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Lecture 5: Forbidding Cycles

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1 Finishing Erdős-Stone Theorem

The Erdős-Stone theorem states the following.

Theorem 1. *Let $r \in \mathbb{N}$ and $\varepsilon > 0$. Then there exists $n_1 \in \mathbb{N}$ such that the following holds. For any graph H with $\chi(H) = r + 1$ and $n \geq n_1$, we have that*

$$ex(n, H) \leq \left(1 - \frac{1}{r} + \varepsilon\right)\binom{n}{2}.$$

The proof of Theorem 1 consists of the following two lemmas. Recall that K_r^p is the complete r-partite graph with p vertices in each class. In other words, $K_r^p = T_r(pr)$, the Turán graph with pr many vertices. It is easy to see that $\chi(K_r^p) = r$.

Lemma 2. *For all $c, \eta > 0$, $n > 8/\eta$, if G is a graph on n vertices with $e(G) \geq (c + \eta)\binom{n}{2}$, then G has a subgraph G' with $n' \geq \frac{1}{2}\sqrt{\eta}n$ vertices such that $\delta(G') \geq cn'$.*

Lemma 3. *Let $r, p \in \mathbb{N}$ and $\varepsilon > 0$. There exists $n_0 \in \mathbb{N}$ such that if $n \geq n_0$ and G is a graph of order n such that*

$$\delta(G) \geq \left(1 - \frac{1}{r} + \varepsilon\right)n,$$

then G contains a copy of K_{r+1}^p as a subgraph.

We have shown Lemma 2 in the last lecture and we will finish the proof of Lemma 3. The overall proof strategy is the following:

We do an induction on r. The induction hypothesis allows us to find a copy of K_r^q, where q is a suitably chosen integer. (q will be much larger than p, and doing this only bumps n_0.) Suppose for contradiction that G does not contain a K_{r+1}^p. Then we can use the minimum degree condition to give a lower bound on the number of edges from U to $\overline{U} := V(G)\backslash U$, where U is the vertex set of K_r^q we found. On the other hand, the fact that G is K_{r+1}^p-free bounds from above the total number of such edges. Conflicting lower and upper bounds will yield the contradiction.

Proof of Lemma 3. We do an induction on r. The base case is $r = 1$. Then we have $\delta(G) \geq \varepsilon n$ and want to show that G contains a copy of K_2^p, or equivalently a bipartite complete graph $K_{p,p}$.

Assume for contradiction that G contains no $K_{p,p}$. Let $U \subseteq V(G)$ be a subset of vertices such that $|U| = q$ where we will choose $q > p$ later. The lower bound on $e(U, \overline{U})$ is easy. Since $\delta(G) \geq \varepsilon n$, we have that

$$\begin{aligned} e(U, \overline{U}) &= \sum_{v \in U} \left|\Gamma(v) \cap \overline{U}\right| \\ &\geq \sum_{v \in U} (d(v) - |U|) \\ &\geq (\varepsilon n - |U|)\,|U| \\ &= \varepsilon n q - q^2. \end{aligned} \tag{1}$$

For the upper bound, for each $v \in \overline{U}$, let $d_U(v) = |\Gamma(v) \cap U|$, the number of neighbours of v in U. Our goal is to show that not too many vertices $v \in \overline{U}$ have very large $d_U(v)$.

Let S be a subset of U such that $|S| = p$. Given S, say a vertex $v \in \overline{U}$ is *completely joined* to S if every vertex in S is adjacent to v. Note that for any S, there can be at most $p-1$ many vertices that are completely joined to S. (Otherwise they form a $K_{p,p}$.) There are $\binom{q}{p}$ many such sets S. Each vertex $v \in \overline{U}$ with $d_U(v) \geq p$ is completely joined to at least one such S.

