MTH742P: Advanced Combinatorics 31/10/2016

Lecture 6: Forbidden Bipartite Cycles and Cliques

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1 Constructing finite projective planes

In the last lecture we showed the following theorem.

Theorem 1 (Reiman 1958)**.**

$$ex(n, C_4) \leq \frac{n}{4}\left(\sqrt{4n-3}+1\right).$$

To get a matching lower bound, we showed that as long as we can construct a finite projective plane of order d, then we have a bipartite graph that is C_4-free and has $\Theta(n^{3/2})$ edges where the number of vertices $n = d^2 + d + 1$. In this lecture we will construct such objects when d is a prime power. This construction is also due to Reiman 1958.

Let q be a prime power, i.e. $q = p^a$ where p is a prime and $a \in \mathbb{N}$. Then there exists a finite field $\mathbb{F}_q$ with q elements. (If q is a prime, then we can simply take the set of integers modulo q. Otherwise to construct $\mathbb{F}_q$ requires a little bit of algebra. We won't do it here but it can be found in almost any algebra textbooks.) In fact, for what we are going to do below, it is enough to simply think q as a prime number.

We build a projective plane of order q as follows. Let $V = \mathbb{F}_q^3$, the vector space of size q^3 over the field $\mathbb{F}_q$. So an element $\mathbf{x} \in V$ has the form of (a, b, c) where $a, b, c \in \mathbb{F}_q$. Recall that a linear subspace S is defined by the following three properties.

1. $\mathbf{0} \in S$;
2. If $\mathbf{x}, \mathbf{y} \in S$, then $\mathbf{x} + \mathbf{y} \in S$;
3. If $\mathbf{x} \in S$, then $c \cdot \mathbf{x} \in S$ for any $c \in \mathbb{F}_q$.

Let P be the set of all 1-dimensional subspaces of V. That is, if $p \in P$, then p is of the form

$$p = \{c \cdot \mathbf{x} \mid c \in \mathbb{F}_q\},$$

where $\mathbf{x}$ is some fixed element in $\mathbb{F}_q^3$. Let L be the set of all 2-dimensional subspaces of V. Any element $\ell \in L$ has the form

$$\ell = \{a \cdot \mathbf{x_1} + b \cdot \mathbf{x_2} \mid a, b \in \mathbb{F}_q\},$$

where $\mathbf{x_1}, \mathbf{x_2}$ are two fixed element in $\mathbb{F}_q^3$ that are not linearly dependent. For any $p \in P$ and $\ell \in L$, p and ℓ are incident if the 1-dimensional space of p is contained in the 2-dimensional space of ℓ. Then we claim that (P, L) is a finite projective plane of order q.

Let us first calculate $|P|$ and $|L|$. Note that $|V \backslash \{\mathbf{0}\}| = q^3 - 1$, and

$$\{U \backslash \{\mathbf{0}\} \mid U \text{ is a 1-dimensional subspace of } V\}$$

is a partition of $V \backslash \{\mathbf{0}\}$ into $|P|$ sets, each of size $q - 1$. Hence,

$$|P|\,(q-1) = |V \backslash \{0\}| = q^3 - 1,$$

so $|P| = (q^3 - 1)/(q - 1) = q^2 + q + 1$.

For $|L|$, let's count the number N of ordered linearly independent vectors $(\mathbf{x}, \mathbf{y})$ in V. There are $q^3 - 1$ choices for $\mathbf{x}$ and fixing $\mathbf{x}$, there are $q^3 - q$ many choices for $\mathbf{y}$. Thus, $N = (q^3 - 1)(q^3 - q)$. Each such pair is a basis of exactly one element of L, and every element of L has $(q^2 - 1)(q^2 - q)$ ordered pairs of vectors that form a basis. Hence,

$$|L|\,(q^2 - 1)(q^2 - q) = N = (q^3 - 1)(q^3 - q).$$

It implies that

$$|L| = \frac{(q^3 - 1)(q^3 - q)}{(q^2 - 1)(q^2 - q)} = \frac{q(q^3 - 1)(q^2 - 1)}{q(q^2 - 1)(q - 1)} = q^2 + q + 1.$$

Hence we see that $|P| = |L|$.

