Advanced Combinatorics - 2016 Fall
Exercise 6

The first two questions require the basic probabilistic method. The next two require a little bit of deduction in addition.
Comments and corrections are welcome.

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1. If there is a real $0 \le p \le 1$ such that
$$\binom{n}{k} p^{\binom{k}{2}} + \binom{n}{\ell}(1-p)^{\binom{\ell}{2}} < 1,$$
then the Ramsey number $R(k, \ell)$ satisfies that $R(k, \ell) > n$.

   Using this, show that
$$R(4, \ell) \ge \Omega(\ell^{3/2} (\log \ell)^{-3/2}).$$
Hint: instead of the uniform distribution, consider choosing each edge with probability p.

2. Let $k \ge 4$ and H be a k-uniform hypergraph with at most $4^{k-1}/3^k$ edges. Prove that there is a 4-colouring of the vertices of H so that in every edge all four colours are represented.

3. Let S be a finite collection of binary strings. For example, S may contain elements like 000, 010101, etc. Assume that no member of S is a prefix of another. Let N_i denote the number of strings of length i in S. Show that
$$\sum_{i=1}^{\infty} \frac{N_i}{2^i} \le 1.$$
Hint: design a probability trial to generate elements from S.

4. A *planar graph* is one so that we can draw it on a plane without any crossings. By Euler's formula, for a planar graph G, $m \le 3n - 6$, where n is the number of vertices and m is the number of edges.

   On the other hand, if a graph (not necessarily planar) is drawn on a plane, then there are at least $m - 3n + 6 > m - 3n$ many crossings.

   Using the fact above, show that if $m \ge 4n$, then drawing G on a plane has at least $\frac{m^3}{64n^2}$ many crossings.
Hint: consider a random subgraph by choosing each vertex with probability p for some p.