Let

$$N := \left|\{(v, S) \mid v \in \overline{U},\ S \subset U \text{ and } |S| = p,\ v \text{ is completely joined to } S\}\right|.$$

Then we have that

$$N \geq \left|\{v \in \overline{U} \mid d_U(v) \geq p\}\right|.$$

On the other hand, we have that

$$N \leq \binom{q}{p}(p-1).$$

Hence,

$$\left|\{x \in \overline{U} \mid d_U(v) \geq p\}\right| \leq \binom{q}{p}(p-1).$$

This gives us the upper bound on $e(U, \overline{U})$:

$$\begin{aligned} e(U, \overline{U}) &= \sum_{v \in \overline{U}} |\Gamma(v) \cap U| = \sum_{v \in \overline{U}} d_U(v) \\ &\leq \binom{q}{p}(p-1)\,|U| + \left(\left|\overline{U}\right| - \binom{q}{p}(p-1)\right) p \\ &\leq \binom{q}{p}(p-1)q + \left(n - \binom{q}{p}(p-1)\right) p \\ &= pn + \binom{q}{p}(p-1)(q-p). \end{aligned} \tag{2}$$

Combining (1) and (2) gives:

$$pn + \binom{q}{p}(p-1)(q-p) \geq \varepsilon nq - q^2.$$

However, this cannot hold if we pick $\varepsilon q > p$ and n sufficiently large. Contradiction. This finishes the base case.

For the induction step, the overall strategy is exactly the same as the base case, except that we need tweak a few details. Let $r \geq 2$ and the lemma holds with $r-1$. Since $r \geq 2$,

$$\left(1 - \frac{1}{r} + \varepsilon\right) n > \left(1 - \frac{1}{r-1} + \varepsilon\right) n.$$

Thus by the induction hypothesis, if n is large enough, then G contains a copy of K_r^q where we set q so that $q > p$ and $\varepsilon rq > p$. Let U be the vertex set of this copy of K_r^q. Then $|U| = qr$.

For the lower bound,

$$\begin{aligned} e(U, \overline{U}) &= \sum_{v \in U} \left|\Gamma(v) \cap \overline{U}\right| \\ &\geq \sum_{v \in U} (d(v) - |U|) \\ &\geq \left(\left(1 - \frac{1}{r} + \varepsilon\right) n - |U|\right) |U| \\ &= (r - 1 + \varepsilon r)\, qn - r^2 q^2. \end{aligned} \tag{3}$$

For the upper bound, we cannot simply choose S with $|S| = q$. Instead, call a subset S *special* if S contains exactly p vertices from each of the r classes of U. Again, call $v \in \overline{U}$ completely joined to S if every vertex of S is adjacent to v. Let

$$N := \left|\{(v, S) \mid v \in \overline{U},\ S \text{ is special},\ v \text{ is completely joined to } S\}\right|.$$

Then we have that

$$N \geq \left|\{v \in \overline{U} \mid d_U(v) \geq (r-1)q + p\}\right|.$$

This is counting from the vertex side: if $d_U(v) \geq (r-1)q + p$, then v is adjacent to at least one special S. On the other hand, we have that

$$N \leq \binom{q}{p}^r (p-1).$$

This is counting from the special sets side. There are exactly $\binom{q}{p}^r$ many special sets, and each special set is completely joined by at most $(p-1)$ many vertices as otherwise G is not K_{r+1}^p-free. Hence,

$$\left|\{v \in \overline{U} \mid d_U(v) \geq (r-1)q + p\}\right| \leq \binom{q}{p}^r (p-1).$$

Thus we have our upper bound:

$$
\begin{aligned}
e(U,\overline{U}) &= \sum_{v\in\overline{U}} |\Gamma(v)\cap U| = \sum_{v\in\overline{U}} d_U(v) \\
&\leq \binom{q}{p}^r (p-1)\,|U| + \left(\left|\overline{U}\right| - \binom{q}{p}^r (p-1)\right)((r-1)q+p) \\
&\leq \binom{q}{p}^r (p-1)qr + \left(n - \binom{q}{p}^r (p-1)\right)((r-1)q+p) \\
&= ((r-1)q+p)n + \binom{q}{p}^r (p-1)(q-p). \qquad (4)
\end{aligned}
$$

Combining (3) and (4) gives:

$$((r-1)q+p)n + c_1 \geq ((r-1)q+\varepsilon rq)nq - c_2,$$

where $c_1 = \binom{q}{p}^r (p-1)(q-p)$ and $c_2 = r^2q^2$. In other words,

$$(\varepsilon rq - p)n \leq c_1 + c_2. \qquad (5)$$

Note that c_1 and c_2 are independent of n. Moreover, we have chosen q so that $\varepsilon rq > p$. Hence if n sufficiently large, (5) cannot hold. This is a contradiction and finishes the proof. □

2 Asymptotics and the Big O notation

The Erdős-Stone theorem gives us an exact answer to $\pi(H)$ for any graph H. Namely,

$$\pi(H) = 1 - \frac{1}{\chi(H)-1},$$

where $\chi(H) \geq 2$ is the chromatic number of H.