Then we verify the axioms. Let us repeat the axioms here.

1. For any $p \in P$, there are exactly $q + 1$ elements $\ell \in L$ such that p is incident to ℓ;
2. For any $\ell \in L$, there are exactly $q + 1$ elements $p \in P$ such that ℓ is incident to p;
3. For any $p_1, p_2 \in P$, there is a unique $\ell \in L$ such that ℓ is incident to both p_1 and p_2;
4. For any $\ell_1, \ell_2 \in L$, there is a unique $p \in P$ such that p is incident to both ℓ_1 and ℓ_2;

We verify axiom 2 first. Thus we just need to count the number of 1-dimensional subspaces of a fixed 2-dimensional subspace, say W, in V. There are $q^2 - 1$ many elements in $W \backslash \{0\}$, but for each element, there are $q - 1$ many element (including itself) that spans the same subspace. Hence there are $(q^2 - 1)/(q - 1) = q + 1$ many such 1-dimensional subspaces. This also implies that G is regular on L's side with degree $q + 1$.

For axiom 1, we need to count the number of 2-dimensional subspaces that contain a fixed 1-dimensional subspace in V. Clearly this number is the same for every fixed 1-dimensional subspace, say Δ. Hence the bipartite graph G is regular on P's side. However axiom 1 already tells us that G is regular on L's side with degree $q + 1$. Thus,

$$e(G) = |L|\,(q + 1) = \Delta\,|P|\,.$$

We have shown that $|P| = |L|$, implying that $\Delta = q + 1$ as well.

For axiom 3, we need to show that any two 1-dimensional subspaces of V spans a unique 2-dimensional subspace of V. This is straightforward.

For axiom 4, we need to show that the intersection of two 2-dimensional subspaces is a 1-dimensional subspace. It is not hard to verify that the intersection of linear subspaces is still a linear subspace. To count the dimension, say the two subspaces are W_1 and W_2. Let $W_1 + W_2$ be the span of $W_1 \cup W_2$. Since W_1 and W_2 are distinct, $\dim(W_1 + W_2) \geq 3$. On the other hand, $W_1 + W_2$ is a subspace of V, and so $\dim(W_1 + W_2) \leq \dim(V) \leq 3$, implying $\dim(W_1 + W_2) = 3$. Thus we see that

$$\begin{aligned}\dim(W_1 \cap W_2) &= \dim(W_1) + \dim(W_2) - \dim(W_1 + W_2) \\ &= 2 + 2 - 3 = 1.\end{aligned}$$

We conclude that (P, L) constructed above is indeed a finite projective plane of order q. This is called the *Desarguesian projective plane of order* q. When $q = 2$, it is also called the *Fano plane*, with $|P| = |L| = 7$.

Thus, we see that if $n = 2(q^2 + q + 1)$ for some prime power q, then $ex(n, C_4) \geq \frac{n}{4}(\sqrt{2n-3} + 1) = \Omega(n^{3/2})$. The lower bound is about $1/\sqrt{2}$ of the upper bound in Theorem 1. We still need to deal with those n's that do not have this form. In that case, we will choose some $n_1 < n$ of the above form but close to n, and then use the fact that $ex(n, C_4) \geq ex(n_1, C_4) = \Omega(n_1^{3/2})$. If $n_1 = \Theta(n)$, then although the constant will increase, the order of magnitude will still be correct!

Let's do this in detail. By Bertrand's postulate, for any real number $x > 2$, there exists a prime number p such that $x < p < 2x$. If $n \in \mathbb{N}$ with $n \geq 17$, then choose a prime number p such that $\sqrt{n}/4 < p < \sqrt{n}/2$ such that $n_1 = 2(p^2 + p + 1)$. Then,

$$n_1 = 2(p^2 + p + 1) < 2(n/4 + \sqrt{n}/2 + 1) = n/2 + \sqrt{n} + 2 < n,$$

and

$$n_1 = 2(p^2 + p + 1) > 2n/16 = n/8.$$

Thus $n_1 = \Theta(n)$.

