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Lecture 9: Alteration

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# 1 Alteration

Sometimes merely randomization is not enough to show what we want. A common strategy is then to alter the instance we construct via randomness. For example, if the expectation of the parameter $X$ of a random object is $\mathbb{E} X$, then we have an instance with parameter $\leq \mathbb{E} X$. However, what we are after might be a "perfect" object with $X = 0$. Thus, what we do is to "correct bad properties". Once again, this is best shown by examples.

An *independent set* of a graph is the complement of a clique; that is, $S$ is an independent set, if there is no edge between vertices of $S$.

**Theorem 1.** *Let $G = (V, E)$ be a graph of $n$ vertices and $m \geq n/2$ edges. Then $G$ contains an independent set of size at least $\frac{n^2}{4m}$.*

*Proof.* We choose vertices independently and with probability $p$ to get a subset $S \subseteq V$. Let $X = |S|$. Then $\mathbb{E} X = np$ by making an indicator variable for each vertex and using linearity of expectation.

For each $e \in E$, let $Y_e$ be the indicator variable of the event that both endpoints of $e$ is occupied. Thus for $e = (u, v)$,

$$\mathbb{E} Y_e = \Pr(u \in S,\ v \in S) = p^2.$$

Let $Y = \sum_{e \in E} Y_e$. Thus, an independent set is one with $Y = 0$. By linearity of expectation, we have that

$$\mathbb{E} Y = \sum_{e \in E} \mathbb{E} Y_e = mp^2.$$

Clearly a random subset will not give us an independent set. However, we can use the alteration method. What we are going to do is to "fix" these occupied edges. A simple fix is to just unselect a vertex of the occupied edge. Thus we need to remove at most one vertex from each chosen edge, leaving an independent set of size at least $X - Y$. Again, by linearity of expectation,

$$\mathbb{E}(X - Y) = np - mp^2.$$

Thus, there exist an independent set $I$ of size at least $np - mp^2$. Maximizing this function yield $p = \frac{n}{2m}$ (here we use $m > n/2$) and $np - mp^2 = \frac{n^2}{4m}$. □

Theorem 1 shows the basic idea of alteration. We want to construct a "perfect" object, and a random choice usually leaves $X$ many "faults". Thus, we perform some deterministic operations to fix these faults and find a (usually smaller) perfect object.

# 2 Ramsey number revisited

Let us revisit the Ramsey numbers. Recall that $R(k, \ell)$ is the smallest size $n$ such that any 2-colouring of $K_n$ must contain a blue clique of size $k$ or a red clique of size $\ell$. Using the basic method, we have shown that $R(k, k) > 2^{\lfloor k/2 \rfloor}$. In fact, if we do a more careful analysis, the bound using the basic method is

$$R(k, k) > \frac{1}{e\sqrt{2}}(1 + o(1))k2^{k/2}. \tag{1}$$

We will show that using alteration, we can get a different bound, that leads to a slight improvement of the one above.

**Theorem 2.** *For any integer $n$, $R(k, k) > n - \binom{n}{k}2^{1-\binom{k}{2}}$.*

*Proof.* As before, consider a uniformly random 2-colouring of $K_n$. Let $X$ be the number of monochromatic cliques of size $k$. Thus, $X = \sum_{S \subset [n],\, |S|=k} X_S$, where $X_S$ is the indicator variable that $S$ is monochromatic for a subset $S$ of vertices of size $k$. Then we have that

$$\mathbb{E}\, X_S = \frac{2}{2^{\binom{k}{2}}},$$

as there are two such colourings among $2^{\binom{k}{2}}$ total possibilities. Due to linearity of expectation,

$$\mathbb{E}\, X = \sum_{S \subset [n],\, |S|=k} \mathbb{E}\, X_S = \binom{n}{k} 2^{1-\binom{k}{2}}.$$

Here comes the idea of alteration. We know that there must exist a colouring such that the number of monochromatic cliques is at most $\mathbb{E}\, X = \binom{n}{k}2^{1-\binom{k}{2}}$. Thus, to fix these "undesired" events, we can keep this colouring, but remove one vertex from each such clique to ensure that the final graph does not have any monochromatic cliques. It is easy to see that we need to remove at most one vertex for every clique, and thus the final graph has size at least $n - \binom{n}{k}2^{1-\binom{k}{2}}$. This finishes the proof. $\square$