When $\chi(H) > 2$, then we know that $\pi(H)$ is strictly positive. Thus, from the fact that

$$\lim_{n\to\infty} \frac{ex(n,H)}{\binom{n}{2}} = \pi(H),$$

we can deduce that

$$\lim_{n\to\infty} \frac{ex(n,H)}{\pi(H)/2\cdot n^2} = 1.$$

In this case, we say that $\pi(H)/2 \cdot n^2$ is asymptotic to $ex(n,H)$.

When talking about these limiting behaviours, the Big O notations are usually useful. For two non-negative functions $f(n)$ and $g(n)$,

1. we write $f = O(g)$ if there exists constants n_0 and C such that $f(n) \le Cg(n)$ for all $n > n_0$;
2. we write $f = \Omega(g)$ if there exists constants n_0 and C such that $f(n) \ge Cg(n)$ for all $n > n_0$;
3. we write $f = \Theta(g)$ if there exists constants n_0, C_1, and C_2, such that $C_1 g(n) \le f(n) \le C_2 g(n)$ for all $n > n_0$.

If $f = \Theta(g)$, we say f and g have the same *order of magnitude*. Moreover, say $f = o(g)$ if for any $C > 0$, there exists n_C such that $f(n) \le Cg(n)$ for all $n > n_C$.

There are three kind of answers in extremal graph theory:

1. We can determine the *exact* answer to $ex(n, H)$. For example, Mantel's theorem states that $ex(n, K_3) = \lfloor n^2/4 \rfloor$.
2. We can determine the *asymptotic* of $ex(n, H)$. For example, Erdős-Stone theorem states that if $\chi(H) \ge 3$, then $\pi(H)/2 \cdot n^2$ is asymptotic to $ex(n, H)$. Another example: $(k-1)/2 \cdot n$ is asymptotic to $ex(n, P_k)$.
3. We can determine the *order of magnitude* of $ex(n, H)$. For example, Erdős-Stone theorem states that if $\chi(H) \ge 3$, then $ex(n, H) = \Theta(n^2)$. Another example: $ex(n, P_k) = \Theta(n)$.

Thus, if $\chi(H) \ge 3$, then Erdős-Stone theorem can give satisfactory answers to the second and the third questions. However, if $\chi(H) = 2$, then all we know is that $ex(n, H) = o(n^2)$. Next we will look at the simplest bipartite graph C_4.

3 Forbidding a cycle of length 4

We will look at $ex(n, C_4)$ in this section. There is no obvious guess, and the trivial lower bound is that $ex(n, C_4) \ge n-1$ as trees have $n-1$ edges. So we know that $ex(n, C_4) = \Omega(n)$. However, the correct order of magnitude is $\Theta(n^{3/2})$.

First let's show an upper bound, due to Reiman 1958, answering a question raised by Erdős in 1938.

Theorem 4 (Reiman 1958)**.**

$$ex(n, C_4) \le \frac{n}{4}\left(\sqrt{4n-3} + 1\right).$$

Proof. Let G be a C_4-free graph with n vertices and m edges. Let N be the number of paths of length 2 in G.

On one hand, each vertex v is the middle vertex of $\binom{d(v)}{2}$ paths of length 2, and so

$$N = \sum_{v \in V(G)} \binom{d(v)}{2}.$$

On the other hand, each unordered pair of vertices are the endpoints of at most one path of length 2. Indeed, for a pair (u, v) if there are two different paths $u - x - v$ and $u - y - v$, then we have a C_4: $u - x - v - y - u$. Hence we see that

$$N \leq \binom{n}{2}.$$

Put these two facts together, we have that

$$\sum_{v \in V(G)} \binom{d(v)}{2} = N \leq \binom{n}{2}.$$

We can rewrite the left hand side as

$$\sum_{v \in V(G)} \binom{d(v)}{2} = \sum_{v \in V(G)} \frac{d(v)(d(v)-1)}{2} = \frac{1}{2} \sum_{v \in V(G)} d(v)^2 - \frac{1}{2} \sum_{v \in V(G)} d(v).$$

Similar to the proof of Mantel's theorem, we have that

$$\sum_{v \in V(G)} d(v) = 2m,$$

and

$$\sum_{v \in V(G)} d(v)^2 \geq \frac{1}{n} \left(\sum_{v \in V(G)} d(v) \right)^2 = \frac{4m^2}{n}.$$