In fact, we see that if $n \geq 17$, then

$$ex(n, C_4) \geq ex(n_1, C_4) = (p^2 + p + 1)(p + 1) > n^{3/2}/64.$$

If $2 \leq n \leq 17$, then $ex(n, C_4) \geq n - 1$ and one can verify that $\frac{n^{3/2}}{64} \leq n - 1$. Thus we can conclude that for any $n \geq 2$,

$$ex(n, C_4) \geq \frac{1}{64} n^{3/2}.$$

This demonstrates one of the advantage of the Big Oh notation — under the Big Oh notation, we only need to consider large enough n's.

For example, assume we want to bound a function $f(n)$, and we can show that $f(n) \leq n^{3/2}$ when $n > n_0$ for some large enough n_0. Then we immediately know that $f(n) = O(n^{3/2})$. Let

$$C := \max_{1 \leq n \leq n_0} f(n).$$

Then we see that for all $n \geq 1$,

$$f(n) \leq C n^{3/2}.$$

Thus in the Big Oh notation, there is no real difference between considering all integers n and considering only large enough integers.

2 Forbidding even cycles

Another extreme bound along the same vein is forbidden even cycles. (Recall that forbidding an odd cycle, the number of edges can still be as large as roughly $n^2/4$.) The following result is due to Bondy and Simonovits 1974.

Theorem 2. *Let $t \geq 2$. Then there exists a constant $c > 0$ such that*

$$ex(n, C_{2t}) \leq cn^{1+1/t}.$$

The proof is rather complicated. We will show a weaker result, for graphs that do not contain any cycle of length at most $2t$. Indeed, the length of shortest cycle is a graph G is called its *girth*.

Theorem 3. *Let $t \geq 2$ and G be a graph of order n. If G has girth at least $2t + 1$, then*

$$e(G) \leq n\left(n^{1/t} + 1\right) = n^{1+1/t} + n.$$

Proof. Suppose, for contradiction, that $e(G) > n\left(n^{1/t} + 1\right)$. The average degree of G is

$$D := \frac{\sum_{v \in V} d(v)}{n} = \frac{2e(G)}{n} > 2\left(n^{1/t} + 1\right).$$

The 4th question in Exercise 4 shows that there exists a subgraph G' of G such that the minimum degree of G', $\delta(G') \geq D/2 > n^{1/t} + 1$.

On the other hand, the girth of G' is also at least $2t+1$. Pick an arbitrary vertex v in G', the neighbourhood of v of distance t must be a tree, as otherwise we have a cycle of length at most $2t$.

To be specific, let

$$N_\ell(v) := \{u \mid u \in V,\ \mathrm{dist}(u, v) = \ell\},$$

where $\mathrm{dist}(u, v)$ is the shortest length of path from u to v. Notice that if $u \in N_\ell(v)$ for any $2 \leq \ell \leq t$, then u is not adjacent to any vertex in $N_k(v)$ for any $k \leq \ell - 2$ (if so, then the distance should be shorter). Moreover, u can be adjacent to only one of $N_{\ell-1}(v)$, since otherwise we create a cycle of length $2\ell \leq 2t$.

Thus, we see that $|N_{\ell+1}(v)| \geq (\delta(G') - 1)\,|N_\ell(v)| > n^{1/t}\,|N_\ell(v)|$. This implies that $|N_t(v)| \geq (n^{1/t})^t = n$, which contradicts to that G' is a subgraph of G. $\square$

3 Forbidding bicliques

Notice that C_4 is also a $K_{2,2}$. Thus we can also generalize Theorem 1 to forbidden bipartite complete graphs. The argument originates from Kővári, Sós, and Turán 1954.

We need to first prove a few facts about convex functions for Jensen's inequality. We will need it in place of the Cauchy-Schwarz inequality.

Definition 1. *A function $f(x)$ is said to be* convex *if for any $\lambda \in [0,1]$ and any x, y, we have that*

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y).$$

Lemma 4 (Jensen's inequality)**.** *If $f(x)$ is convex, then for any $\lambda_1, \cdots, \lambda_n \geq 0$ with $\sum_{i=1}^n \lambda_i = 1$ and $x_1, \cdots, x_n \in \mathbb{R}$, we have*

$$f\left(\sum_{i=1}^n \lambda_i x_i\right) \leq \sum_{i=1}^n \lambda_i f(x_i).$$

Proof. We prove it by induction on n. For $n = 1$ it is trivial. For $n = 2$ it is Definition 1.

