Advanced Combinatorics - 2016 Fall
Exercise 5

The second part of Q3 would need some finite projective plane knowledge and should be doable after lecture 6. All other questions can be attempted after lecture 5.
Comments and corrections are welcome.

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1. Show that a C_4-free graph with 7 vertices has at most 26 edges.

2. Let G be a graph with n vertices. Show that we can cover all edges of G with a collection of a triangles of G and b edges of G, for some non-negative integers a and b such that $a + b \leq \lfloor n^2/4 \rfloor$.
   (By "cover" we mean that edges can be used multiple times, but all edges have to be used.)

   Hint: Do an induction on n. As an example, when G is triangle-free, we can pick $b = \lfloor n^2/4 \rfloor$ and $a = 0$.

3. Let H be a fixed bipartite graph and $n \in \mathbb{N}$. Let $\textit{bi-ex}(n, H)$ be the maximum number of edges in an H-free bipartite graph with vertex classes L and R such that $|L| = |R| = n$. Show that

   $$\textit{bi-ex}(n, C_4) \leq \frac{1}{2} n(\sqrt{4n-3} + 1).$$

   Moreover, construct an infinite family of graphs such that the equality holds.

4. (a) Let S be a subset of $\{1, 2, \ldots, \sqrt{n}\}$ with the property that no integer (not necessarily in S) can be written as the sum of two elements of S in more than one way. (It is okay that some integer is not the sum of any two elements of S.) Use the fact that

   $$ex(n, C_4) \leq \frac{n}{4}(\sqrt{4n-3} + 1)$$

   to prove that

   $$|S| \leq c_0 \sqrt{n},$$

   for some positive constant c_0.

   Hint: you need to construct an appropriate bipartite graph and then apply Erdős's Theorem.

   (b) Construct such an S whose order of magnitude is as large as you can.
   (The best possible is $\sqrt{n}$, but the construction is fairly complicated. A $\Omega(\log n)$ construction should be good enough.)