There is still some calculus to do to find the best $n$ to optimize the bound in Theorem 2. Since $\binom{n}{k} \leq \left(\frac{ne}{k}\right)^k$, Theorem 2 implies that

$$R(k, k) > n - \left(\frac{ne}{k}\right)^k 2^{1-\binom{k}{2}}.$$

Taking the derivative of $n$ on the right hand side yields

$$1 - kn^{k-1}\left(\frac{e}{k}\right)^k 2^{1-\binom{k}{2}}.$$

It implies that $n = \frac{k}{e} \cdot 2^{k/2} \cdot (2e)^{-1/k}$ maximizes $n - \left(\frac{ne}{k}\right)^k 2^{1-\binom{k}{2}}$. Indeed, if we use the tighter Stirling's approximation, we will find out that $n = (1-o(1))\frac{k}{e} \cdot 2^{k/2}$ maximizes $n - \binom{n}{k}2^{1-\binom{k}{2}}$, in which case

$$R(k,k) > \frac{1}{e}(1+o(1))k2^{k/2}.$$

This is a $\sqrt{2}$ improvement upon (1). Later we will see yet another $\sqrt{2}$ improvement using the Lovász Local Lemma.

For off-diagonal Ramsey numbers, the alteration method is in fact stronger. Recall that using the basic method, we can get a bound $R(k,\ell) > n$ if there exists $p \in (0,1)$ such that

$$\binom{n}{k}p^{\binom{k}{2}} + \binom{n}{\ell}(1-p)^{\binom{\ell}{2}} < 1.$$

**Theorem 3.** *For any integer $n$ and $p \in (0,1)$,*

$$R(k,\ell) > n - \binom{n}{k}p^{\binom{k}{2}} - \binom{n}{\ell}(1-p)^{\binom{\ell}{2}}.$$

*Proof.* Consider a random colouring such that every edge is coloured blue with probability $p$ and red with probability $1-p$ independently. Let $X$ be the number of blue cliques of size $k$ plus the number of red cliques of size $\ell$ ("bad" objects). Then by linearity of expectation again,

$$\mathbb{E}\,X = \binom{n}{k}p^{\binom{k}{2}} + \binom{n}{\ell}(1-p)^{\binom{\ell}{2}}.$$

Thus there exists a colouring such that $X$ is at most $\mathbb{E}\,X$. Using the alteration idea, we keep the colouring and remove at most one vertex from each such "bad" clique to obtain a complete graph of size at least $n - \mathbb{E}\,X$ and there is no blue $K_k$ nor red $K_\ell$ under this 2-colouring. $\square$

Comparing Theorem 3 with the bound derived from the basic method can get quite complicated. However, usually Theorem 3 is an improvement. For example, the basic method yields $R(4,k) \geq \Omega\left(\left(\frac{k}{\log k}\right)^{3/2}\right)$ whereas Theorem 3 yields $R(4,k) \geq \Omega\left(\left(\frac{k}{\log k}\right)^{2}\right)$.

# 3 Girth and the chromatic number

The next theorem shows the surprising power of the probabilistic method. We have constructed graphs of chromatic number $k$ that are not isomorphic to $K_k$. However, those graphs have girth 3. One natural question is: does the chromatic number obey some upper bound if the girth is large enough? Intuitively this is not at all obvious — high girth means locally tree-like, and trees have chromatic number 2! Thus, for a graph of high girth to have high chromatic number, we cannot hope it is because of some local properties.

The answer to this question is no. For any two integers $\ell$ and $k$, there exists a graph of girth $> \ell$ and chromatic number $> k$. The construction is via alteration after randomly choosing edges.

Here we need to construct a random graph. We use the Erdő-Renyi random graph $G(n, p)$, which is constructed on $n$ vertices by choosing every possible edge with probability $p$ independently. In other words, we start with $K_n$ and then remove edges with probability $1 - p$ independently.

In order to obtain high chromatic number, we will look at the maximum independent set in the graph. Let $\alpha(G)$ be its size. Suppose that $\chi(G) = k$ and there is a proper $k$-colouring. Vertices of a particular colour form an independent set. Let $n_i$ be the number of vertices coloured $i$. Hence $n_i \leq \alpha(G)$, which implies that $n = \sum_{i=1}^{k} n_i \leq k\alpha(G)$. In other words,

$$\chi(G) \geq \frac{n}{\alpha(G)}. \tag{2}$$

We will use this fact later.

Another ingredient we will need is the Markov inequality.

**Theorem 4.** *Let $X$ be a non-negative random variable. Then for any $C > 0$,*

$$\Pr(X \geq C) \leq \frac{\mathbb{E}\, X}{C}.$$

*Proof.* Let $I_{X \geq C}$ be the indicator variable for the event $X \geq C$. Since $X$ is non-negative,

$$X \geq C I_{X \geq C}.$$

Take the expectation,

$$\mathbb{E}\, X \geq C\, \mathbb{E}\, I_{X \geq C} = C \Pr(X \geq C).$$

Rearranging finishes the proof. □

Now we are ready to prove the main result.