For the induction step, if $\lambda_n = 0$, then it is the same as the case of $n-1$. Otherwise $\lambda_n > 0$ and we have that

$$\begin{aligned} f\left(\sum_{i=1}^n \lambda_i x_i\right) &= f\left(\lambda_n x_n + (1-\lambda_n)\sum_{i=1}^{n-1} \frac{\lambda_i}{1-\lambda_n} x_i\right) \\ &\leq \lambda_n f(x_n) + (1-\lambda_n) f\left(\sum_{i=1}^{n-1} \frac{\lambda_i}{1-\lambda_n} x_i\right) && \text{(by Def 1)} \\ &\leq \lambda_n f(x_n) + (1-\lambda_n) \sum_{i=1}^{n-1} \frac{\lambda_i}{1-\lambda_n} f(x_i) && \text{(by IH)} \\ &= \sum_{i=1}^n \lambda_i f(x_i). \end{aligned}$$

□

In particular, Lemma 4 implies that if we fix $\sum_{i=1}^n x_i = S$, then the sum $\frac{1}{n}\sum_{i=1}^n f(x_i)$ is minimized at $x_i = \frac{S}{n}$ where $f(x)$ is convex (apply Lemma 4 with $\lambda_i = 1/n$).

To verify convexity, we need the following.

Lemma 5. *If $f(x)$ is twice differentiable with $f''(x) \geq 0$, then $f(x)$ is convex.*

This can be easily proved using the Mean Value Theorem.

We extend the definition of $\binom{n}{t}$ for a fixed integer t. If $x \geq t$ is not necessarily an integer, let

$$h_t(x) := \begin{cases} \frac{x(x-1)(x-2)\cdots(x-t+1)}{t!} & \text{if } x \geq t-1; \\ 0 & \text{otherwise.} \end{cases}$$

Note that if x is an integer then $h_t(x)$ coincides with the normal $\binom{n}{t}$. We can verify that $h_t(x)$ is convex for all positive x using Lemma 5.

Theorem 6. *Let $t \geq 2$. Then there exists a constant $c > 0$ such that*

$$ex(n, K_{t,t}) \leq cn^{2-1/t}.$$

Proof. Let G be a graph without $K_{t,t}$ of order n and m edges. The neighbourhood of vertex v contains $\binom{d(v)}{t}$ many t-tuples of vertices. Let's count such t-tuples over the neighbourhoods of all vertices. Note that any particular t-tuple can be counted at most $t-1$ times in this way, since the lack of $K_{t,t}$. It implies that,

$$\sum_{v \in V} \binom{d(v)}{t} \leq (t-1)\binom{n}{t}.$$

The left hand side is a sum of functions $\sum_{v \in V} h_t(d(v))$. Due to the convexity of $h_t(x)$ and Lemma 4, the left-hand side is minimized if all degrees are equal, $d(v) = 2m/n$. Note that $2m/n$ is the average degree of G and it should be much larger than t, which is a constant. Therefore,

$$\sum_{v \in V} \binom{d(v)}{t} \geq nh_t(2m/n).$$

It is not hard to see that for any two $x \geq t \geq 1$,

$$h_t(x) = \frac{x(x-1)(x-2)\cdots(x-t+1)}{t!} \geq \frac{(x-t)^t}{t!};$$
$$h_t(x) = \frac{x(x-1)(x-2)\cdots(x-t+1)}{t!} \leq \frac{x^t}{t!}.$$

Put these facts together,

$$n \cdot \frac{(2m/n - t)^t}{t!} \leq nh_t(2m/n) \leq \sum_{v \in V} \binom{d(v)}{t} \leq (t-1)\binom{n}{t} \leq (t-1)\frac{n^t}{t!}.$$

Simplifying the equation, we get,

$$2m/n - t \leq (t-1)^{1/t} n^{1-1/t},$$

or equivalently,

$$m \leq \frac{(t-1)^{1/t}}{2} n^{2-1/t} + \frac{tn}{2}. \qquad \square$$