U} := V(G)\backslash U$, where $U$ is the vertex set of $K_t^q$ we found. On the other hand, the fact that $G$ is $K_{r+1}^p$-free bounds from above the total number of such edges. Conflicting lower and upper bounds will yield the contradiction.

*Proof of Lemma 6.* We do an induction on $r$. The base case is $r = 1$. Then we have $\delta(G) \geq \varepsilon n$ and want to show that $G$ contains a copy of $K_2^p$, or equivalently a bipartite complete graph $K_{p,p}$.

Assume for contradiction that $G$ contains no $K_{p,p}$. Let $U \subseteq V(G)$ be a subset of vertices such that $|U| = q$ where we will choose $q > p$ later. The lower bound on $e(U, \overline{U})$ is easy. Since $\delta(G) \geq \varepsilon n$, we have that

$$\begin{aligned} e(U, \overline{U}) &= \sum_{v \in U} \left|\Gamma(v) \cap \overline{U}\right| \\ &\geq \sum_{v \in U} (d(v) - |U|) \\ &\geq (\varepsilon n - |U|)\,|U| \\ &= \varepsilon n q - q^2. \end{aligned} \tag{1}$$

For the upper bound, for each $v \in \overline{U}$, let $d_U(v) = |\Gamma(v) \cap U|$, the number of neighbours of $v$ in $U$. Our goal is to show that not too many vertices $v \in \overline{U}$ have very large $d_U(v)$.

Let $S$ be a subset of $U$ such that $|S| = p$. Given $S$, say a vertex $v \in \overline{U}$ is *completely joined* to $S$ if every vertex in $S$ is adjacent to $v$. Note that for any $S$, there can be at most $p-1$ many vertices that are completely joined to $S$. (Otherwise they form a $K_{p,p}$.) There are $\binom{q}{p}$ many such sets $S$. Each vertex $v \in \overline{U}$ with $d_U(v) \geq p$ is completely joined to at least one such $S$.

Let

$$N := \left|\{(v,S) \mid v \in \overline{U},\ S \subset U \text{ and } |S| = p,\ v \text{ is completely joined to } S\}\right|.$$

Then we have that

$$N \geq \left|\{v \in \overline{U} \mid d_U(v) \geq p\}\right|.$$

On the other hand, we have that

$$N \leq \binom{q}{p}(p-1).$$

Hence,

$$\left|\{x \in \overline{U} \mid d_U(v) \geq p\}\right| \leq \binom{q}{p}(p-1).$$

This gives us the upper bound on $e(U,\overline{U})$:

$$\begin{aligned} e(U,\overline{U}) &= \sum_{v \in \overline{U}} |\Gamma(v) \cap U| = \sum_{v \in \overline{U}} d_U(v) \\ &\leq \binom{q}{p}(p-1)\,|U| + \left(\left|\overline{U}\right| - \binom{q}{p}(p-1)\right)p \\ &\leq \binom{q}{p}(p-1)q + \left(n - \binom{q}{p}(p-1)\right)p \\ &= pn + \binom{q}{p}(p-1)(q-p). \end{aligned} \tag{2}$$

Combining (1) and (2) gives:

$$pn + \binom{q}{p}(p-1)(q-p) \geq \varepsilon nq - q^2.$$

However, this cannot hold if we pick $\varepsilon q > p$ and $n$ sufficiently large. Contradiction. This finishes the base case.

For the induction step, the overall strategy is exactly the same as the base case, except that we need tweak a few details. Let $r \geq 2$ and the lemma holds with $r-1$. Since $r \geq 2$,

$$\left(1 - \frac{1}{r} + \varepsilon\right)n > \left(1 - \frac{1}{r-1} + \varepsilon\right)n.$$

Thus by the induction hypothesis, if $n$ is large enough, then $G$ contains a copy of $K_r^q$ where we set $q$ so that $q > p$ and $\varepsilon rq > p$. Let $U$ be the vertex set of this copy of $K_r^q$. Then $|U| = qr$.

For the lower bound,

$$\begin{aligned}
e(U,\overline{U}) &= \sum_{v\in U} \left|\Gamma(v)\cap \overline{U}\right| \\
&\geq \sum_{v\in U} (d(v) - |U|) \\
&\geq \left(\left(1-\frac{1}{r}+\varepsilon\right)n - |U|\right)|U| \\
&= (r-1+\varepsilon r)\,qn - r^2q^2. \qquad (3)
\end{aligned}$$

For the upper bound, we cannot simply choose $S$ with $|S| = q$. Instead, call a subset $S$ *special* if $S$ contains exactly $p$ vertices from each of the $r$ classes of $U$. Again, call $v \in \overline{U}$ completely joined to $S$ if every vertex of $S$ is adjacent to $v$. Let

$$N := \left|\{(v,S) \mid v\in \overline{U},\ S \text{ is special},\ v \text{ is completely joined to } S\}\right|.$$

Then we have that

$$N \geq \left|\{v\in\overline{U} \mid d_U(v) \geq (r-1)q+p\}\right|.$$

This is counting from the vertex side: if $d_U(v) \geq (r-1)q+p$, then $v$ is adjacent to at least one special $S$. On the other hand, we have that

$$N \leq \binom{q}{p}^r (p-1).$$

This is counting from the special sets side. There are exactly $\binom{q}{p}^r$ many special sets, and each special set is completely joined by at most $(p-1)$ many vertices as otherwise $G$ is not $K^p_{r+1}$-free. Hence,

$$\left|\{v\in\overline{U} \mid d_U(v) \geq (r-1)q+p\}\right| \leq \binom{q}{p}^r (p-1).$$

Thus we have our upper bound:

$$\begin{aligned}
e(U,\overline{U}) &= \sum_{v\in\overline{U}} |\Gamma(v)\cap U| = \sum_{v\in\overline{U}} d_U(v) \\
&\leq \binom{q}{p}^r (p-1)\,|U| + \left(\left|\overline{U}\right| - \binom{q}{p}^r (p-1)\right)((r-1)q+p) \\
&\leq \binom{q}{p}^r (p-1)qr + \left(n - \binom{q}{p}^r (p-1)\right)((r-1)q+p) \\
&= ((r-1)q+p)n + \binom{q}{p}^r (p-1)(q-p). \qquad (4)
\end{aligned}$$

Combining (3) and (4) gives:

$$((r-1)q+p)n+c_1 \geq ((r-1)q+\varepsilon rq)nq-c_2,$$

where $c_1 = \binom{q}{p}^r(p-1)(q-p)$ and $c_2 = r^2q^2$. In other words,

$$(\varepsilon rq-p)n \leq c_1+c_2. \tag{5}$$

Note that $c_1$ and $c_2$ are independent of $n$. Moreover, we have chosen $q$ so that $\varepsilon rq > p$. Hence if $n$ sufficiently large, (5) cannot hold. This is a contradiction and finishes the proof. $\square$