**Theorem 5** (Erdős 1959)**.** *For any two integers $\ell$ and $k$, there exists a graph $G$ with* $\mathrm{girth}(G) > \ell$ *and* $\chi(G) > k$.

*Proof.* Consider a random graph $G = (V, E)$ drawn from $G(n, p)$ with $p = n^{\theta - 1}$ where $\theta < 1/\ell$. Thus $np = n^{\theta}$. The reason behind this choice will become clear soon. Let $X$ be the number of cycles of sizes at most $\ell$. For an ordered set $C = \{v_1, \cdots, v_t\}$, let $X_C$ be the indicator variable of the event that $C$ is a cycle; that is, $v_i v_{i+1} \in E$ for all $1 \leq i \leq t$ where $v_{t+1} = v_1$ by convention. As edges a chosen independently, we have that

$$\mathbb{E}\, X_C = \Pr(X_C = 1) = p^t.$$

Each cycle of size $t$ in $G$ corresponds to $2t$ ordered set $C$ ($t$ choices for the start and 2 choices for the direction). Hence,

$$\mathbb{E}\,X = \sum_{C,\,|C|\le \ell} \frac{1}{2\,|C|}\,\mathbb{E}\,X_C.$$

The number of ordered set $C$ of size $t$ is $n(n-1)(n-2)\cdots(n-t+1) \le n^t$ ($n$ choices for the first, $n-1$ for the second, and so on). Recall that $\theta\ell < 1$, implying

$$\begin{aligned}\mathbb{E}\,X &= \sum_{t=1}^{\ell} n(n-1)(n-2)\cdots(n-t+1)\cdot \frac{p^t}{2t} \\ &\le \sum_{t=1}^{\ell} \frac{(np)^t}{2t} = \sum_{t=1}^{\ell} \frac{n^{\theta t}}{2t} < \ell n^{\theta\ell} = o(n).\end{aligned}$$

To get a graph with girth $> \ell$, one way is to remove one vertex from each cycle, leaving a graph with at least $n - \ell n^{\theta\ell}$ vertices. Instead, we use the Markov inequality here; that is, by Theorem 4,

$$\Pr(X \ge n/2) \le \frac{\mathbb{E}\,X}{n/2} = 2\ell n^{\theta\ell-1} = o(1).$$

Thus, there is a sufficiently large $n$ such that $\Pr(X \ge n/2) \le 1/4$.

On the other hand, we want to obtain a graph with large chromatic number. We will upper bound $\alpha(G)$ and use (2). This is fine because when we remove vertices, the size of maximum independent set will not increase. If we bound the chromatic number directly instead, it may go down when we remove some vertices.

Set $t = \lceil 3/p \cdot \log n \rceil \ge 3/p \cdot \log n$. Then $ne^{-p(t-1)/2} < 1$ and

$$\begin{aligned}\Pr(\alpha(G) \ge t) &\le \binom{n}{t}(1-p)^{\binom{t}{2}} \le n^t e^{-pt(t-1)/2} \\ &= \left(ne^{-p(t-1)/2}\right)^t = o(1).\end{aligned}$$

Again, there is a sufficiently large $n$ such that $\Pr(\alpha(G) \ge t) \le 1/4$.

By a union bound,

$$\begin{aligned}\Pr(X < n/2 \wedge \alpha(G) < t) &= 1 - \Pr(X \ge n/2 \vee \alpha(G) \ge t) \\ &\ge 1 - \Pr(X \ge n/2) - \Pr(\alpha(G) \ge t) \\ &\ge 1 - 1/2 = 0.5 > 0.\end{aligned}$$

In particular, there exists a graph $G$ such that $X < n/2$ and $\alpha(G) < t = \lceil 3/p \cdot \log n \rceil$. For each cycle of length at most $\ell$, we remove at most one vertex to destroy it and get a new graph $G'$ such that $\mathrm{girth}(G') > \ell$. The size of $G'$ is at least $n - X \ge n/2$. Moreover, $\alpha(G') \le \alpha(G) < \lceil 3/p \cdot \log n \rceil$. By (2),

$$\chi(G') \ge \frac{|G'|}{\alpha(G')} \ge \frac{n/2}{3/p \cdot \log n} = \frac{n^\theta}{6 \log n}.$$

Thus, when $n$ is sufficiently large, $\chi(G') > k$. $\square$