The second inequality is due to the Cauchy-Schwarz inequality. Putting everything together, we get that

$$\frac{n(n-1)}{2} \geq \sum_{v \in V(G)} \binom{d(v)}{2} \geq \frac{2m^2}{n} - m,$$

or equivalently,

$$4m^2 - 2nm - n^2(n-1) \leq 0.$$

Solving this quadratic inequality, we get

$$m \leq \frac{n}{4} \left(\sqrt{4n-3} + 1 \right). \qquad \square$$

Theorem 4 gives us the desired upper bound. However, the lower bound is not as easy. We will construct a C_4-free graph using projective planes later.

The situation is much less clear for general k. There are upper bounds of the form $ex(n, C_{2k}) = O(n^{1+1/k})$, but the matching lower bound $ex(n, C_{2k}) = \Omega(n^{1+1/k})$ is only known for C_6 and C_{10}.

4 Finite projective planes

Let $d \in \mathbb{N}$. A *finite projective plane of order* d is a pair (P, L), where P and L are disjoint finite sets together with a symmetric relation I (called 'incidence') between elements of P and elements of L, satisfying the following axioms.

1. For any element $p \in P$, there are exactly $d+1$ elements $\ell \in L$ such that p is incident to ℓ;

2. For any element $\ell \in L$, there are exactly $d+1$ elements $p \in P$ such that ℓ is incident to p;

3. For any two elements $p_1, p_2 \in P$, there is a unique elements $\ell \in L$ such that ℓ is incident to both p_1 and p_2;

4. For any two elements $\ell_1, \ell_2 \in L$, there is a unique elements $p \in P$ such that p is incident to both ℓ_1 and ℓ_2;

You should imagine P as points, and L as lines. Then incidence is that a point is on a line.

If (P, L) is a finite projective plane of order d, then we can construct a bipartite graph G from (P, L) as follows: Let P and L be two vertex classes of the bipartite graph. If $p \in P$ is incident to $\ell \in L$, then draw the edge (p, ℓ). This is a C_4-free graph due to the 3rd or the 4th axiom. If there is a C_4, then there are two points p_1, p_2 that have two common neighbours ℓ_1 and ℓ_2, which violates the 3rd axiom.

What we are going to argue is that the graph G has many edges. Since the graph G is $(d+1)$-regular, $e(G) = (d+1)|P| = (d+1)|L|$. Thus we just need to determine $|P| = |L|$.

First we make the following observation.

Lemma 5. *Let* (P, L) *be a finite projective plane of order* d. *Then*

$$|P| = |L| = d^2 + d + 1.$$

Proof. Let G be the bipartite graph obtained from (P, L). Let N be the number of paths of length 2 in G with middle vertex in P. For each $p \in P$, there are exactly $\binom{d+1}{2}$ paths of length 2 in G with middle vertex p, so

$$N = |P| \binom{d+1}{2}.$$

On the other hand, for any (unordered) pair of distinct elements $\ell_1, \ell_2 \in L$, there is exactly one path of length 2 in G whose middle vertex is in P and whose vertices are ℓ_1 and ℓ_2, so

$$N = \binom{|L|}{2}.$$

Therefore we have that

$$|P|\binom{d+1}{2} = N = \binom{|L|}{2}.$$

But we know that $|P| = |L|$. It implies that

$$\frac{|P|}{2} \cdot d(d+1) = \frac{|P|}{2}(|P|-1).$$

Cancelling $\frac{|P|}{2}$ yields

$$|P| - 1 = d^2 + d.$$

This finishes the proof. $\square$

If $d \in \mathbb{N}$ and such a finite projective plane of order d exists, then we can construct such a bipartite C_4-free graph G with $n = 2(d^2 + d + 1)$ vertices and $e(G) = (d^2 + d + 1)(d + 1)$ edges. Since

$$d^2 + d + 1 - n/2 = 0,$$

we have that

$$d = \frac{-1 + \sqrt{2n-3}}{2}.$$

Thus,

$$e(G) = n/2(d+1) = \frac{n}{4}\left(\sqrt{2n-3} + 1\right).$$

This is roughly $1/\sqrt{2}$ of the upper bound we get in Theorem 4.

Of course, all of the above is assuming that such a finite projective plane does exist. The next step is to construct these objects when d is a prime power.