Advanced Combinatorics - 2016 Fall
Exercise 1

You should be able to attempt the following problems after lecture 1.
Comments and corrections are welcome.

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1. Prove that for each $n \in \mathbb{N}$, if G is a triangle free graph with n vertices and $e(G) = \lfloor n^2/4 \rfloor$ then G is isomorphic to $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$.

2. Show that if G is a graph with n vertices, m edges, and t triangles, then
$$t \geq \frac{m}{3n}(4m - n^2).$$
Find an infinite family of graphs such that the above equality holds; that is,
$$t = \frac{m}{3n}(4m - n^2).$$

3. For each integer $n \geq 3$, what is the maximum possible number of edges in a graph with n vertices which contains exactly one triangle. Prove your answer.

4. Suppose that n and d are positive integers. Suppose that $x_1, \cdots, x_n$ are vectors in $\mathbb{R}^d$ with $|x_i| > 1$ for all $1 \leq i \leq n$. Show that the number of pairs (i, j) with $i < j$ and $|x_i + x_j| < 1$ is at most $\lfloor n^2/4 \rfloor$.

   (Here, $|x|$ is the Euclidean length of the vector $x \in \mathbb{R}^d$; namely
$$|x| = \sqrt{\sum_{k=1}^{d} x(k)^2},$$
   where $x(k)$ is the kth entry of the vector x.)