

Advanced Combinatorics - 2016 Fall

Exercise 8

Questions 4 and 5 will need the Lovász Local Lemma.
Comments and corrections are welcome.

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1. Prove that there exists a two-edge-colouring of K_n with at most

$$\binom{n}{a} 2^{1-\binom{a}{2}}$$

monochromatic K_a .

2. Using the alteration method, prove that the Ramsey number $R(4, k)$ satisfies

$$R(4, k) \geq \Omega((k/\log k)^2)$$

3. An subset S of vertices in a hypergraph $H = (V, E)$ is *independent* if there is no $e \in E$ such that $e \subseteq S$. In other words, S does not completely contain any (hyper-)edge.

Prove that every 3-uniform hypergraph with n vertices and $m \geq n/3$ edges contains an independent set of size at least

$$\frac{2n^{3/2}}{3\sqrt{3m}}.$$

4. Let $G = (V, E)$ be a graph. Associate each $v \in V$ a list $S(v)$ of colours of size at least $10d$ for some $d \geq 1$. Moreover, suppose that for each $v \in V$ and $c \in S(v)$, there are at most d neighbours u of v such that $c \in S(u)$.

Prove that there is a proper colouring of G assigning to each vertex v a colour from its list $S(v)$.

5. Let $G = (V, E)$ be a cycle of length $11n$, and $V = V_1 \cup V_2 \cup \dots \cup V_n$ be an arbitrary partition of its $11n$ vertices; that is $V_i \cap V_j = \emptyset$ for any $1 \leq i \neq j \leq n$. Moreover, $|V_i| = 11$ for every $i \in [n]$.

Prove that there exists an independent set of G that contains precisely one vertex from each V_i .