Advanced Combinatorics - 2016 Fall
Exercise 4

Comments and corrections are welcome.

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1. Let $r, n \in \mathbb{N}$ with $r \geq 2$. Let G be a graph with n vertices and with $\delta(G) \geq \lfloor (1 - \frac{1}{r})n \rfloor + 1$. Show that G contains a K_{r+1}.

2. Let $n \geq r \geq 2$ be two positive integers. Prove that
$$\left(1 - \frac{1}{r}\right) \frac{n^2}{2} \geq t_r(n) \geq \left(1 - \frac{1}{r}\right) \binom{n}{2}.$$

3. Let P be the Petersen graph, as shown in Figure 1. Show that $\pi(P) = 1/2$.

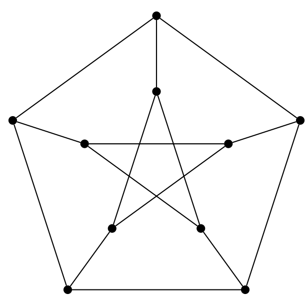

Figure 1: The Petersen graph

4. Let G be a graph with average degree D. That is,
$$\frac{1}{|V(G)|} \sum_{v \in V(G)} d(v) = D.$$
Show that G has a graph of minimum degree at least $D/2$.

5. We know that $\pi(A) = 1 - \frac{1}{r}$ if $\chi(A) = r + 1$. Thus $\lim_{n \to \infty} \frac{ex(n, K_{r+1})}{ex(n, A)} = 1$.

   Q: For each integer $r > 1$, find a graph A with $\pi(A) = 1 - \frac{1}{r}$, but $ex(n, A) > ex(n, K_{r+1})$.

6. We have seen that $\pi(K_{r+1}) = 1 - \frac{1}{r}$. On the other hand, it is not necessary to contain K_{r+1} to have density $1 - \frac{1}{r}$.

   Q: For each integer $r > 1$, find a graph B which is K_{r+1}-free but has $\pi(B) = 1 - \frac{1}{r}$.